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TOPOLOGICAL ENTROPY OF A SEQUENCE OF MONOTONE MAPS ON CIRCLES

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ABSTRACT. In this paper, we prove that the topological entropy of a sequence of equi-continuous monotone maps $f_{1,\infty} = \{f_i\}_{i=1}^{\infty}$ on circles is $h(f_{1,\infty}) = \limsup_{n\to\infty} \frac{1}{n} \log \prod_{i=1}^{n} |\deg f_i|$. As applications, we give the estimation of the entropies for some skew products on annular and torus. We also show that a diffeomorphism f on a smooth 2-dimensional closed manifold and its extension on the unit tangent bundle have the same entropy.

1. Introduction

The concept of topological entropy was originally introduced by Adler, Konheim, and Mcandrew [1] as an invariant of topological conjugacy and a numerical measure for the complexity of a dynamical system. Later, Bowen [2] and Dinaburg [3] gave an equivalent definition when the space under consideration is metrizable. We can see [12] for the definition and main properties of it. With the development of the study of nonautonomous dynamical systems, recently, Kolyada and Snoha [7] introduced and studied the notion of topological entropy for a sequence of endomorphisms of a compact topological space. For other recent results about entropy one can see [4], [9], [11], etc.

The systems on circle play an important role in the study of one dimensional dynamical systems. In [5] and [12] the authors studied the entropies of homeomorphism and monotone continuous map on circle respectively. Our purpose is to study the topological entropy of a sequence

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of monotone maps on circles. In section 2, by estimating the cardinal of the spanning set and the separated set, we prove that the topological entropy of a sequence of equi-continuous monotone maps $f_{1,\infty} = \{f_i\}_{i=1}^{\infty}$ is $h(f_{1,\infty}) = \limsup_{n\to\infty} \frac{1}{n} \log \prod_{i=1}^{n} |\deg f_i|$. In section 3, as applications, we give the estimation of the entropies for some skew products on annular and torus. We also show that a C^1 diffeomorphism f on a smooth 2-dimensional closed manifold M and its extension $D^{\sharp}f$ on the unit tangent bundle SM have the same entropy, i.e., $h(f) = h(D^{\sharp}f)$.

Let (X,d) be a compact metric space and $\{f_i\}_{i=1}^{\infty}$ a sequence of continuous maps on X. The identity map on X will be denoted by Id. Let $\mathbf{N}$ be the set of all positive integers. For any $i \in \mathbf{N}$, let $f_i^0 = Id$ and for any $i,\ n \in \mathbf{N}$, let

$$f_i^n = f_{i+(n-1)} \circ \cdots \circ f_{i+1} \circ f_i, \quad f_i^{-n} = (f_i^n)^{-1} = f_i^{-1} \circ f_{i+1}^{-1} \circ \cdots \circ f_{i+(n-1)}^{-1}.$$

(f^{-1} will be applied to sets, we don't assume that the maps f_i are invertible). Denote by $f_{1,\infty}$ the sequence $\{f_i\}_{i=1}^{\infty}$ and the dynamical system $(X, \{f_i\}_{i=1}^{\infty})$. Finally, denote by $f_{1,\infty}^{[n]}$ the sequence of maps $\left\{f_i^{[n]} = f_{(i-1)n+1}^n\right\}_{i=1}^{\infty}$.

Let $\{f_i\}_{i=1}^{\infty}$ be a sequence of continuous maps of compact metric space (X,d). For any $n \in \mathbf{N}$, define a new metric d_n on X by

$$d_n(x,y) := \max_{0\le i\le n-1} d(f_1^i(x), f_1^i(y)).$$

For any $\varepsilon > 0$, a subset $E \subset X$ is said to be an $(n, f_{1,\infty}, \varepsilon)$ *spanning set* of X, if for any $x \in X$, there exists $y \in E$ such that $d_n(x,y) \le \varepsilon$. Let $r(n, f_{1,\infty}, \varepsilon)$ denote the smallest cardinality of any $(n, f_{1,\infty}, \varepsilon)$-spanning set of X. A subset $F \subset X$ is said to be an $(n, f_{1,\infty}, \varepsilon)$-*separated set* of X, if $x, y \in F, x \ne y$, implies $d_n(x,y) > \varepsilon$. Let $s(n, f_{1,\infty}, \varepsilon)$ denote the largest cardinality of any $(n, f_{1,\infty}, \varepsilon)$-separated set of X. It's easy to prove that (similar to the proof for the autonomous system in [12])

$$r(n, f_{1,\infty}, \varepsilon) \le s(n, f_{1,\infty}, \varepsilon) \le r(n, f_{1,\infty}, \frac{\varepsilon}{2}).$$

DEFINITION 1.1. Let $f_{1,\infty} = \{f_i\}_{i=1}^{\infty}$ be a sequence of continuous maps of compact metric space (X,d), then the *topological entropy* of $f_{1,\infty}$ is defined by

$$h(f_{1,\infty}) = \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log r(n, f_{1,\infty}, \varepsilon) = \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log s(n, f_{1,\infty}, \varepsilon).$$

Furthermore, we can see the equivalent definition using open covers in [7].

Let S^1 be a circle with the "geodesic" metric, in which S^1 has length 1 and the distance between two points is the length of the shortest path joining them. Let $f : S^1 \to S^1$ be a continuous surjective map and $F : \mathbf{R}^1 \to \mathbf{R}^1$ a lift of f, we say f is monotone if F is monotone. Denote by $\deg f$ the degree of f (see [13]).

2. The main result

The main result of this paper is:

THEOREM 2.1. *Let $f_{1,\infty} = \{f_i\}_{i=1}^{\infty}$ be a sequence of equi-continuous monotone maps of S^1. Then*

$$h(f_{1,\infty}) = \limsup_{n\to\infty} \frac{1}{n} \log \prod_{i=1}^{n} |\deg f_i|.$$

We will prove this theorem using the idea in [6]. Let $f : S^1 \to S^1$ be a continuous monotone map, $|\deg f| = k$. Then for any $x \in S^1$, $f^{-1}(x)$ is a set consist of k points, denote $f^{-1}(x) = \{x_1, x_2, \ldots, x_n\}$. Let $\alpha_{f,1} = (x_1, x_2), \ldots, \alpha_{f,k-1} = (x_{k-1}, x_k), \alpha_{f,k} = (x_k, x_1)$. Then we get a finite partition $\xi_f = \{\alpha_{f,1}, \alpha_{f,2}, \ldots, \alpha_{f,k}\}$ of S^1, where $f(\overline{\alpha_{f,i}}) = S^1$ and $\alpha_{f,i} \cap \alpha_{f,j} = \emptyset$ for $1 \le i \ne j \le k$.

LEMMA 2.2. *Let $f_{1,\infty} = \{f_i\}_{i=1}^{\infty}$ be a sequence of equi-continuous monotone maps of S^1. Then there exists a constant $a > 0$, such that for every $f_i (i \ge 1)$ and any partition $\xi_{f_i} = \{\alpha_{f_i,1}, \alpha_{f_i,2}, \ldots, \alpha_{f_i,k_i}\}$ of S^1 defined as above, we have*

$$\operatorname{diam} \alpha_{f_i,j} \ge a, \ 1 \le j \le k_i,$$

where $k_i = |\deg f_i|$.

Proof. Since $\{f_i\}_{i=1}^{\infty}$ is equi-continuous, then for $\varepsilon = \frac{1}{2}$, there exists a constant $a > 0$ such that

$$d(x, y) < a \Longrightarrow d(f_i(x), f_i(y)) < \varepsilon, \ \forall i \in \mathbf{N}, \ x, y \in S^1.$$

Note that for every $f_i (i \ge 1)$ and any partition $\xi_{f_i} = \{\alpha_{f_i,1}, \alpha_{f_i,2}, \ldots, \alpha_{f_i,k_i}\}$ of S^1 defined as above, $f_i(\overline{\alpha_{f_i,j}}) = S^1 (1 \le j \le k_i)$ and $\operatorname{diam} S^1 = 1$, we have $\operatorname{diam} \alpha_{f_i,j} \ge a, \ 1 \le j \le k_i$. □

Lemma 2.3. *Let $f_{1,\infty} = \{f_i\}_{i=1}^{\infty}$ be a sequence of equi-continuous monotone maps of S^1, $\{\xi_{f_i} = \{\alpha_{f_i,1}, \alpha_{f_i,2}, \dots, \alpha_{f_i,k_i}\}\}_{i=1}^{\infty}$ be any sequence of partitions of S^1 defined as above. Then for the new sequence of partitions of S^1*

$$\left\{\xi_{f_1^n} = \{f_1^{-(n-1)}(\alpha_{f_n,j}) \mid \alpha_{f_n,j} \in \xi_{f_n},\ 1 \le j \le k_n\}\right\}_{n=1}^{\infty},$$

we have

$$h(f_{1,\infty}) \le \limsup_{n\to\infty} \frac{1}{n} \log \operatorname{card} \xi_{f_1^n} \le h(f_{1,\infty}) + \log 2.$$

Proof. For any $x \in S^1$, let $B_d(x,\varepsilon) = \{y \in S^1 \mid d(x,y) < \varepsilon\}$. By the definition of $\xi_{f_1^n}$, for any given $n \in \mathbf{N}$, there are $n-1$ new partitions of S^1: $\xi'_{f_i} = \{\alpha'_{f_i,1}, \alpha'_{f_i,2}, \dots, \alpha'_{f_i,k_i}\}$, $1 \le i \le n-1$, such that

$$\xi_{f_1^n} = \left\{\bigcap_{i=1}^{n} f_1^{-(i-1)}(\alpha'_{f_i,j}) \mid \alpha'_{f_i,j} \in \xi'_{f_i},\ 1 \le j \le k_i\right\}.$$

by Lemma 2.2, diam $\alpha'_{f_i,j} \ge a$ for any $1 \le i \le n-1$, $1 \le j \le k_i$.

Let $0 < \varepsilon < \frac{a}{2}$(the meaning of a is in Lemma 2.2), and E be an $(n, f_{1,\infty}, \varepsilon)$-spanning set of minimal cardinality of S^1. It can be seen that for any $x \in E$ and $0 \le i \le n-1$, the ε-neighborhood $\overline{B_d(f_1^i(x), \varepsilon)}$ of $f_1^i(x)$ intersects at most 2 elements of ξ'_{f_i}. So $\overline{B_{d_n}(x,\varepsilon)}$ intersects at most 2^n elements of $\xi_{f_1^n}$. By the definition of spanning set, $\bigcup_{x\in E} \overline{B_{d_n}(x,\varepsilon)} = S^1$, then card $\xi_{f_1^n} \le 2^n$card E. Therefore,

$$\limsup_{n\to\infty} \frac{1}{n} \log \operatorname{card} \xi_{f_1^n} \le h(f_{1,\infty}) + \log 2. \tag{1}$$

Now we take an arbitrary $0 < \varepsilon < \frac{1}{2}$ and choose an $(n, f_{1,\infty}, \varepsilon)$-separated set F of maximal cardinality of S^1. By the definition of separated set, for any $\alpha \in \xi_{f_1^n}$ and any two adjacent points x, y in $\alpha \cap F$, there exists j with $0 \le j \le n-1$ such that $d(f_1^j(x), f_1^j(y)) > \varepsilon$. Since f_1^j is monotone on α, then $f_1^j(x)$ and $f_1^j(y)$ are also two adjacent points. Hence, for each $0 \le j \le n-1$, there are at most $M = [\frac{1}{\varepsilon}] + 1$ pairs adjacent points which are more than ε apart in $f_1^j(\alpha \cap F)$. We claim that there are at most $nM + 1$ points in $\alpha \cap F$. In fact, if there are $nM + 2$ points in $\alpha \cap F$, then there are at least $nM + 1$ pairs adjacent points. As mentioned above, for any two adjacent points x, y in $\alpha \cap F$, there exists j with $0 \le j \le n-1$ such that $d(f_1^j(x), f_1^j(y)) > \varepsilon$. This implies that there exists at least one $0 \le s \le n-1$ such that $d(f_1^s(x), f_1^s(y)) > \varepsilon$ for at least $M+1$ pairs adjacent points. This contradicts with the definition of M.

In such a way, we have $\text{card}(\alpha \cap F) \le nM + 1$. Hence, $\text{card}F \le (nM+1)\,\text{card}\,\xi_{f_1^n}$. Furthermore, we have

$$\frac{1}{n}\log s(n, f_{1,\infty}, \varepsilon) \le \frac{1}{n}\log \text{card}\,\xi_{f_1^n} + \frac{1}{n}\log(nM+1).$$

Letting $n \to \infty$, we have

$$\limsup_{n\to\infty}\frac{1}{n}\log s(n, f_{1,\infty}, \varepsilon) \le \limsup_{n\to\infty}\frac{1}{n}\log \text{card}\,\xi_{f_1^n}.$$

Taking limits as ε goes to 0 establish the following inequality:

$$h(f_{1,\infty}) \le \limsup_{n\to\infty}\frac{1}{n}\log \text{card}\,\xi_{f_1^n}. \tag{2}$$

Then (1) and (2) yields

$$h(f_{1,\infty}) \le \limsup_{n\to\infty}\frac{1}{n}\log \text{card}\,\xi_{f_1^n} \le h(f_{1,\infty}) + \log 2.$$

□

Let $f_{1,\infty} = \{f_i\}_{i=1}^{\infty}$ be a sequence of equi-continuous monotone maps of S^1. It is easy to see that for any $m \in \mathbf{N}$, the sequence $f_{1,\infty}^{[m]}$ still be a sequence of equi-continuous monotone maps of S^1. For simplification, we denote $g_{1,\infty} = f_{1,\infty}^{[m]}$, i.e., $g_{1,\infty} = \{g_i\}_{i=1}^{\infty}$, where $g_i = f_i^{[m]}$, $i \in \mathbf{N}$. Accordingly, we can construct a new sequence of finite partitions of S^1 $\{\xi_{g_1^n}\}_{n=1}^{\infty}$ from the sequence of finite partitions of S^1 $\{\xi_{f_i}\}_{i=1}^{\infty}$, where $\xi_{g_1^n} = \xi_{f_1^{nm}}$, $n \in \mathbf{N}$.

LEMMA 2.4. *Let m be any given positive integer. Then for the sequence of maps $g_{1,\infty}$ defined as above and the relevant sequence of partition $\{\xi_{g_1^n}\}_{n=1}^{\infty}$, we have*

$$\limsup_{n\to\infty}\frac{1}{n}\log \text{card}\,\xi_{g_1^n} = m\limsup_{n\to\infty}\frac{1}{n}\log \text{card}\,\xi_{f_1^n}.$$

Proof. By Lemma 2.2, for any $i \in \mathbf{N}$, $\text{card}\,\xi_{f_i} \le N := [\frac{1}{a}] + 1$. Then, for any positive integer $n = lm + j$, $0 \le j \le m-1$, we have

$$\text{card}\,\xi_{f_1^{lm}} \le \text{card}\,\xi_{f_1^n} \le N^m \text{card}\,\xi_{f_1^{lm}}.$$

So

$$\begin{aligned}\limsup_{n\to\infty}\frac{1}{n}\log\operatorname{card}\xi_{f_1^n} &= \limsup_{l\to\infty}\frac{1}{lm}\log\operatorname{card}\xi_{f_1^{lm}}\\ &= \frac{1}{m}\limsup_{l\to\infty}\frac{1}{l}\log\operatorname{card}\xi_{g_1^l}.\end{aligned}$$

Therefore,

$$\limsup_{n\to\infty}\frac{1}{n}\log\operatorname{card}\xi_{g_1^n} = m\limsup_{n\to\infty}\frac{1}{n}\log\operatorname{card}\xi_{f_1^n}.$$

□

LEMMA 2.5. ([7]). *If $f_{1,\infty}=\{f_i\}_{i=1}^{\infty}$ is a sequence of equi-continuous maps on a compact metric space, then for any $m\in\mathbf{N}$, we have*

$$h(f_{1,\infty}^{[m]}) = m\cdot h(f_{1,\infty}).$$

Proof of Theorem 2.1. For any $\varepsilon>0$, take $m\in\mathbf{N}$ such that $\frac{\log 2}{m}<\varepsilon$. Since $f_{1,\infty}$ is a sequence of monotone equi-continuous maps on S^1, as mentioned above, it is easy to see that $g_{1,\infty}=f_{1,\infty}^{[m]}$ is also a sequence of equi-continuous monotone maps on S^1. By Lemma 2.3, we get

$$h(f_{1,\infty}^{[m]}) \le \limsup_{n\to\infty}\frac{1}{n}\log\operatorname{card}\xi_{g_1^n} \le h(f_{1,\infty}^{[m]})+\log 2.$$

Using Lemmas 2.4 and 2.5, and notice the way m is taken, we get

$$h(f_{1,\infty}) \le \limsup_{n\to\infty}\frac{1}{n}\log\operatorname{card}\xi_{f_1^n} \le h(f_{1,\infty})+\varepsilon.$$

Since ε is arbitrary, noting that $\operatorname{card}\xi_{f_1^n}=\prod_{i=1}^{n}|\deg f_i|$, we get immediately

$$h(f_{1,\infty}) = \limsup_{n\to\infty}\frac{1}{n}\log\prod_{i=1}^{n}|\deg f_i|.$$

□

COROLLARY 2.6. *If $f_{1,\infty}=\{f_i\}_{i=1}^{\infty}$ is a sequence of equi-continuous monotone maps of S^1, and the absolute values of the degrees of the mappings are the same, denote it by k, then $h(f_{1,\infty})=\log k$.*

In particular, (Theorem in [5]) If $f:S^1\to S^1$ is a continuous monotone map, then $h(f)=\log|\deg f|$.

COROLLARY 2.7. *If every element of the sequence $\{f_i\}_{i=1}^{\infty}$ on S^1 is chosen from a set consisted of finite continuous monotone maps, then*

$$h(f_{1,\infty}) = \limsup_{n\to\infty} \frac{1}{n} \log \prod_{i=1}^{n} |\deg f_i|.$$

Proof. It is only to note that the continuous map on compact space is uniformly continuous, and finite uniformly continuous maps are equi-continuous. □

COROLLARY 2.8. *Let f be an expansive map of S^1, i.e., f be of C^1, and for every lift $F : R^1 \to R^1$ of it, $|F'(x)| > 1$, $\forall\ x \in \mathbf{R}$. If $\{f_i\}_{i=1}^{\infty}$ are generated by sufficiently small C^1-perturbation of f, then $h(f_{1,\infty}) = \log|\deg f|$.*

Proof. Note that the expansive map of S^1 is strictly monotone and structurally stable ([13]). Also note the degree of the mapping is an invariant of topological conjugacy. Therefore, if every element of $\{f_i\}_{i=1}^{\infty}$ is chosen from the sufficiently small C^1-neighborhood of f, then $\{f_i\}_{i=1}^{\infty}$ must be a sequence of equi-continuous monotone mappings, and $\deg f_i = \deg f$, $\forall\ i \in \mathbf{N}$. From Lemma 2.6, we have $h(f_{1,\infty}) = \log|\deg f|$. □

3. Applications

PROPOSITION 3.1. ([2]). *Let X, Y be compact metric spaces, $F : X \to X$, $f : Y \to Y$ be continuous maps, $\pi : X \to Y$ be a surjective continuous map, and satisfy $\pi \circ F = f \circ \pi$, that is, f and F are topological semi-conjugate and f is the factor of F. Then*

$$h(f) \le h(F) \le h(f) + \sup_{y\in Y} h(F, \pi^{-1}(y)).$$

Let X, Y be compact metric spaces. A continuous map $F : X\times Y \to X \times Y$ is called a *skew-product*, if there exist a continuous map f of X and a set of continuous maps $\{g_x \mid x \in X\}$ of Y which depend on x continuously, such that $F(x,y) = (f(x), g_x(y)), \forall\ x \in X,\ y \in Y$. By Proposition 3.1, we can get that: for the skew-product $F : X \times Y \to X \times Y$, we have

$$h(f) \le h(F) \le h(f) + \sup_{x\in X} h(F, \pi^{-1}(x)),$$

where $\pi : X \times Y \to X,\ (x,y) \mapsto x$ is the natural projection.

PROPOSITION 3.2. ([10]). *If f is a piecewise monotone continuous self-map of I, then*

$$h(f) = \lim_{n\to\infty} \frac{1}{n} \log C_n,$$

where C_n denotes the number of pieces of monotonicity of f^n.

COROLLARY 3.3. (1) *Let $F(x,y) = (f(x), g_x(y))$ be a skew product of annular $I \times S^1$. If f is piecewise monotone, $\{g_x \mid x \in I\}$ is a sequence of equi-continuous monotone maps, then*

$$\lim_{n\to\infty} \frac{1}{n} \log C_n \le h(F)$$

$$\le \lim_{n\to\infty} \frac{1}{n} \log C_n + \sup_{x\in I} \limsup_{n\to\infty} \frac{1}{n} \log \prod_{i=0}^{n-1} |\deg g_{f^i(x)}|.$$

(2) *Let $F(x,y) = (f(x), g_x(y))$ be a skew product of torus $S^1 \times S^1$. If $\{f\} \cup \{g_x \mid x \in S^1\}$ is a sequence of equi-continuous monotone maps, then*

$$\log|\deg f| \le h(F) \le \log|\deg f| + \sup_{x\in S^1} \limsup_{n\to\infty} \frac{1}{n} \log \prod_{i=0}^{n-1} |\deg g_{f^i(x)}|.$$

Proof. Firstly, note that for any skew product $F : X \times Y \to X \times Y$, and any $x \in X$, we have

$$h(F, \pi^{-1}(x)) = h(\{g_{f^{i-1}(x)}\}_{i=1}^{\infty}).$$

From Propositions 3.1, 3.2 and Theorem 2.1, we can get (1). From Proposition 3.1, Corollary 2.6 and Theorem 2.1, we can get (2). □

Let (M, ρ) be a smooth 2-dimensional closed manifold (i.e., M is compact and without boundary), TM be the tangent bundle of M. We denote $|\cdot|$, $\|\cdot\|$ and $d(\cdot,\cdot)$, respectively, the norm on TM, the operator norm and the metric on M induced by the Riemannian metric. Denote by $SM = \bigcup_{x\in M} S_xM$ the unit tangent bundle of M, where $S_xM = \{u \in T_xM \mid |u| = 1\}$. Note that SM is a compact metric space and its metric d can be derived from ρ. That is, the restriction of d on S_xM is consistent with the restriction of the metric of T_xM, which derived from the inner product ρ_x, on S_xM.

Let $f : M \to M$ be a C^1 diffeomorphism, $Df : TM \to TM$ be the tangent map of f. Let $D^\sharp f : SM \to SM$, $u \mapsto \frac{Df(x)u}{|Df(x)u|}$, $u \in T_xM$. Then $(SM, D^\sharp f)$ is a compact topological system, we also call it the *extension* of f on the unit tangent bundle. One can see [8] for some connections of the dynamics between f and its extension $D^\sharp f$.

PROPOSITION 3.4. *Let $f : M \to M$ be a C^1 diffeomorphism on a smooth two-dimensional closed Riemannian manifold M , and $D^\sharp f$ be its extension on the unit tangent bundle SM. Then*

$$h(f) = h(D^\sharp f).$$

Proof. Let $\pi : SM \to M,\ u \mapsto x,\ u \in S_xM$ be the natural projection. It is easy to verify that $\pi \circ D^\sharp f = f \circ \pi$. By Proposition 3.1, we have

$$h(f) \le h(D^\sharp f) \le h(f) + \sup_{x\in M} h(D^\sharp f, \pi^{-1}(x)). \tag{3}$$

Since M is compact, f is a C^1 diffeomorphism, then we can take

$$M = \max_{x\in M} \|Df(x)\|, \quad m = \min_{x\in M} \|Df(x)\|.$$

For any $x \in M,\ u, v \in S_xM$, we have

$$\begin{aligned}
& d(D^\sharp f(x)u, D^\sharp f(x)v) \\
= & \left|D^\sharp f(x)u - D^\sharp f(x)v\right| \\
= & \left|\frac{Df(x)u}{|Df(x)u|} - \frac{Df(x)v}{|Df(x)v|}\right| \\
= & \frac{1}{|Df(x)u| \cdot |Df(x)v|} \, ||Df(x)v| \cdot Df(x)u - |Df(x)u| \cdot Df(x)v| \\
\le & \frac{1}{m^2} \, ||Df(x)v| \cdot [Df(x)(u-v)] - [|Df(x)u| - |Df(x)v|] \cdot Df(x)v| \\
\le & \frac{1}{m^2}[M^2(u-v) + M|Df(x)(u-v)|] \\
\le & \frac{2M^2}{m^2}|u-v|.
\end{aligned}$$

This shows that $\{D^\sharp f(x) \mid x \in M\}$ are equi-continuous with respect to d.

Since $D^\sharp f(x) : S_xM \to S_{f(x)}M$ is a homeomorphism, then it is monotone and $|\deg D^\sharp f(x)| = 1$. Hence, from Theorem 2.1 and Corollary 2.6, we have $h(D^\sharp f, \pi^{-1}(x)) = 0$ for any $x \in M$. Therefore, from (3) we have

$$h(f) = h(D^\sharp f).$$

□

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