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GENERALIZED BROWNIAN MOTIONS WITH APPLICATION TO FINANCE

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ABSTRACT. Let $X = (X_t, t \in [0,T])$ be a generalized Brownian motion(gBm) determined by mean function $a(t)$ and variance function $b(t)$. Let $L^2(\tilde{\mu})$ denote the Hilbert space of square integrable functionals of $\tilde{X} = (X_t - a(t),\, t \in [0,T])$. In this paper we consider a class of nonlinear functionals of X of the form $F(\cdot + a)$ with $F \in L^2(\tilde{\mu})$ and discuss their analysis. Firstly, it is shown that such functionals do not enjoy, in general, the square integrability and Malliavin differentiability. Secondly, we establish regularity conditions on F for which $F(\cdot + a)$ is in $L^2(\tilde{\mu})$ and has its Malliavin derivative. Finally we apply these results to compute the price and the hedging portfolio of a contingent claim in our financial market model based on a gBm X.

1. Introduction

Let $(B(t) = B(t,w);\, t \in [0,T])$ be a 1-dimensional Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $[0,T]$ such that $B(0,w) = 0$ $a.s.$ $\mathbb{P}$. In the Black-Scholes model (see [6]), it is assumed that the log price of underlying asset follows a process $Y(t) = \theta t + \sigma B(t),\ t \in [0,T]$ where θ is a constant drift and σ is a constant volatility. Let b be a strictly increasing and absolutely continuous function on $[0,T]$ with $b(0) = 0$. For the process $Y(t)$, we consider a deterministic time-changed process:

$$Z(t) = \theta b(t) + \sigma B(b(t)). \tag{1.1}$$

We note that the process $Y = (Y(t),\, t \in [0,T])$ is homogeneous in time, while the process $Z = (Z(t),\, t \in [0,T])$ is inhomogeneous in time.

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It is interesting to note that $b(t)$ can be considered as a new time scale which reflects the trading activity in financial market. The exact form of the time change can be calibrated to reproduce the term structure of option prices. Thus the process Z can be used to model the term structure of volatility in the option pricing theory.

It is easy to see that the process Z can be written in a different form:

$$Z(t) = \theta b(t) + \int_0^t \sqrt{b'(s)}d\hat{B}(s), \tag{1.2}$$

where $\hat{B}$ is a Brownian motion w.r.t. the filtration $\mathcal{F}_{b(t)} = \sigma(B(s)), s \leq b(t)$. It is indeed the solution to the stochastic differential equation

$$dZ(t) = \theta b'(t)dt + \sqrt{b'(t)}d\hat{B}(t). \tag{1.3}$$

We note from (1.2) and (1.3) that the process Z is a semimartimgale as well as a continuous diffusion process.

From (1.1), we may introduce a stochastic process of the form:

$$X(t) = a(t) + B(b(t)), \ t \in [0,T], \tag{1.4}$$

where a is of bounded variation function on $[0,T]$ with $a(0) = 0$ (cf. [8]).

In [1, 3] such a process X has been used to study generalized analytic Feyman integrals and generalized analytic Fourier- Feynman transforms. In [2] the conditional function space integral of functionals of the process X was studied. In this paper we take such a process X as a model in finance and then study some stochastic calculus for the process X and its application to finance.

The organization of this paper is as follows. In Section 2 we first discuss the existence of such a process X. In Section 3 we consider a class of nonlinear functionals of X of the form $F(\cdot + a)$ with $F \in L^2(\tilde{\mu})$ and then discuss their square integrability. In Section 4 we discuss the Malliavin differentiability for such functionals of X. In Section 5 we apply our results to compute the price and the hedging portfolio for a contingent claim in a market model based on X.

2. Preliminaries

Let $(C_0[0,T], \mathcal{B}(C_0[0,T]), m_w)$ be the Wiener measure space. Let $a \in C_0[0,T]$ be of bounded variation on $[0,T]$ and let $b \in C_0[0,T]$ be strictly increasing and of bounded variation on $[0,T]$. Then by Theorem 14.2 ([13], p.187), there exists a Gaussian measure μ on $(C_0[0,T], \mathcal{B}(C_0[0,T]))$

such that the coordinate process defined as $X(t,x) = x(t)$ is a continuous additive process on $(\Omega, \mathcal{B}, P) \equiv (C_0[0,T], \mathcal{B}(C_0[0,T]), \mu)$ and $[0,T]$, on which the probability distribution of $X(t_2,\cdot) - X(t_1,\cdot)$, $t_1 < t_2$ is normally distributed with mean $a(t_2) - a(t_1)$ and variance $b(t_2) - b(t_1)$. Let $\tilde{\mu}$ denote the Gaussian measure for the case $a \equiv 0$.

The stochastic process $X = (X_t, t \in [0,T])$ will be referred to as *the generalized Brownian motion*(gBm) determined by mean function $a(t)$ and variance function $b(t)$. We will write the probability space $(C_0[0,T], \mathcal{B}(C_0[0,T]), \mu)$ as $C_{a,b}[0,T] \equiv (C_{a,b}[0,T], \mathcal{B}(C_{a,b}[0,T]), \mu)$. Let $C_{0,b}[0,T] \equiv C_b[0,T]$. Let $E[F]$(resp. $\tilde{E}(F)$) denote the expectation of a functional F on the space $C_{a,b}[0,T]$ (resp. $C_b[0,T]$).

Let $X \sim \text{gBm}(a(t), b(t))$ denote the generalized Brownian motion determined by $a(t)$ and $b(t)$. Then we note that X is an L^2-process and $\mathcal{B}(C_{a,b}[0,T]) = \sigma(X)$ (the smallest σ-algebra for which each X_t is measurable). We consider two Hilbert spaces associated to X. The one is the *non-linear Hilbert space* of X: $L^2(\mu) = L^2(C_{a,b}[0,T], \sigma(X), \mu)$. The other one is the non-linear Hilbert space of $\tilde{X} = (\tilde{X} \equiv X_t - a(t),\ t \in [0,T])$: $L^2(\tilde{\mu}) = L^2(C_b([0,T], \sigma(\tilde{X}), \tilde{\mu})$. Each element of $L^2(\mu)$ (resp. $L^2(\tilde{\mu})$) is called $L^2(\mu)$*-functional* (resp. $L^2(\tilde{\mu})$*-functional*).

We now introduce three function Hilbert spaces $L_b^2[0,T]$, $L_{a,b}^2[0,T]$ and $\hat{L}_b^2[0,T]^n$. We will assume throughout this paper that $a(t)$ is an absolutely continuous function on $[0,T]$ with $a(0) = 0$ and $a'(t) \in L^2[0,T]$, and $b(t)$ is a differentiable function with $b(0) = 0$ and $b_1 \le b'(t) \le b_2$ $(b_1, b_2 > 0)$ for all $t \in [0,T]$.

Let $L_b^2[0,T]$ denote the Hilbert space of real valued Lebesgue-Stieltjes square integrable functions on $[0,T]$ with respect to b equipped with the inner product $\langle f, g\rangle_b = \int_0^T f(s)g(s)db(s)$ and the norm $\|\cdot\|_b = \sqrt{\langle\cdot,\cdot\rangle_b}$. In particular, if $b(t) = t$ on $[0,T]$, we put $L_b^2[0,T] = L^2[0,T]$ and $\langle\cdot,\cdot\rangle_b = \langle\cdot,\cdot\rangle$ and $\|\cdot\|_b = \|\cdot\|$. For $E \subset [0,T]$, 1_E will denote the indicator function of the set E.

Let $L_{a,b}^2([0,T])$ denote the Hilbert space $L_b^2([0,T])$ with inner product

$$\langle f, g\rangle_{a,b} = \langle f, g\rangle_b + \left\langle \int_0^T f da, \int_0^T g da \right\rangle_{\mathbb{R}},$$

where $\langle\cdot,\cdot\rangle_{\mathbb{R}}$ is the inner product on $\mathbb{R}$. We note that the three norms $\|\cdot\|$, $\|\cdot\|_b$ and $\|\cdot\|_{a,b}$ are equivalent. Hence the spaces $L^2[0,T]$, $L_b^2[0,T]$ and $L_{a,b}^2([0,T])$ coincide as sets and furthermore, $(L_{a,b}^2([0,T]), \langle\cdot,\cdot\rangle_{a,b})$ is a separable Hilbert space.

Let $\mathcal{S}$ be the set of all step functions on $[0,T]$, $f=\sum_{j=0}^{n-1}c_j 1_{(t_j,t_{j+1}]}$, where $0=t_0<t_1<t_2<\cdots<t_n=T$ and $c_j\in\mathbb{R}$. Let $H(X)$ be the closed linear subspace of $L^2(\mu)$ spanned by X. We define $\theta:\mathcal{S}\to H(X)$ by $\theta(f)=\sum_{j=0}^{n-1}c_j(X_{t_{j+1}}-X_{t_j})=\sum_{j=0}^{n-1}c_j[(\tilde{X}_{t_{j+1}}-\tilde{X}_{t_j})+(a(t_{j+1})-a(t_j))]$. For all $f,g\in\mathcal{S}$, we have $E[\theta(f)]=\int_0^T f da$, $E[\theta(f)\theta(g)]=\langle f,g\rangle_{a,b}$. Thus the correspondence $f\mapsto\theta(f)$ is an inner product preserving mapping from $\mathcal{S}$ to $H(X)$. Hence θ is uniquely extended to $L^2_{a,b}[0,T]$. For $f\in L^2_{a,b}[0,T]$, we call $\theta(f)$ *the stochastic integral* of f against X and write it as

$$\theta(f)=\int_0^T f dX,\quad\text{or}\quad \int_0^T f d\tilde{X}+\int_0^T f da. \tag{2.1}$$

It has been shown (see [7]) that $L^2_{a,b}([0,T])\cong H(X)$.

Let $\hat{L}^2_b[0,T]^n$ denote the Hilbert space of real-valued Lebesgue-Stieltjes square integrable symmetric functions on $\mathbb{R}^n$ with respect to $\prod_{j=1}^n db(t_j)$ equipped with the inner product

$$\langle f_n,g_n\rangle_{b,n}=\int_{[0,T]^n} f_n(t_1,\ldots,t_n)g_n(t_1,\ldots,t_n)\prod_{i=1}^n db(t_i)$$

and the norm $\|\cdot\|_{b,n}=\sqrt{\langle\cdot,\cdot\rangle_{b,n}}$.

3. Square integrable functionals of X

For each $\sigma>0$, the Hermite polynomial H_{n,σ^2} of degree n with parameter σ^2 is defined by

$$H_{n,\sigma^2}(x)=(-\sigma^2)^n\exp\left(\frac{x^2}{2\sigma^2}\right)\frac{d^n}{dx^n}\exp\left(-\frac{x^2}{2\sigma^2}\right),\ n\geq 1, \tag{3.1}$$

and $H_{0,\sigma^2}(x)=1$. We note from [7] that the mapping $f\mapsto\int_0^T f d\tilde{X}$ is a linear isometry from $L^2_b[0,T]$ into $L^2(\tilde{\mu})$. For $f\in L^2_b[0,T]$ and $n\geq 0$, we put $I_n(f^{\otimes n})(\tilde{x})=H_{n,\|f\|_b^2}(\int_0^T f d\tilde{X})$. Then for $f\in L^2_b[0,T]$, we have

$$I_n(f^{\otimes n})(\tilde{x}+a)=\sum_{k=0}^n\binom{n}{k}I_k(g_n^k)(\tilde{x}), \tag{3.2}$$

where $g_n^k=f^{\hat{\otimes}k}\cdot(\int_0^T f da)^{n-k}$. The mapping $I_n: f^{\otimes n}\mapsto I_n(f^{\otimes n})(\cdot)$ can be extended to a linear continuous mapping from $\hat{L}^2_b[0,T]^n$ to $L^2(\tilde{\mu})$(see [4]). We denote this extension by $I_n(f_n)$. The image $I_n(f_n)$ of $f_n\in$

$\hat{L}_b^2[0,T]^n$ is indeed *the multiple Wiener integral* with respect to the process $\tilde{X}$.

The following lemma shows that the mapping $f^{\otimes n} \mapsto I_n(f^{\otimes n})(\cdot + a)$ can be extended from $\hat{L}_b^2[0,T]^n$ to $L^2(\tilde{\mu})$. For $f_n \in \hat{L}_b^2[0,T]^n$, we denote by $K_n(f_n)(\cdot)$ the image of f_n under this extension.

LEMMA 3.1. *For $f_n \in \hat{L}_b^2[0,T]^n$, we have*

$$K_n(f_n)(\cdot) = \sum_{k=0}^{n} \binom{n}{k} I_k(g_n^k)(\cdot), \tag{3.3}$$

where $g_n^k = (f_n^{(n-k)}, a^{\otimes(n-k)})$, which indicates the following integral :

$$\int_{[0,T]^{n-k}} f_n(t_1, \ldots, t_k, s_1, \ldots, s_{n-k}) \prod_{j=1}^{n-k} da(s_j).$$

Proof. Let $\mathcal{E}_n$ be the linear space spanned by $E_n = \{f^{\hat{\otimes} n} | \ f \in L_b^2[0,T]\}$. Define the mapping K_n on $\mathcal{E}_n$ by

$$K_n(\phi_n) = \sum_{n=0}^{\infty} \binom{n}{k} I_k(g_n^k), \ \ \phi_n \in \mathcal{E}_n,$$

where $g_n^k = (\phi_n^{(n-k)}, a^{\otimes(n-k)})$. Now we extend K_n linearly and continuously from the space $\hat{L}_b^2[0,T]^n$ to the space $L^2(\tilde{\mu})$ in the following way: Let $f_n \in \hat{L}_b^2[0,T]^n$. Since $\mathcal{E}_n$ is dense in $\hat{L}_b^2[0,T]^n$, we take a sequence $\{\phi_{n_j}\}_{j=1}^{\infty}$ in $\mathcal{E}_n$ such that $\phi_{n_j} \to f_n$ in $\hat{L}_b^2[0,T]^n$ as $j \to \infty$. We observe that

$$\begin{aligned} \|K_n(\phi_{n_j}) - K_n(\phi_{n_k})\|_{L^2(\tilde{\mu})}^2 &= \left\| \sum_{l=0}^{n} \binom{n}{l} I_l(g_{n_j}^l - g_{n_k}^l) \right\|_{L^2(\tilde{\mu})}^2 \\ &= \sum_{l=0}^{n} \binom{n}{l}^2 l! \, \|g_{n_j}^l - g_{n_k}^l\|_{b,l}^2 \\ &\le \sum_{l=0}^{n} \binom{n}{l}^2 l! \, \left\|\frac{da}{db}\right\|_b^{2(n-l)} \|\phi_{n_j}^l - \phi_{n_k}^l\|_{b,n}^2, \end{aligned}$$

where $g_{n_j}^l = (\phi_{n_j}^{n-l}, a^{\otimes(n-l)})$. Hence the sequence $\{K_n(\phi_{n_j})\}_{j=1}^{\infty}$ is Cauchy in $L^2(\tilde{\mu})$. Define

$$K_n(f_n) = \lim_{j\to\infty} K_n(\phi_{n_j}) \text{ in } L^2(\tilde{\mu}).$$

It can be shown that the limit is independent of the choice of a sequence $\{\phi_{n_j}\}_{j=0}^{\infty}$ and so the extension K_n is well-defined and is a continuous linear mapping from the space $\hat{L}_b^2[0,T]^n$ to $L^2(\tilde{\mu})$. □

By the Wiener-Ito decomposition theorem (see [4], [5]), any functional $F \in L^2(\tilde{\mu})$ can be represented uniquely as

$$F = \sum_{n=0}^{\infty} I_n(f_n), \; f_n \in \hat{L}_b^2[0,T]^n. \tag{3.4}$$

In view of this fact, we shall write $F \sim (f_n)$ for notational simplicity. The $L^2(\tilde{\mu})$-norm of F is given by

$$\|F\|_1 = \left(\sum_{n=0}^{\infty} n! \|f_n\|_{b,n}^2\right)^{\frac{1}{2}}.$$

If $F \sim (f_n) \in L^2(\tilde{\mu})$, then by Lemma 3.1, we obtain the formal sum:

$$F(\cdot + a) \equiv \sum_{n=0}^{\infty} K_n(f_n)(\cdot) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} \binom{n}{k} I_k(g_n^k)(\cdot),$$

where $g_n^k = (f_n^{(n-k)}, a^{\otimes(n-k)})$, $k \geq 1$. We shall give an example of $L^2(\tilde{\mu})$-functional F for which the formal sum $F(\cdot + a)$ is not in $L^2(\tilde{\mu})$ for some $a \in C_0[0,T]$(see Example 3.2 below). Due to this example, we need to impose an appropriate regularity on the space $L^2(\tilde{\mu})$ so that each $F(\cdot + a)$ is in $L^2(\tilde{\mu})$(see Theorem 3.3 below). For this, define the following weighted Hilbert space (see [11]): for $\alpha \geq 1$, let

(3.5)

$$L_\alpha^2 \equiv L_\alpha^2(\mu) = \left\{F \in L^2(\tilde{\mu}) : F \sim (f_n) \text{ and } \sum_{n=0}^{\infty} n! \alpha^{2n} \|f_n\|_{b,n}^2 < \infty\right\}.$$

It is easy to see that $L_1^2 = L^2(\tilde{\mu})$ and L_α^2 is a Hilbert space under the norm

$$\|F\|_\alpha = \left(\sum_{n=0}^{\infty} n! \|f_n\|_{b,n,\alpha}^2\right)^{\frac{1}{2}}, \tag{3.6}$$

where $\|f_n\|_{b,n,\alpha} = \|\alpha^n f_n\|_{b,n}$, $f_n \in \hat{L}_b^2[0,T]^n$. For any $1 < \alpha_1 < \alpha_2$, we have the continuous inclusion: $L_{\alpha_2}^2 \subset L_{\alpha_1}^2 \subset L_1^2$.

The following example shows that there exists an $F \sim (f_n) \in L^2(\tilde{\mu})$ such that $F_a(\cdot) \equiv F(\cdot + a) \notin L^2(\tilde{\mu})$.

EXAMPLE 3.2. Let $F \sim (f_n)$, where $f_n = \dfrac{1}{n\sqrt{n!b(T)^n}} 1_{[0,T]}^{\otimes n}$ for $n \geq 1$ and $f_0 = 0$. Then we have

$$\sum_{n=0}^{\infty} n! \, \|f_n\|_{b,n}^2 = \sum_{n=0}^{\infty} n! \left(\frac{1}{n\sqrt{n!b(T)^n}} \right)^2 b(T)^n = \sum_{n=0}^{\infty} \frac{1}{n^2} < \infty,$$

so that $F \in L^2(\tilde{\mu})$. We now show that F_a may not be in $L^2(\tilde{\mu})$. We note that $F_a = \sum_{k=0}^{\infty} I_k(h_k)$, where $h_k = \sum_{n=k}^{\infty} \binom{n}{k} g_n^k$ and $g_n^k = \dfrac{1}{n\sqrt{n!b(T)^n}} a(T)^{n-k} 1_{[0,T]}^{\otimes k}$, $k \geq 1$. Take $a \in C_0[0,T]$ with $a(T) > 0$. Then $\|h_k\|_{b,k} > \|\binom{k+1}{k} g_{k+1}^k\|_{b,k}$. Hence we have

$$\begin{aligned}
\sum_{k=0}^{\infty} k! \, \|h_k\|_{b,k}^2 &> \sum_{k=0}^{\infty} k! \, \|(k+1) g_{k+1}^k\|_{b,k}^2 \\
&= \sum_{k=0}^{\infty} k! \left(\frac{a(T)}{\sqrt{(k+1)!b(T)^{k+1}}} \right)^2 b(T)^k \\
&= \frac{a(T)^2}{b(T)} \sum_{k=0}^{\infty} \frac{1}{k+1},
\end{aligned}$$

so that $F_a \notin L^2(\tilde{\mu})$.

The following theorem gives the regularity condition on F for which F_a belongs to $L^2(\tilde{\mu})$.

THEOREM 3.3. *Let $F \sim (f_n) \in L^2(\tilde{\mu})$ and let $F_a(\tilde{x}) = F(\tilde{x} + a)$ for $a \in C_0[0,T]$. If (f_n) is in L_α^2 for some $\alpha > 1$, then $F_a \in L^2(\tilde{\mu})$ and its chaos expansion is given by $F_a = \sum_{k=0}^{\infty} I_k(h_k)$, where $h_k = \sum_{n=k}^{\infty} \binom{n}{k} g_n^k$ and $g_n^k = (f_n^{(n-k)}, a^{\otimes(n-k)})$, $k \geq 1$.*

Proof. Let $F \sim (f_n) \in L^2(\tilde{\mu})$ with $(f_n) \in L_\alpha^2$. Then, by using Lemma 3.1 and changing the order of summation, we obtain $F_a = \sum_{k=0}^{\infty} I_k(h_k)$, where $h_k = \sum_{n=k}^{\infty} \binom{n}{k} g_n^k$ and $g_n^k = (f_n^{(n-k)}, a^{\otimes(n-k)})$, $k \geq 0$. In order to justify this assertion, it suffices to show that $\sum_{k=0}^{\infty} k! \|h_k\|_{b.k}^2 < \infty$. By using the Hölder inequality and the assumption that $\alpha > 1$, we see that

$$
\begin{aligned}
&\sum_{k=0}^{\infty} k!\|h_k\|_{b,k}^2 \\
&\le \sum_{k=0}^{\infty} k!\Big(\sum_{n=k}^{\infty}\binom{n}{k}\|g_n^k\|_{b,k}\Big)^2 \\
&= \sum_{k=0}^{\infty} k!\Big(\sum_{n=0}^{\infty}\binom{n+k}{k}\|g_{n+k}^k\|_{b,k}\Big)^2 \\
&\le \sum_{k=0}^{\infty} k!\Big(\sum_{n=0}^{\infty}\binom{n+k}{k}\|f_{n+k}\|_{b,n+k}\Big\|\frac{da}{db}\Big\|_b^n\Big)^2 \\
&= \sum_{k=0}^{\infty} k!\Big(\sum_{n=0}^{\infty}\sqrt{(n+k)!}\alpha^k\|f_{n+k}\|_{b,n+k} \times \Big(\frac{1}{\alpha}\Big)^k\frac{\sqrt{(n+k)!}}{n!k!}\Big\|\frac{da}{db}\Big\|_b^n\Big)^2 \\
&\le \sum_{k=0}^{\infty} k!\sum_{n=0}^{\infty}(n+k)!\|f_{n+k}\|_{b,n+k,\alpha}^2\sum_{n=0}^{\infty}\frac{(n+k)!}{(n!k!)^2}\Big(\frac{1}{\alpha}\Big)^{2k}\Big\|\frac{da}{db}\Big\|_b^{2n} \\
&\le \|F\|_\alpha^2 \times \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{k=0}^{\infty}\frac{(n+k)!}{n!k!}\Big(\frac{1}{\alpha}\Big)^{2k}\Big\|\frac{da}{db}\Big\|_b^{2n} \\
&= \|F\|_\alpha^2 \times \sum_{n=0}^{\infty}\frac{1}{n!}(1-\alpha^{-2})^{-(n+1)}\Big\|\frac{da}{db}\Big\|_b^{2n} \\
&= \|F\|_\alpha^2 \times (1-\alpha^{-2})^{-1}\exp\Big(\Big\|\frac{da}{db}\Big\|_b^2\cdot(1-\alpha^{-2})^{-1}\Big) < \infty.
\end{aligned}
$$

This completes the proof. □

4. Malliavin differentiability of functionals of X

Let $L_b^2([0,T]\times C_b[0,T])$ denote the set of all stochastic processes $u(t,x)$ on $C_b[0,T]$ and $[0,T]$ such that $E[\int_0^T u^2(t,x)db(t)] < \infty$. Let ψ be a continuously differentiable function with bounded derivative and $G \sim (g_n) \in L^2(\tilde{\mu})$. Then the functional $F = \psi(G)$ is called *Malliavin differentiable* if

$$
D_tF = D_t\psi(G) = \psi'(G)\sum_{n=1}^{\infty} nI_{n-1}(g_n(\cdot,t)) \tag{4.1}
$$

converges in $L^2(\tilde{\mu})$ for almost everywhere t with respect to db. The set of all such Malliavin differentiable functionals with being in $L^2(\tilde{\mu})$ is denoted by $\mathcal{D}$. For $\alpha \geq 1$, let $\mathbb{D}^{1,2}_\alpha$ denote the set of all $F \sim (f_n) \in L^2(\tilde{\mu})$ such that $\sum_{n=0}^\infty nn!\|f_n\|^2_{b,n,\alpha} < \infty$. Put $\mathbb{D}^{1,2}_1 = \mathbb{D}^{1,2}$. It is well-known (see [4], [9]) that $\mathbb{D}^{1,2}$ is indeed the completion of $\mathcal{D}$ with respect to the Hilbertian norm $\|F\|_{1,2} = (E[F]^2 + E[\|D_tF\|^2_b])^{\frac{1}{2}}$.

DEFINITION 4.1. Let $F \in \mathbb{D}^{1,2}$, so that there exists a sequence $\{F_n\} \subset \mathcal{D}$ such that $F_n \to F$ in $L^2(\tilde{\mu})$ and $\{D_tF_n\}$ is convergent in $L^2_b([0,T] \times C_b[0,T])$. Then we define $D_tF = lim_{n\to\infty} D_tF_n$. The limit $D_tF \in L^2_b([0,T] \times C_b[0,T])$ is called the *Malliavin derivative* of F.

The following example shows that there exists an $F \sim (f_n) \in L^2(\tilde{\mu})$ such that $F \in \mathbb{D}^{1,2}$ but $F_a \notin \mathbb{D}^{1,2}$ for some $a \in C_0[0,T]$.

EXAMPLE 4.2. Let $F \sim (f_n)$ where $f_n = \dfrac{1}{n\sqrt{nn!b(T)^n}} 1^{\otimes n}_{[0,T]}$ for $n \geq 1$ and $f_0 = 0$. Then we have

$$\sum_{n=0}^\infty nn!\|f_n\|^2_{b,n} = \sum_{n=0}^\infty nn!\left(\frac{1}{n\sqrt{nn!b(T)^n}}\right)^2 b(T)^n = \sum_{n=0}^\infty \frac{1}{n^2} < \infty,$$

so that $F \in \mathbb{D}^{1,2}$. Take $a \in C_0[0,T]$ with $a(T) > 0$. Then we note that $F_a = \sum_{k=0}^\infty I_k(h_k)$, where $h_k = \sum_{n=k}^\infty \binom{n}{k} g^k_n$ and $g^k_n = \dfrac{a(T)^{n-k}}{\sqrt{n^3 n! b(T)^n}} 1^{\otimes k}_{[0,T]}$, $k \geq 1$. Then $\|h_k\|_{b,k} > \|\binom{k+1}{k} g^k_{k+1}\|_{b,k}$. Hence we have

$$\begin{aligned}
\sum_{k=0}^\infty kk!\|h_k\|^2_{b,k} &> \sum_{k=0}^\infty kk!\|(k+1)g^k_{k+1}\|^2_{b,k} \\
&= \sum_{k=0}^\infty kk!\left(\frac{a(T)}{\sqrt{(k+1)(k+1)!b(T)^{k+1}}}\right)^2 b(T)^k \\
&= \frac{a(T)^2}{b(T)} \sum_{k=0}^\infty \frac{k}{(k+1)^2} = \infty,
\end{aligned}$$

so that $F_a \notin \mathbb{D}^{1,2}$.

The following theorem gives the regularity condition on F for which F_a belongs to $\mathbb{D}^{1,2}$.

THEOREM 4.3. *Let $F \sim (f_n) \in L^2(\tilde{\mu})$ and let $\alpha \in \mathbb{R}$ such that $\alpha > \left(1 + \left\|\frac{da}{db}\right\|^2_b\right)^{\frac{1}{2}}$. If $(f_n) \in \mathbb{D}^{1,2}_\alpha$, then $F_a \in \mathbb{D}^{1,2}$, and hence F_a has* *the following integral representation :*

$$F_a = E[F_a] + \int_0^T \tilde{E}[D_t F_a | \mathcal{F}_t] d\tilde{X}(t). \tag{4.2}$$

Proof. By Theorem 3.3, we have $F_a(\cdot) \in L^2(\tilde{\mu})$ and $F_a = \sum_{k=0}^{\infty} I_k(h_k)$, where $h_k = \sum_{n=k}^{\infty} \binom{n}{k} g_n^k$ and $g_n^k = (f_n^{(n-k)}, a^{\otimes(n-k)})$, and hence $F \in L^2(\mu)$. To prove that $F_a \in \mathbb{D}^{1,2}$, it suffices to show that $\sum_{k=0}^{\infty} kk! \|h_k\|_{b,k}^2 < \infty$. By using the inequality $\frac{k}{(n-k)!} \leq n$, we have

$$\begin{aligned}
&\sum_{k=0}^{\infty} kk! \|h_k\|_{b,k}^2 \\
&\leq \sum_{k=0}^{\infty} kk! \Big(\sum_{n=k}^{\infty} \binom{n}{k} \|g_n^k\|_{b,k} \Big)^2 \\
&\leq \sum_{k=0}^{\infty} kk! \Big(\sum_{n=k}^{\infty} \binom{n}{k} \|f_n\|_{b,n} \Big\| \frac{da}{db} \Big\|_b^{n-k} \Big)^2 \\
&= \sum_{k=0}^{\infty} kk! \Big(\sum_{n=k}^{\infty} \binom{n}{k} \|f_n\|_{b,n} \sqrt{\alpha^2 - 1}^{\,n-k} \Big(\frac{\| \frac{da}{db} \|_b}{\sqrt{\alpha^2 - 1}} \Big)^{n-k} \Big)^2 \\
&\leq \sum_{k=0}^{\infty} \Big(\sum_{n=k}^{\infty} kk! \binom{n}{k}^2 (\alpha^2 - 1)^{n-k} \|f_n\|_{b,n}^2 \times C \Big) \\
&\leq \sum_{n=0}^{\infty} \Big(\sum_{k=0}^{n} nn! \binom{n}{k} \|f_n\|_{b,n}^2 (\alpha^2 - 1)^{n-k} \times C \Big) \\
&= \sum_{n=0}^{\infty} nn! \|f_n\|_{b,n}^2 \alpha^{2n} \times C \\
&= \sum_{n=0}^{\infty} nn! \|f_n\|_{b,n,\alpha}^2 \times C,
\end{aligned}$$

where $C = \sum_{n=0}^{\infty} \left(\frac{\| \frac{da}{db} \|_b^2}{\alpha^2 - 1} \right)^n$. Hence it follows that $F_a \in \mathbb{D}^{1,2}$, so that by the Clark-Ocone theorem (see [9]), we have the representation (4.2). □

EXAMPLE 4.4. For $f \in L_b^2[0, T]$, consider a functional given by $F(x) = \exp(\int_0^T f dX(x) - \frac{1}{2}\|f\|_b^2)$. Let $G(\tilde{x}) = \exp(\int_0^T f d\tilde{X}(x) - \frac{1}{2}\|f\|_b^2)$. Then we see that $G \sim (\frac{f^{\otimes n}}{n!})$ and $F(x) = G(\tilde{x} + a) = G_a(\tilde{x})$. We observe that for all $\alpha \geq 1$, we have

$$\sum_{n=0}^{\infty} nn! \left\| \frac{f^{\otimes n}}{n!} \right\|_{b,n,\alpha}^2 = \alpha^2 \|f\|_b^2 \times \exp(\alpha^2 \|f\|_b^2) < \infty.$$

Thus by using Theorem 4.3, we have $G_a(\cdot) \in \mathbb{D}^{1,2}$. Since $G_a(\cdot) = \sum_{k=0}^{\infty} I_k(h_k)(\cdot)$, where $h_k = \sum_{n=k}^{\infty} \binom{n}{k} \frac{1}{n!} f^{\otimes k} \cdot (\int_0^T f da)^{n-k}$, the Malliavin derivative of G_a is given by $D_t G_a(\tilde{x}) = G_a(\tilde{x}) f(t)$.

5. Application to finance

In this section we apply our results obtained above to compute the price and the hedging portfolio for a contigent claim in a financial market based on gBm.

Let $a(t)$ and $b(t)$ be as in Section 2. We consider a financial market consisting of a stock $S(t)$ and a bond $A(t)$ whose prices are modeled by, respectively,

(5.1)

$$A(t) = \exp\left(\int_0^t r(s) ds \right) \text{ and } S(t) = S_0 \exp\left(\int_0^t r(s) ds + X(t) \right),$$

where $X(t) \sim \mathrm{gBm}(a(t), b(t))$ and $r(t)$ is a continuous compound interest rate at time t. It can be shown the the stock price process $S(t)$ is risk neutral under the measure P if and only if $a(t) = -\frac{1}{2} b(t)$. Under this risk neutral model, we see that $\left\| \frac{da}{db} \right\|_b^2 = b(T)/4$.

A portfolio $\phi(t) = (\phi_1(t), \phi_2(t))$ is defined as a pair of $\mathcal{F}_t$-adapted stochastic processes, which gives the number of units of $A(t)$ and $S(t)$ invested at time t, respectively. The market value of the portfolio at time t is given by

$$V(t) = \phi_1(t) A(t) + \phi_2(t) S(t), 0 \leq t \leq T. \tag{5.2}$$

A portfolio ϕ is called *self-financing* if $dV(t) = \phi_1(t) dA(t) + \phi_2(t) dS(t)$. From now on we assume that ϕ has to be self-financing. Let $\hat{V}(t) = V(t)/A(t)$. Then by the integration by part formula, we have

$$d\hat{V}(t) = \phi_2(t) d\hat{S}(t), \tag{5.3}$$

where $\hat{S}(t) = S(t)/A(t)$. Since $\hat{S}(t) = S_0 \exp(X(t))$, it follows from the Ito lemma that $d\hat{S}(t) = \hat{S}(t)d\tilde{X}(t)$. Thus by (5.3) we have $d\hat{V}(t) = \phi_2(t)\hat{S}(t)d\tilde{X}(t)$, so that

$$\hat{V}(t) = V(0) + \int_0^t \phi_2(t)\hat{S}(t)d\tilde{X}(t). \tag{5.4}$$

Let F be an $\mathcal{F}_T$-measurable(i.e. a contingent claim)such that $F(x) = G(\tilde{x}+a)$ with $G \in L^2(\tilde{\mu})$. We note that $\hat{F} \equiv F/A(T) = G_a/A(T) \equiv \hat{G}_a$. Then if $G \in \mathbb{D}_\alpha^{1,2}$ with $\alpha > (1+b(T)/4)^{1/2}$, by Theorem 4.3, we have

$$\hat{F} = E[\hat{F}] + \int_0^T \tilde{E}[D_t\hat{G}_a|\mathcal{F}_t]d\tilde{X}(t). \tag{5.5}$$

One problem in option pricing theory is to determine an initial value $V(0)$ and to find a portfolio $\phi(t)$ such that $V(T) = F$, a.s.. Such a portfolio $\phi(t)$ is called the *hedging portfolio* for the contingent claim F.

The following theorem gives an answer for this problem in our market model based on a gBm X.

THEOREM 5.1. *Let F be an $\mathcal{F}_T$-measurable(i.e. a contingent claim) such that $F(x) = G(\tilde{x}+a)$ with $G \in L^2(\tilde{\mu})$. If $G \in \mathbb{D}_\alpha^{1,2}$ with $\alpha > (1+b(T)/4)^{1/2}$ then the price of the claim F at $t = 0$ is given by $E[\hat{F}] = E[A(T)^{-1}F]$. And the hedging portfolio $\phi(t) = (\phi_1(t), \phi_2(t))$ for the contingent claim F is given by*

$$\phi_1(t) = \frac{V(t) - \phi_2(t)S(t)}{A(t)} \tag{5.6}$$

$$\phi_2(t) = \exp\Big(-\int_t^T r(s)ds\Big) S(t)^{-1}\, \tilde{E}[D_tG_a|\mathcal{F}_t]. \tag{5.7}$$

Proof. We need to find $V(0)$ and $\phi(t)$ such that $V(T) = F$ a.s.. So from (5.4) and (5.5), $\hat{F}$ has two representations. Hence by uniqueness the proof follows. □

EXAMPLE 5.2. Let F be a contingent claim of which the payoff is given by

$$F(x) = (L(T,x) - K)^+,\ x \in C_{a,b}[0,T],$$

where K is the strike price and $L(T,x) = \exp\{\frac{1}{T}\int_0^T \log S(t)dt\}$. Such a contingent claim is called the *geometric average rate call option* (see [12], p.224). We shall determine the price of F and find its hedging portfolio, by using Theorem 5.1.

We first observe that $\frac{1}{T}\int_0^T logS(t)dt = C + \int_0^T f(t)dX(t) - \frac{1}{2}\|f\|_b^2$, where $C = logS_0 + \frac{1}{T}\int_0^T \int_0^t r(s)dsdt + \frac{1}{2}\|f\|_b^2$ and $f(t) = \frac{1}{T}(T-t)$. Then $L(T,x) = e^C \times \exp(\int_0^T f(t)dX(t) - \frac{1}{2}\|f\|_b^2)$.

We put $M(T,\tilde{x}) = e^C \times \exp(\int_0^T fd\tilde{X} - \frac{1}{2}\|f\|_b^2)$. Then $M(T,\tilde{x}+a) \equiv M_a(T,\tilde{x}) = L(T,x)$. It follows from Example 4.4 that $M_a(T,\cdot) \in \mathbb{D}^{1,2}$ and $D_t M_a(T,\tilde{x}) = f(t)M_a(T,\tilde{x})$. Let $\psi(u) = (u-K)^+$, $u \in \mathbb{R}$, and let $G(\tilde{x}) \equiv \psi(M_a(T,\tilde{x}))$. Let us take a sequence $\{\psi_n\}$ of continuously differentiable functions with bounded derivative, converging to ψ such that

$$\psi_n(u) = \psi(u) \text{ for } |u-K| \geq \frac{1}{n}, \quad \text{and } |\psi_n'(u)| \leq 1 \text{ for all } u \in \mathbb{R}.$$

Then each $G_n(\cdot) \equiv \psi_n(M_a(T,\cdot))$ is in $\mathbb{D}^{1,2}$ and we have

$$D_t G_n(\cdot) = \psi_n'(M_a(T,\cdot))D_t M_a(T,\cdot).$$

Now we show that $G_n \to G$ in $\mathbb{D}^{1,2}$. We first shall show that $G_n \to G$ in $L^2(\tilde{\mu})$. We note that $|G_n(\tilde{x})| \leq |M_a(T,\tilde{x})|$ and $M_a(T,\cdot) \in L^2(\tilde{\mu})$. Since $G_n(\tilde{x}) \to G(\tilde{x})$ pointwise, it follows from the dominated convergence theorem that $G_n \to G$ in $L^2(\tilde{\mu})$. Let $A = \{\tilde{x} : M_a(T,\tilde{x}) > K\}$. We next shall show that $D_t G_n(\cdot) \to 1_A(\cdot)D_t M_a(T,\cdot)$ in $L_b^2([0,T] \times C_b[0,T])$. We note that $|D_t G_n(\tilde{x})| \leq |D_t M_a(T,\tilde{x})|$ and $D_t M_a(T,\tilde{x}) \in L_b^2([0,T] \times C_b[0,T])$. Since $D_t G_n(\tilde{x}) \to 1_A(\tilde{x})D_t M_a(T,\tilde{x})$ pointwise, it follows from the dominated convergence theorem that $D_t G_n(\cdot) \to 1_A(\cdot)D_t M_a(T,\cdot)$ in $L_b^2([0,T] \times C_b[0,T])$. From these two norm convergences, we conclude that $G_n \to G$ in $\mathbb{D}^{1,2}$, and the Malliavin derivative of G is given by $D_t G(\tilde{x}) = 1_A(\tilde{x})f(t)M(T,\tilde{x})$.

By the result of Theorem 5.1, we obtain the hedging portfolio of stock:

$$\begin{aligned}\phi_2(t) = \exp\Big(-\int_t^T r(u)du\Big)S(t)^{-1}\Big(\frac{T-t}{T}\Big) \\ \times \tilde{E}[M_a(T,\cdot)1_{(K,\infty)}(M_a(T,\cdot))|\mathcal{F}_t],\end{aligned}$$

and the price of F is given by $V(0) = \exp(-\int_0^T r(s)ds)E[F]$. Through simple calculations, it can be shown that

$$\begin{aligned}V(0) = \exp\Big(-\int_0^T r(s)ds\Big) \\ \times \{S_0 \exp(R(T) + m(T) + \frac{1}{2}\sigma^2(T))N(d_1) - KN(d_2)\},\end{aligned}$$

where

$$\begin{aligned}
R(T) &= \frac{1}{T}\int_0^T \left(\int_0^s r(u)du\right)ds,\\
m(T) &= E\left[\int_0^T f(s)dX(s)\right] = \int_0^T -\frac{1}{2}\left(1-\frac{s}{T}\right)db(s),\\
\sigma^2(T) &= \mathrm{Var}\left[\int_0^T f(s)d\tilde{X}(s)\right] = \int_0^T \left(1-\frac{s}{T}\right)^2 db(s),\\
d_1 &= d_2 + \sigma(T),\\
d_2 &= \frac{m(T)+R(T)+\log(S_0/K)}{\sigma(T)},\\
N(x) &= \int_{-\infty}^x \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{u^2}{2}\right)du,\ x\in\mathbb{R}.
\end{aligned}$$

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