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EQUATIONS $AX = Y$ AND $Ax = y$ IN ALG$\mathcal{L}$

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Abstract. Let $\mathcal{L}$ be a subspace lattice on a Hilbert space $\mathcal{H}$ and X and Y be operators acting on a Hilbert space $\mathcal{H}$. Let P be the projection onto $\overline{\mathcal{R}(X)}$, where $\mathcal{R}X$ is the range of X. If $PE = EP$ for each $E \in \mathcal{L}$, then there exists an operator A in Alg$\mathcal{L}$ such that $AX = Y$ if and only if

$$\sup\{\|E^{\perp}Yf\|/\|E^{\perp}Xf\| : f \in \mathcal{H},\ E \in \mathcal{L}\} = K < \infty.$$

Moreover, if the necessary condition holds, then we may choose an operator A such that $AX = Y$ and $\|A\| = K$.

Let x and y be vectors in $\mathcal{H}$ and let P_x be the projection onto the singlely generated space by x. If $P_xE = EP_x$ for each $E \in \mathcal{L}$, then the assertion that there exists an operator A in Alg$\mathcal{L}$ such that $Ax = y$ is equivalent to the condition

$$K_0 := \sup\{\|E^{\perp}y\|/\|E^{\perp}x\| :\ E \in \mathcal{L}\} < \infty.$$

Moreover, we may choose an operator A such that $\|A\| = K_0$ whose norm is K_0 under this case.

1. Introduction

Let $\mathcal{H}$ be a Hilbert space and $\mathcal{B}(\mathcal{H})$ be the class of all bounded operators acting on $\mathcal{H}$. A subspace is a closed linear manifold and we identify the subspace with the orthogonal projection whose range it is. If $\mathcal{L}$ is a collection of projections that is closed under the operations of meet and join, then it is a lattice. A *subspace lattice* $\mathcal{L}$ is a strongly closed lattice of orthogonal projections on $\mathcal{H}$ containing the trivial projections 0 and I. The symbol Alg$\mathcal{L}$ denotes the algebra of bounded operators on

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$\mathcal{H}$ that leave invariant all projections in $\mathcal{L}$; Alg$\mathcal{L}$ is a weakly closed subalgebra of $\mathcal{B}(\mathcal{H})$. A lattice $\mathcal{L}$ is a *commutative subspace lattice*, or CSL, if the projections in $\mathcal{L}$ all commute; in this case, Alg$\mathcal{L}$ is called a *CSL algebra*. Let $x_1, \ldots, x_n$ be vectors of $\mathcal{H}$. Then the generating space by $\{x_1, \ldots, x_n\}$ is the set $\{\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n : \alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{C}\}$. Let M be a subset of $\mathcal{H}$. Then $\overline{M}$ means the closure of M and $\overline{M}^{\perp}$ the orthogonal complement of $\overline{M}$. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{C}$ be the set of complex numbers.

Let X and Y be operators acting on $\mathcal{H}$ and $\mathcal{A}$ be a subalgebra of $\mathcal{B}(\mathcal{H})$. An interpolation problem for $\mathcal{A}$ is that given two operators X and Y, under what conditions can we be sure that there is an operator A in $\mathcal{A}$ such that $AX = Y$? And we consider the above problem for finite and countable operators X_i and $Y_i (i = 1, 2, \ldots)$.

Given two vectors x and y in $\mathcal{H}$, when is there an operator A in $\mathcal{A}$ that maps x to y? In [3], the problem is studied under the conditions that $\mathcal{L}$ is a commutative subspace lattice on $\mathcal{H}$ and X is an operator of rank 0 or 1 acting on $\mathcal{H}$. Let $\mathcal{R}(X)$ be the range of X. In [5] and [6], authors considered it when $\mathcal{R}(X)$ is dense in $\mathcal{H}$ and have investigated the interpolation problem for Alg$\mathcal{L}$ with strong desire for finding weaker conditions than the dense range of X. In this paper, we obtain a condition that we wanted to find. We introduce a history of interpolation problems to know how it is developed.

The simplest case of the operator interpolation problem relaxes all restrictions on A, requiring it simply to be a bounded operator. In this case, the existence of A is nicely characterized by the well-known factorization theorem.

THEOREM A. (R. G. Douglas [2]) *Let X and Y be bounded operators acting on a Hilbert space $\mathcal{H}$. Then the following statements are equivalent:*

(1) $\mathcal{R}(Y^*) \subseteq \mathcal{R}(X^*)$;

(2) $Y^*Y \leq \lambda^2 X^*X$ *for some* $\lambda \geq 0$;

(3) *there exists a bounded operator A on $\mathcal{H}$ so that $AX = Y$.*

Moreover, if (1), (2), *and* (3) *are valid, then there exists a unique operator A so that*

(a) $\|A\|^2 = \inf\{\mu : Y^*Y \leq \mu X^*X\}$

(b) $\ker Y^* = \ker A^*$ *and*

(c) $\mathcal{R}(A^*) \subseteq \mathcal{R}(X^-)$.

We need to look at the proof of Theorem A carefully. Then we know that the image of A on $\overline{\mathcal{R}(X)}^{\perp}$ is 0 from the proof of (3) by (2).

In [8], Katsoulis, Moore and Trent found a necessary and sufficient condition of the existence of an interpolation operator A for a nest algebra. That is, the following:

$$(*) \qquad \sup\{\|E^{\perp}Yf\|/\|E^{\perp}Xf\| : f \in \mathcal{H} \text{ and } E \in \mathcal{N}\} = K < \infty,$$

where we use the convention $\frac{0}{0} = 0$, when necessary.

In [3], Hopenwasser found a result that, if X is rank-one or zero, then the condition $(*)$ is also sufficient for a commutative subspace lattice on $\mathcal{H}$. In [10], Moore and Trent showed that the condition $(*)$ is a necessary and sufficient condition that there exists an operator A in Alg$\mathcal{L}$ such that $AX = Y$ where $\mathcal{L}$ is a commutative subspace lattice. In [5], Jo and Kang obtained a necessary and sufficient condition for the interpolation problem in Alg$\mathcal{L}$.

THEOREM B. (Jo, Kang [5]) *Let $\mathcal{L}$ be a subspace lattice on $\mathcal{H}$. Let X and Y be operators in $\mathcal{B}(\mathcal{H})$. Assume that $\mathcal{R}(X)$ is dense in $\mathcal{H}$. Then the following statements are equivalent*:

(1) *There exists an operator A in Alg$\mathcal{L}$ such that $AX = Y$ and every E in $\mathcal{L}$ reduces A.*

(2) $\sup\{\|\sum_{i=1}^{n} E_i Y f_i\|/\|\sum_{i=1}^{n} E_i X f_i\| : n \in \mathbb{N}, E \in \mathcal{L}$ *and* $f_i \in \mathcal{H}\} < \infty.$

In [6], the authors showed the condition $(*)$ is a necessary and sufficient condition for the interpolation problem in Alg$\mathcal{L}$ when $\mathcal{R}(X)$ is dense in $\mathcal{H}$ and $\mathcal{L}$ is a subspace lattice on $\mathcal{H}$.

2. The equation $AX = Y$ in Alg$\mathcal{L}$

We use the convention $\frac{0}{0} = 0$, when necessary.

THEOREM 2.1. *Let $\mathcal{L}$ be a subspace lattice on $\mathcal{H}$ and let X and Y be operators acting on $\mathcal{H}$. Let P be the projection onto the $\overline{\mathcal{R}(X)}$. If $PE = EP$ for each $E \in \mathcal{L}$, then the following are equivalent*:

(1) *There exists an operator A in Alg$\mathcal{L}$ such that $AX = Y$.*

(2) $\sup\{\|E^{\perp}Yf\|/\|E^{\perp}Xf\| : f \in \mathcal{H},\ E \in \mathcal{L}\} = K < \infty.$

Proof. Assume that $\sup\{\|E^{\perp}Yf\|/\|E^{\perp}Xf\| : f \in \mathcal{H}, E \in \mathcal{L}\} = K < \infty$. Then for each E in $\mathcal{L}$, there exists an operator A_E in $\mathcal{B}(\mathcal{H})$ such that $A_E(E^{\perp}X) = E^{\perp}Y$ and $\|A_E\| \leq K$ by Theorem A. In particular, if

$E = 0$, then we have an operator A_0 in $\mathcal{B}(\mathcal{H})$ such that $A_0X = Y$ and $\|A_0\| \leq K$. So $A_E(E^\perp X) = E^\perp Y = E^\perp A_0 X$. Since $PE = EP$ for each $E \in \mathcal{L}$, $A_E E^\perp = E^\perp A_0$ on $\mathcal{H}$. For, since $A_E(E^\perp X) = E^\perp Y = E^\perp(A_0 X)$, $A_E E^\perp = E^\perp A_0$ on $\overline{\mathcal{R}(X)}$ for each E in $\mathcal{L}$. Let E be in $\mathcal{L}$ and $f \in \overline{\mathcal{R}(X)}^\perp$. Then for any $g \in \mathcal{H}$, $\langle f, E^\perp X g\rangle = \langle f, E^\perp P X g\rangle = \langle f, P E^\perp X g\rangle = 0$. Also by the definition of A_0, $E^\perp A_0 f = 0$. Hence $A_E E^\perp h = 0 = E^\perp A_0 h$ for $h \in (\overline{\mathcal{R}(X)}^\perp \cap E^\perp) + (\overline{\mathcal{R}(X)}^\perp \cap E)$. Since $PE = EP$, $P^\perp E = EP^\perp$, $P^\perp E$ and $P^\perp E^\perp$ are projections onto $\overline{\mathcal{R}(X)}^\perp$ $\cap E$ and $\overline{\mathcal{R}(X)}^\perp \cap E^\perp$, respectively. Hence

$$\begin{aligned}(\overline{\mathcal{R}(X)}^\perp \cap E^\perp) + (\overline{\mathcal{R}(X)}^\perp \cap E) &= P^\perp E^\perp + P^\perp E \\ &= P^\perp E^\perp \oplus P^\perp E \\ &= P^\perp(E^\perp \oplus E) = P^\perp.\end{aligned}$$

Therefore $A_E E^\perp = E^\perp A_0$ on $\mathcal{H}$.

For each E in $\mathcal{L}$,

$$E^\perp A_0 E^\perp = A_E E^\perp E^\perp = A_E E^\perp = E^\perp A_0 .$$

So A_0 is an operator in Alg$\mathcal{L}$.

If (1) is assumed, then $AX = Y$ and $E^\perp A E^\perp = E^\perp A$ for all $E \in \mathcal{L}$. So for any vector f, since $E^\perp A E^\perp X = E^\perp Y$, $E^\perp A E^\perp X f = E^\perp Y f$. Hence

$$\|E^\perp Y f\| \leq \|E^\perp A\| \|E^\perp X f\| \leq \|A\| \|E^\perp X f\|$$

for all $E \in \mathcal{L}$ and all $f \in \mathcal{H}$. So

$$\sup\{\|E^\perp Y f\| / \|E^\perp X f\| : f \in \mathcal{H},\ E \in \mathcal{L}\} < \infty. \qquad \square$$

In [10], authors induced the second condition of Theorem 2.1 for the interpolation problem in Alg$\mathcal{L}$ under the condition that $\mathcal{R}(X)$ is dense in $\mathcal{H}$ and $\mathcal{L}$ is a commutative subspace lattice. We proved it without the dense range condition and commutative condition of $\mathcal{L}$ in Theorem 2.1. We give two examples to compare a difference between the Theorem in [10] and Theorem 2.1.

Let P be the projection onto the $\overline{\mathcal{R}(X)}$.

EXAMPLE 1. Let $\mathcal{H}$ be a Hilbert space with an orthonormal basis $\{e_1, e_2, e_3, \dots\}$. Let $\mathcal{L}_\infty$ be the subspace lattice generated by $\{[e_{2k-1}],$

$[e_{2k-1}, e_{2k}, e_{2k+1}] : k = 1, 2, \dots\}$. Then Alg$\mathcal{L}_\infty$ is the collection of all matrices of the following form

$$\begin{pmatrix} * & * & & & & & & & \\ & * & & & & & & & \\ & * & * & * & & & & & \\ & & & * & & & & & \\ & & & * & * & * & & & \\ & & & & & * & & & \\ & & & & & * & * & * & \\ & & & & & & & * & \\ & & & & & & & & \ddots \end{pmatrix},$$

where non-starred entries are zero.

Alg$\mathcal{L}_\infty$ is called a *tridiagonal algebra.* If $\mathcal{R}(X)$ is dense, then $P = I$. But if X is an operator and $PE = EP$ for every E in $\mathcal{L}$, then all of off-diagonal entries are zero.

EXAMPLE 2. Let $\mathcal{H} = [e_1, e_2, e_3, \dots]$ and let $E_0 = 0$ and $E_n = [e_1, e_2, \dots, e_n]$. Let $\mathcal{L}$ be the subspace lattice generated by $\{E_n \mid n = 0, 1, 2, \dots\}$. Then $A \in$ Alg$\mathcal{L}$ iff A has the form

$$\begin{pmatrix} * & * & * & * & * & * & \\ * & * & * & * & * & * & \\ * & * & * & * & * & * & \\ & & & * & * & * & \\ & & & * & * & * & \\ & & & * & * & * & \\ & & & & & & \ddots \end{pmatrix},$$

where non-starred entries are zero.

If $\mathcal{R}(X)$ is dense, then $P = I$. And if $PE = EP$ for every E in $\mathcal{L}$, then P has the following form

$$\begin{pmatrix} * & * & * & & & & \\ * & * & * & & & & \\ * & * & * & & & & \\ & & & * & * & * & \\ & & & * & * & * & \\ & & & * & * & * & \\ & & & & & & \ddots \end{pmatrix},$$

where non-starred entries are zero.

THEOREM 2.2. *Let $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ be bounded operators acting on $\mathcal{H}$. Let P_j be the projection onto $\overline{\mathcal{R}(X_j)}$ for all $j = 1, 2, \ldots, n$. If $P_k E = E P_k$ for some k in $\{1, 2, \ldots, n\}$, then the following are equivalent:*

(1) *There exists an operator A in Alg$\mathcal{L}$ such that $AX_i = Y_i$ for $i = 1, 2, \ldots, n$.*

(2) $\sup\{\|E^{\perp}(\sum_{i=1}^{n} Y_i f_i)\| / \|E^{\perp}(\sum_{i=1}^{n} X_i f_i)\| : f_i \in \mathcal{H},\ E \in \mathcal{L}\} = K < \infty.$

THEOREM 2.3. *Let X_i and Y_i be bounded operators acting on $\mathcal{H}$ for all $i = 1, 2, \ldots$. Let P_j be the projection onto $\overline{\mathcal{R}(X_j)}$ for all $j = 1, 2, \ldots$. If $P_k E = E P_k$ for some k in $\mathbb{N}$, then the following are equivalent:*

(1) *There exists an operator A in Alg$\mathcal{L}$ such that $AX_i = Y_i$ for $i = 1, 2, \ldots$.*

(2) $\sup\{\|E^{\perp}(\sum_{i=1}^{m} Y_i f_i)\| / \|E^{\perp}(\sum_{i=1}^{m} X_i f_i)\| : f_i \in \mathcal{H},\ E \in \mathcal{L},\ m \in \mathbb{N}\} = K < \infty.$

Proof. If we assume that there is an operator A in Alg$\mathcal{L}$ such that $AX_i = Y_i$ for $i = 1, 2, \ldots$, then for $f_i \in \mathcal{H}$, $E \in \mathcal{L}$ and $m \in \mathbb{N}$,

$$\begin{aligned} \|E^{\perp}(\sum_{i=1}^{m} Y_i f_i)\| = \|E^{\perp}(\sum_{i=1}^{m} AX_i f_i)\| &= \|E^{\perp}A(\sum_{i=1}^{m} X_i f_i)\| \\ &\leq \|E^{\perp}AE^{\perp}(\sum_{i=1}^{m} X_i f_i)\| \\ &\leq \|E^{\perp}A\|\|E^{\perp}(\sum_{i=1}^{m} X_i f_i)\| \end{aligned}$$

$$\leq \|A\| \|E^{\perp}(\sum_{i=1}^{m} X_i f_i)\|.$$

If $\|E^{\perp}(\sum_{i=1}^{m} X_i f_i)\|$ is not zero, then

$$\|E^{\perp}(\sum_{i=1}^{m} Y_i f_i)\| / \|E^{\perp}(\sum_{i=1}^{m} X_i f_i)\| < \|A\|.$$

Otherwise, we use the convention $\frac{0}{0} = 0$. So

$$\sup\{\|E^{\perp}(\sum_{i=1}^{m} Y_i f_i)\| / \|E^{\perp}(\sum_{i=1}^{m} X_i f_i)\| : f_i \in \mathcal{H},\ E \in \mathcal{L},\ m \in \mathbb{N}\} < \infty.$$

Conversely, we assume that

$$\sup\{\|E^{\perp}(\sum_{i=1}^{m} Y_i f_i)\| / \|E^{\perp}(\sum_{i=1}^{m} X_i f_i)\| : f_i \in \mathcal{H},\ E \in \mathcal{L}, m \in \mathbb{N}\}$$
$$= K < \infty.$$

Then for each E in $\mathcal{L}$, there exists an operator A_E in $\mathcal{B}(\mathcal{H})$ such that $A_E(E^{\perp}X_i) = E^{\perp}Y_i$ for $i = 1, 2, \ldots$ and $\|A_E\| \leq K$ by Theorem A. In particular, if $E = 0$, then we have an operator A_0 in $\mathcal{B}(\mathcal{H})$ such that $A_0 X_i = Y_i$ for $i = 1, 2, \ldots$ and $\|A_0\| \leq K$. So $A_E(E^{\perp}X_i) = E^{\perp}Y_i = E^{\perp}A_0 X_i$. Since $P_k E = E P_k$ for some $k \in \{1, 2, \ldots\}$, for each $E \in \mathcal{L}$, $A_E E^{\perp} = E^{\perp} A_0$ on $\mathcal{H}$. For, since $A_E(E^{\perp}X_i) = E^{\perp}Y_i = E^{\perp}(A_0 X_i)$, $A_E E^{\perp} = E^{\perp} A_0$ on $\overline{\mathcal{R}(X_i)}$ for each E in $\mathcal{L}$ and $i = 1, 2, \ldots$. We assumed that $P_k E = E P_k$ for some $k \in \{1, 2, \ldots\}$. Let E be in $\mathcal{L}$ and $f \in \overline{\mathcal{R}(X_k)}^{\perp}$. Then for any $g \in \mathcal{H}$, $\langle f, E^{\perp} X_k g\rangle = \langle f, E^{\perp} P_k X_k g\rangle = \langle f, P_k E^{\perp} X_k g\rangle = 0$. Also by the definition of A_0, $E^{\perp} A_0 f = 0$. Hence $A_E E^{\perp} f = 0 = E^{\perp} A_0 f$ for $f \in (\overline{\mathcal{R}(X_k)}^{\perp} \cap E^{\perp}) + (\overline{\mathcal{R}(X_k)}^{\perp} \cap E)$. Since $P_k E = E P_k$, $P_k^{\perp} E = E P_k^{\perp}$, $P_k^{\perp} E$ and $P_k^{\perp} E^{\perp}$ are projections onto $\overline{\mathcal{R}(X_k)}^{\perp} \cap E$ and $\overline{\mathcal{R}(X_k)}^{\perp} \cap E^{\perp}$, respectively. Hence

$$\begin{aligned}(\overline{\mathcal{R}(X_k)}^{\perp} \cap E^{\perp}) + (\overline{\mathcal{R}(X_k)}^{\perp} \cap E) &= P_k^{\perp} E^{\perp} + P_k^{\perp} E \\ &= P_k^{\perp} E^{\perp} \oplus P_k^{\perp} E \\ &= P_k^{\perp}(E^{\perp} \oplus E) = P_k^{\perp}.\end{aligned}$$

Therefore $A_E E^{\perp} = E^{\perp} A_0$ on $\mathcal{H}$.

For each E in $\mathcal{L}$,

$$E^{\perp} A_0 E^{\perp} = A_E E^{\perp} E^{\perp} = A_E E^{\perp} = E^{\perp} A_0 .$$

So A_0 is an operator in Alg$\mathcal{L}$. □

3. The equation $Ax = y$ in Alg$\mathcal{L}$

Assume that x and y are vectors in $\mathcal{H}$ and A is an operator in Alg$\mathcal{L}$ such that $Ax = y$. Then $\|E^\perp y\| = \|E^\perp Ax\| = \|E^\perp AE^\perp x\| \leq \|A\|\|E^\perp x\|$ for all $E \in \mathcal{L}$. If, for convenience, we adopt the convention that a fraction whose numerator and denominator are both zero is equal to zero, then the above inequality may be stated in the form

$$\sup_{E\in\mathcal{L}} \frac{\|E^\perp y\|}{\|E^\perp x\|} \leq \|A\|.$$

We consider the above fact when $\mathcal{L}$ is a subspace lattice without the commutative condition.

LEMMA 3.1. *Let X and Y be operators in $\mathcal{B}(\mathcal{H})$ and let x and y be vectors in $\mathcal{H}$. Then the following are equivalent:*

(1) *There exists an operator A in $\mathcal{B}(\mathcal{H})$ such that $AXx = Yy$.*

(2) $\|Yy\| \leq \lambda_0\|Xx\|$, *where* $\lambda_0 = \inf\{\lambda : \|Yy\| \leq \lambda\|Xx\|\}$.

Moreover, if condition (2) *holds, we may choose an operator A such that* $\|A\| = \lambda_0$.

Proof. Assume that

$$\|Yy\| \leq \lambda_0\|Xx\|,$$

where $\lambda_0 = \inf\{\lambda : \|Yy\| \leq \lambda\|Xx\|\}$. Let $\mathcal{M} = \{\alpha Xx : \alpha \in \mathbb{C}\}$. Then $\mathcal{M}$ is a linear manifold. Define $A : \mathcal{M} \to \mathcal{H}$ by $A(\alpha Xx) = \alpha Yy$. Then A is well-defined by the assumption. Define $Ag = 0$ for all g in $\overline{\mathcal{M}}^\perp$. Then A is an operator acting on $\mathcal{H}$ and $AXx = Yy$.

If $Xx \neq 0$, then

$$\|A\| = \frac{\|Yy\|}{\|Xx\|} \leq \lambda_0$$

Since $\|Yy\| = \|AXx\| \leq \|A\|\|Xx\|$, $\lambda_0 \leq \|A\|$. So $\|A\| = \lambda_0$.

The proof of the converse is obvious. □

LEMMA 3.2. *Let $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ be vectors in $\mathcal{H}$ and let X and Y be operators in $\mathcal{B}(\mathcal{H})$. Then the following are equivalent:*

(1) *There exists an operator A in $\mathcal{B}(\mathcal{H})$ such that $AXx_i = Yy_i$ for $i = 1, 2, \ldots, n$.*

(2) $\sup\{\|\sum_{i=1}^n \alpha_i Yy_i\|/\|\sum_{i=1}^n \alpha_i Xx_i\| : \alpha_i \in \mathbb{C}\} = \lambda < \infty$.

Moreover, if condition (2) *holds, we may choose an operator A such that* $\|A\| = \lambda$.

Proof. Assume that

$$\sup\{\|\sum_{i=1}^{n}\alpha_i Y y_i\|/\|\sum_{i=1}^{n}\alpha_i X x_i\| : \alpha_i \in \mathbb{C}\} = \lambda < \infty.$$

Then

$$(**) \qquad \|\sum_{i=1}^{n}\alpha_i Y y_i\| \leq \lambda \|\sum_{i=1}^{n}\alpha_i X x_i\| \quad \text{for } \alpha_i \in \mathbb{C}.$$

Let $\mathcal{N}_X = \{\sum_{i=1}^{n}\alpha_i X x_i : \alpha_i \in \mathbb{C}\}$. Then $\mathcal{N}_X$ is a linear manifold. Define $A : \mathcal{N}_X \to \mathcal{H}$ by

$$A(\sum_{i=1}^{n}\alpha_i X x_i) = \sum_{i=1}^{n}\alpha_i Y y_i.$$

Then A is well-defined by $(**)$. Define $Ag = 0$ for all g in $\overline{\mathcal{N}_X}^{\perp}$. Then A is an operator in $\mathcal{B}(\mathcal{H})$ and $AXx_i = Yy_i (i = 1, 2, \ldots, n)$. If $\sum_{i=1}^{n}\alpha_i X x_i \neq 0$, then

$$\|A\| = \sup\{\|\sum_{i=1}^{n}\alpha_i Y x_i\|/\|\sum_{i=1}^{n}\alpha_i X x_i\| \ : \ \alpha_i \in \mathbb{C}\} \leq \lambda.$$

Since

$$\|\sum_{i=1}^{n}\alpha_i Y y_i\| = \|A(\sum_{i=1}^{n}\alpha_i X x_i)\| \leq \|A\|\|\sum_{i=1}^{n}\alpha_i X x_i\|, \ \lambda \leq \|A\|.$$

So $\|A\| = \lambda$. The proof of the converse is obvious. □

We can extend this fact to the countable case.

LEMMA 3.3. *Let $x_1, x_2, \ldots$ and $y_1, y_2, \ldots$ be vectors in $\mathcal{H}$ and X and Y be operators in $\mathcal{B}(\mathcal{H})$. Then the following are equivalent:*

(1) *There exists an operator A in $\mathcal{B}(\mathcal{H})$ such that $AXx_n = Yy_n$ for $n \in \mathbb{N}$.*

(2) $\sup\{\|\sum_{i=1}^{n}\alpha_i Y y_i\|/\|\sum_{i=1}^{n}\alpha_i X x_i\| : \alpha_i \in \mathbb{C}$ *and* $n \in \mathbb{N}\} = \lambda < \infty$.

Moreover, if condition (2) *holds, we may choose an operator A such that $\|A\| = \lambda$.*

From Lemmas 3.1, 3.2 and 3.3, we can obtain generalized results on the interpolation problems in Alg$\mathcal{L}$.

THEOREM 3.4. *Let $\mathcal{L}$ be a subspace lattice on $\mathcal{H}$ and let x and y be vectors in $\mathcal{H}$. Let P_x be the projection onto $sp(x)$. If $P_xE = EP_x$ for each $E \in \mathcal{L}$, then the following are equivalent:*

(1) *There exists an operator A in Alg$\mathcal{L}$ such that $Ax = y$.*

(2) $\sup\{\|E^\perp y\|/\|E^\perp x\| : E \in \mathcal{L}\} = K_0 < \infty$.

Moreover, if condition (2) holds, we may choose an operator A such that $\|A\| = K_0$.

Proof. Assume that $\sup\{\|E^\perp y\|/\|E^\perp x\| : E \in \mathcal{L}\} = K_0 < \infty$. By Lemma 3.1, for each E in $\mathcal{L}$, there exists an operator A_E in $\mathcal{B}(\mathcal{H})$ such that $A_E(E^\perp x) = E^\perp y$ and $\|A_E\| \leq K_0$. In particular, if $E = 0$, then we have an operator A_0 in $\mathcal{B}(\mathcal{H})$ such that $A_0 x = y$ and $\|A_0\| \leq K_0$. Then for each $E \in \mathcal{L}$, $A_E E^\perp x = E^\perp y = E^\perp A_0 x$. Hence $A_E E^\perp = E^\perp A_0$ on sp(x). And for each E in $\mathcal{L}$, $A_E E^\perp h = 0$ for h in sp$(x)^\perp$.
For, let E be in $\mathcal{L}$ and h be in $sp(x)^\perp$. Then

$$\begin{aligned}\langle E^\perp h, E^\perp x\rangle &= \langle h, E^\perp x\rangle = \langle h, E^\perp P_x x\rangle \\ &= \langle h, P_x E^\perp x\rangle = \langle P_x h, E^\perp x\rangle = 0.\end{aligned}$$

Hence $E^\perp h \in$ sp$(E^\perp x)^\perp$ for $h \in$ sp$(x)^\perp$. Since $A_E\, g = 0$ for all g in sp$(E^\perp x)^\perp$, $A_E E^\perp h = 0$.

By the definition of the operator A_0, $E^\perp A_0 h = 0$ for $h \in$ sp$(x)^\perp$. Hence $A_E E^\perp h = 0 = E^\perp A_0 h$ for $h \in$ sp$(x)^\perp$. So $A_E E^\perp = E^\perp A_0$. Let E be in $\mathcal{L}$. Then

$$E^\perp A_0 E^\perp = A_E E^\perp E^\perp = A_E E^\perp = E^\perp A_0 .$$

So A_0 is an operator in Alg$\mathcal{L}$. Since $A_0 x = y$ and $E^\perp A_0 E^\perp = E^\perp A_0$,

$$\|E^\perp y\| = \|E^\perp A_0 x\| = \|E^\perp A_0 E^\perp x\| \leq \|A_0\|\|E^\perp x\|.$$

So $K_0 \leq \|A_0\|$. Therefore $\|A_0\| = K_0$.

The proof of the converse is obvious. □

THEOREM 3.5. *Let $\mathcal{L}$ be a subspace lattice on $\mathcal{H}$ and let $x_1, x_2, \ldots, x_n$ and $y_1, y_2, \ldots, y_n$ be vectors in $\mathcal{H}$. Let $\mathcal{M} = \{\sum_{i=1}^{n} \alpha_i x_i : \alpha_i \in \mathbb{C}\}$ and $P_\mathcal{M}$ be the projection onto $\mathcal{M}$. If $P_\mathcal{M} E = EP_\mathcal{M}$ for each $E \in \mathcal{L}$, then the following are equivalent:*

(1) *There exists an operator A in Alg$\mathcal{L}$ such that $Ax_i = y_i$ for $i = 1, 2, \dots, n$.*

(2) $\sup\{\|\sum_{i=1}^{n} \alpha_i E^{\perp} y_i\| / \|\sum_{i=1}^{n} \alpha_i E^{\perp} x_i\| : \ E \in \mathcal{L}, \ \alpha_i \in \mathbb{C}\} = K_0 < \infty.$

Moreover, if condition (2) holds, we may choose an operator A such that $\|A\| = K_0$.

Proof. Suppose that

$$\sup\{\|\sum_{i=1}^{n} \alpha_i E^{\perp} y_i\| / \|\sum_{i=1}^{n} \alpha_i E^{\perp} x_i\| : \ E \in \mathcal{L}, \ \alpha_i \in \mathbb{C}\} = K_0 < \infty.$$

Let E be in $\mathcal{L}$. Then by Lemma 3.2, there exists an operator A_E in $\mathcal{B}(\mathcal{H})$ such that $A_E E^{\perp} x_i = E^{\perp} y_i$ for $i = 1, 2, \dots, n$. If $E = 0$, then $A_0 x_i = y_i$ for $i = 1, 2, \dots, n$. So $A_E E^{\perp} x_i = E^{\perp} y_i = E^{\perp} A_0 x_i$ for each $i = 1, 2, \dots, n$. Hence

$$\begin{aligned} A_E E^{\perp}(\sum_{i=1}^{n} \alpha_i x_i) &= \sum_{i=1}^{n} \alpha_i A_E E^{\perp} x_i = \sum_{i=1}^{n} \alpha_i E^{\perp} y_i \\ &= \sum_{i=1}^{n} \alpha_i E^{\perp} A_0 x_i \\ &= E^{\perp} A_0 (\sum_{i=1}^{n} \alpha_i x_i). \end{aligned}$$

Therefore $A_E E^{\perp} = E^{\perp} A_0$ on $\mathcal{M}$.

Let $h \in \mathcal{M}^{\perp}$. Then

$$\begin{aligned} \left\langle E^{\perp} h, \sum_{i=1}^{n} E^{\perp} x_i \right\rangle &= \sum_{i=1}^{n} \langle h, E^{\perp} x_i \rangle \\ &= \sum_{i=1}^{n} \langle h, E^{\perp} P_{\mathcal{M}} x_i \rangle \\ &= \sum_{i=1}^{n} \langle h, P_{\mathcal{M}} E^{\perp} x_i \rangle = \sum_{i=1}^{n} \langle P_{\mathcal{M}} h, E^{\perp} x_i \rangle = 0. \end{aligned}$$

So $E^{\perp} h \in \mathcal{N}^{\perp}$, where $\mathcal{N} = sp\{E^{\perp} x_1, E^{\perp} x_2, \dots, E^{\perp} x_n\}$. Since $A_E g = 0$ for all g in $\mathcal{N}^{\perp}$, $A_E E^{\perp} h = 0$.

Hence $E^\perp A_0 = A_E E^\perp$ for all E in $\mathcal{L}$.
For each $E \in \mathcal{L}$,

$$E^\perp A_0 E^\perp = A_E E^\perp E^\perp = A_E E^\perp = E^\perp A_0.$$

Hence $A_0 \in \text{Alg}\mathcal{L}$. Moreover, since $A_0 x_i = y_i$ for $i = 1, 2, \ldots, n$ and $E^\perp A_0 E^\perp = E^\perp A_0$,

$$\begin{aligned} \|\sum_{i=1}^n \alpha_i E^\perp y_i\| &= \|\sum_{i=1}^n \alpha_i E^\perp A_0 x_i\| \\ &= \|\sum_{i=1}^n \alpha_i E^\perp A_0 E^\perp x_i\| \\ &= \|E^\perp A_0 (\sum_{i=1}^n \alpha_i E^\perp x_i)\| \le \|A_0\| \|\sum_{i=1}^n \alpha_i E^\perp x_i\|. \end{aligned}$$

So $K_0 \le \|A_0\|$. Hence $\|A_0\| = K_0$. □

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