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ON A GENERALIZED TRIF'S MAPPING IN BANACH MODULES OVER A C^*-ALGEBRA

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ABSTRACT. Let X and Y be vector spaces. It is shown that a mapping $f : X \to Y$ satisfies the functional equation

$$(\ddagger)\qquad \begin{aligned} & mn\ {}_{mn-2}C_{k-2} f\left(\frac{x_1 + \cdots + x_{mn}}{mn}\right) \\ & + m\ {}_{mn-2}C_{k-1} \sum_{i=1}^{n} f\left(\frac{x_{mi-m+1} + \cdots + x_{mi}}{m}\right) \\ & = k \sum_{1\le i_1<\cdots<i_k\le mn} f\left(\frac{x_{i_1} + \cdots + x_{i_k}}{k}\right) \end{aligned}$$

if and only if the mapping $f : X \to Y$ is additive, and we prove the Cauchy-Rassias stability of the functional equation ($\ddagger$) in Banach modules over a unital C^*-algebra. Let $\mathcal{A}$ and $\mathcal{B}$ be unital C^*-algebras or Lie JC^*-algebras. As an application, we show that every almost homomorphism $h : \mathcal{A} \to \mathcal{B}$ of $\mathcal{A}$ into $\mathcal{B}$ is a homomorphism when $h(2^d uy) = h(2^d u)h(y)$ or $h(2^d u \circ y) = h(2^d u) \circ h(y)$ for all unitaries $u \in \mathcal{A}$, all $y \in \mathcal{A}$, and $d = 0, 1, 2, \ldots$, and that every almost linear almost multiplicative mapping $h : \mathcal{A} \to \mathcal{B}$ is a homomorphism when $h(2x) = 2h(x)$ for all $x \in \mathcal{A}$.

Moreover, we prove the Cauchy-Rassias stability of homomorphisms in C^*-algebras or in Lie JC^*-algebras, and of Lie JC^*-algebra derivations in Lie JC^*-algebras.

1. Introduction

In 1940, S. M. Ulam [20] raised the following question: Under what conditions does there exist an additive mapping near an approximately

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additive mapping?

Let X and Y be Banach spaces with norms $||\cdot||$ and $\|\cdot\|$, respectively. Hyers [3] showed that if $\epsilon > 0$ and $f : X \to Y$ such that

$$\|f(x+y) - f(x) - f(y)\| \le \epsilon$$

for all $x, y \in X$, then there exists a unique additive mapping $T : X \to Y$ such that

$$\|f(x) - T(x)\| \le \epsilon$$

for all $x \in X$.

Consider $f : X \to Y$ to be a mapping such that $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$. Assume that there exist constants $\epsilon \ge 0$ and $p \in [0, 1)$ such that

$$(*) \qquad \|f(x+y) - f(x) - f(y)\| \le \epsilon(||x||^p + ||y||^p)$$

for all $x, y \in X$. Th. M. Rassias [12] showed that there exists a unique $\mathbb{R}$-linear mapping $T : X \to Y$ such that

$$\|f(x) - T(x)\| \le \frac{2\epsilon}{2 - 2^p}||x||^p$$

for all $x \in X$. The inequality $(*)$ that was introduced for the first time by Th. M. Rassias [12] we call Cauchy-Rassias inequality and the stability of the functional equation *Cauchy-Rassias stability.* This inequality has provided a lot of influence in the development of what is now known as *Hyers–Ulam–Rassias stability* of functional equations. Beginning around the year 1980 the topic of approximate homomorphisms, or the stability of the equation of homomorphism, was taken up by a number of mathematicians (cf. [4], [10], [14]–[18]). Th. M. Rassias [13] during the 27^{th} International Symposium on Functional Equations asked the question whether such a theorem can also be proved for $p \ge 1$. Z. Gajda [1] following the same approach as in Th. M. Rassias [12], gave an affirmative solution to this question for $p > 1$.

Găvruta [2] generalized the Rassias' result: Let G be an abelian group and Y a Banach space. Denote by $\varphi : G \times G \to [0, \infty)$ a function such that

$$\widetilde{\varphi}(x, y) = \sum_{j=0}^{\infty} 2^{-j}\varphi(2^j x, 2^j y) < \infty$$

for all $x, y \in G$. Suppose that $f : G \to Y$ is a mapping satisfying

$$\|f(x+y) - f(x) - f(y)\| \le \varphi(x,y)$$

for all $x, y \in G$. Then there exists a unique additive mapping $T : G \to Y$ such that

$$\|f(x) - T(x)\| \le \frac{1}{2}\widetilde{\varphi}(x,x)$$

for all $x \in G$. C. Park [9] applied the Găvruta's result to linear functional equations in Banach modules over a C^*-algebra.

Jun and Lee [5] proved the following: Denote by $\varphi : X\backslash\{0\}\times X\backslash\{0\} \to [0,\infty)$ a function such that

$$\widetilde{\varphi}(x,y) = \sum_{j=0}^{\infty} 3^{-j}\varphi(3^j x, 3^j y) < \infty$$

for all $x, y \in X \setminus \{0\}$. Suppose that $f : X \to Y$ is a mapping satisfying

$$\|2f(\frac{x+y}{2}) - f(x) - f(y)\| \le \varphi(x,y)$$

for all $x, y \in X \setminus \{0\}$. Then there exists a unique additive mapping $T : X \to Y$ such that

$$\|f(x) - f(0) - T(x)| \le \frac{1}{3}(\widetilde{\varphi}(x,-x) + \widetilde{\varphi}(-x,3x))$$

for all $x \in X \setminus \{0\}$. C. Park and W. Park [11] applied the Jun and Lee's result to the Jensen's equation in Banach modules over a C^*-algebra.

Recently, T. Trif [19, Theorem 2.1] proved that, for vector spaces V and W, a mapping $f : V \to W$ with $f(0) = 0$ satisfies the functional equation

$$\text{(T)}\qquad \begin{aligned} &n\ {}_{n-2}C_{k-2}f\left(\frac{x_1+\cdots+x_n}{n}\right) + {}_{n-2}C_{k-1}\sum_{i=1}^{n} f(x_i) \\ &= k \sum_{1\le i_1<\cdots<i_k\le n} f\left(\frac{x_{i_1}+\cdots+x_{i_k}}{k}\right) \end{aligned}$$

for all $x_1, \ldots, x_n \in V$ if and only if the mapping $f : V \to W$ satisfies the additive Cauchy equation $f(x+y) = f(x)+f(y)$ for all $x, y \in V$. He proved the stability of the functional equation (T) (see [19, Theorems 3.1 and 3.2]).

Throughout this paper, assume that m, n, k are integers with $1 < m < k < mn$, and that s, q are integers with $1 \leq s \leq [\frac{n}{2}]$ and $1 < 2q \leq m$, where [ ] denotes the Gauss symbol.

In this paper, we solve the following functional equation

$$\begin{aligned} & mn\ {}_{mn-2}C_{k-2} f\left(\frac{x_1 + \cdots + x_{mn}}{mn}\right) \\ & + m\ {}_{mn-2}C_{k-1} \sum_{i=1}^{n} f\left(\frac{x_{mi-m+1} + \cdots + x_{mi}}{m}\right) \\ = k & \sum_{1 \leq i_1 < \cdots < i_k \leq mn} f\left(\frac{x_{i_1} + \cdots + x_{i_k}}{k}\right). \end{aligned} \tag{1.i}$$

We moreover prove the Cauchy-Rassias stability of the functional equation (1.i) in Banach modules over a unital C^*-algebra. The main purpose of this paper is to investigate homomorphisms between C^*-algebras and between Lie JC^*-algebras, and to prove their Cauchy-Rassias stability.

2. A generalized Trif's mapping

Throughout this section, assume that X and Y are linear spaces.

LEMMA 2.1. *A mapping $f : X \to Y$ with $f(0) = 0$ satisfies the functional equation (1.i) for all $x_1, \ldots, x_{mn} \in X$ if and only if the mapping $f : X \to Y$ satisfies the additive Cauchy equation $f(x+y) = f(x) + f(y)$ for all $x, y \in X$.*

Proof. Assume that a mapping $f : X \to Y$ with $f(0) = 0$ satisfies the functional equation (1.i).

Replacing x_m and x_{m+1} by x_{m+1} and x_m in (1.i), respectively, we get

$$\begin{aligned} & mn\ {}_{mn-2}C_{k-2} f\left(\frac{x_1 + \cdots + x_{mn}}{mn}\right) \\ & + m\ {}_{mn-2}C_{k-1}\ f\left(\frac{x_1 + \cdots + x_{m-1} + x_{m+1}}{m}\right) \\ & + m\ {}_{mn-2}C_{k-1}\ f\left(\frac{x_m + x_{m+2} + \cdots + x_{2m}}{m}\right) \end{aligned} \tag{2.1}$$

$$+ m\ {}_{mn-2}C_{k-1} \sum_{i=3}^{n} f\left(\frac{x_{mi-m+1} + \cdots + x_{mi}}{m}\right)$$
$$= k \sum_{1 \le i_1 < \cdots < i_k \le mn} f\left(\frac{x_{i_1} + \cdots + x_{i_k}}{k}\right)$$

for all $x_1, \ldots, x_{mn} \in X$. It follows from (1.i) and (2.1) that

$$\begin{aligned} &f\left(\frac{x_1 + \cdots + x_{m-1} + x_m}{m}\right) + f\left(\frac{x_{m+1} + x_{m+2} + \cdots + x_{2m}}{m}\right) \\ &= f\left(\frac{x_1 + \cdots + x_{m-1} + x_{m+1}}{m}\right) + f\left(\frac{x_m + x_{m+2} + \cdots + x_{2m}}{m}\right) \end{aligned} \tag{2.2}$$

for all $x_1, \ldots, x_{2m} \in X$. Letting $x_{m-1} = x_m = x$, $x_{m+1} = x_{m+2} = y$ and $x_1 = \cdots = x_{m-2} = x_{m+3} = \cdots = x_{2m} = 0$ in (2.2), we get

$$f\left(\frac{2x}{m}\right) + f\left(\frac{2y}{m}\right) = 2f\left(\frac{x+y}{m}\right) \tag{2.3}$$

for all $x, y \in X$. Putting $y = 0$ in (2.3), we obtain

$$f\left(\frac{2x}{m}\right) = 2f\left(\frac{x}{m}\right)$$

for all $x \in X$. So

$$2f\left(\frac{x}{m}\right) + 2f\left(\frac{y}{m}\right) = 2f\left(\frac{x+y}{m}\right) \tag{2.4}$$

for all $x, y \in X$. Replacing x and y by mx and my in (2.4), respectively, we get

$$2f(x) + 2f(y) = 2f(x+y)$$

for all $x, y \in X$. Thus the mapping $f : X \to Y$ satisfies the additive Cauchy equation $f(x+y) = f(x) + f(y)$ for all $x, y \in X$.

The converse is obvious. □

3. Cauchy-Rassias stability of the generalized Trif's mapping in Banach modules over a C^*-algebra

Throughout this section, assume that $\mathcal{A}$ is a unital C^*-algebra with norm $|\cdot|$ and unitary group $\mathcal{U}(\mathcal{A})$, and that X and Y are left Banach modules over $\mathcal{A}$ with norms $||\cdot||$ and $\|\cdot\|$, respectively.

Given a mapping $f : X \to Y$, we set

$$D_u f(x_1, \dots, x_{mn}) := mn\ {}_{mn-2}C_{k-2} f\left(\frac{ux_1 + \cdots + ux_{mn}}{mn}\right) + m\ {}_{mn-2}C_{k-1} \sum_{i=1}^{n} f\left(\frac{ux_{mi-m+1} + \cdots + ux_{mi}}{m}\right) - k \sum_{1 \le i_1 < \cdots < i_k \le mn} uf\left(\frac{x_{i_1} + \cdots + x_{i_k}}{k}\right)$$

for all $u \in \mathcal{U}(\mathcal{A})$ and all $x_1, \dots, x_{mn} \in X$.

THEOREM 3.1. *Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ for which there is a function $\varphi : X^{mn} \to [0, \infty)$ such that*

$$\text{(3.i)} \qquad \widetilde{\varphi}(x_1, \dots, x_{mn}) := \sum_{j=0}^{\infty} \frac{1}{2^j} \varphi(2^j x_1, \dots, 2^j x_{mn}) < \infty,$$

$$\text{(3.ii)} \qquad \|D_u f(x_1, \dots, x_{mn})\| \le \varphi(x_1, \dots, x_{mn})$$

for all $u \in \mathcal{U}(\mathcal{A})$ and all $x_1, \dots, x_{mn} \in X$. Then there exists a unique $\mathcal{A}$-linear generalized Trif's mapping $T : X \to Y$ such that

(3.iii)

$$\begin{aligned}
&\|f(x) - T(x)\| \\
&\le \frac{1}{2ms\ {}_{mn-2}C_{k-1}} \widetilde{\varphi}(\underbrace{0, \dots, 0}_{m-2q \text{ times}}, \underbrace{\frac{mx}{q}, \dots, \frac{mx}{q}}_{q \text{ times}}, \underbrace{0, \dots, 0}_{q \text{ times}}, \\
&\quad \underbrace{\frac{mx}{q}, \dots, \frac{mx}{q}}_{q \text{ times}}, \underbrace{0, \dots, 0}_{m-q \text{ times}}, \dots, \underbrace{0, \dots, 0}_{m-2q \text{ times}}, \underbrace{\frac{mx}{q}, \dots, \frac{mx}{q}}_{q \text{ times}}, \underbrace{0, \dots, 0}_{q \text{ times}}, \\
&\quad \underbrace{\frac{mx}{q}, \dots, \frac{mx}{q}}_{q \text{ times}}, \underbrace{0, \dots, 0}_{m-q \text{ times}}, \underbrace{0, \dots, 0}_{mn-2ms \text{ times}}) \\
&\quad + \frac{1}{2ms\ {}_{mn-2}C_{k-1}} \widetilde{\varphi}(\underbrace{0, \dots, 0}_{m-2q \text{ times}}, \underbrace{\frac{mx}{q}, \dots, \frac{mx}{q}}_{q \text{ times}}, \underbrace{\frac{mx}{q}, \dots, \frac{mx}{q}}_{q \text{ times}}, \\
&\quad \underbrace{0, \dots, 0}_{q \text{ times}}, \underbrace{0, \dots, 0}_{m-q \text{ times}}, \dots, \underbrace{0, \dots, 0}_{m-2q \text{ times}}, \underbrace{\frac{mx}{q}, \dots, \frac{mx}{q}}_{q \text{ times}}, \underbrace{\frac{mx}{q}, \dots, \frac{mx}{q}}_{q \text{ times}}, \\
&\quad \underbrace{0, \dots, 0}_{q \text{ times}}, \underbrace{0, \dots, 0}_{m-q \text{ times}}, \underbrace{0, \dots, 0}_{mn-2ms \text{ times}})
\end{aligned}$$

for all $x \in X$*.*

Proof. Let $u = 1 \in \mathcal{U}(\mathcal{A})$. Putting $x_{im-2q+1} = \cdots = x_{im-q} = x$, $x_{im-q+1} = \cdots = x_{im} = 0$, $x_{im+1} = \cdots = x_{im+q} = x$ for $i = 1, 3, \ldots, 2s-1$, and $x_j = 0$ for other indices j in (3.ii), we have

$$\begin{aligned}
&\left\| mn\ {}_{mn-2}C_{k-2} f\left(\frac{2qsx}{mn}\right) + 2ms\ {}_{mn-2}C_{k-1} f\left(\frac{qx}{m}\right) \right.\\
&\left. - k \sum_{1\le i_1<\cdots<i_k\le mn} f\left(\frac{x_{i_1}+\cdots+x_{i_k}}{k}\right)\right\| \\
&\le \varphi(\underbrace{0,\ldots,0}_{m-2q \text{ times}}, \underbrace{x,\ldots,x}_{q \text{ times}}, \underbrace{0,\ldots,0}_{q \text{ times}}, \underbrace{x,\ldots,x}_{q \text{ times}}, \underbrace{0,\ldots,0}_{m-q \text{ times}}, \ldots, \\
&\quad \underbrace{0,\ldots,0}_{m-2q \text{ times}}, \underbrace{x,\ldots,x}_{q \text{ times}}, \underbrace{0,\ldots,0}_{q \text{ times}}, \underbrace{x,\ldots,x}_{q \text{ times}}, \underbrace{0,\ldots,0}_{m-q \text{ times}}, \underbrace{0,\ldots,0}_{mn-2ms \text{ times}})
\end{aligned} \tag{3.1}$$

for all $x \in X$. Putting $x_{im-2q+1} = \cdots = x_{im-q} = x$, $x_{im-q+1} = \cdots = x_{im} = x$ for $i = 1, 3, \ldots, 2s-1$, and $x_j = 0$ for other indices j in (3.ii), we have

$$\begin{aligned}
&\left\| mn\ {}_{mn-2}C_{k-2} f\left(\frac{2qsx}{mn}\right) + ms\ {}_{mn-2}C_{k-1} f\left(\frac{2qx}{m}\right) \right.\\
&\left. - k \sum_{1\le i_1<\cdots<i_k\le mn} f\left(\frac{x_{i_1}+\cdots+x_{i_k}}{k}\right)\right\| \\
&\le \varphi(\underbrace{0,\ldots,0}_{m-2q \text{ times}}, \underbrace{x,\ldots,x}_{q \text{ times}}, \underbrace{x,\ldots,x}_{q \text{ times}}, \underbrace{0,\ldots,0}_{q \text{ times}}, \underbrace{0,\ldots,0}_{m-q \text{ times}}, \ldots, \\
&\quad \underbrace{0,\ldots,0}_{m-2q \text{ times}}, \underbrace{x,\ldots,x}_{q \text{ times}}, \underbrace{x,\ldots,x}_{q \text{ times}}, \underbrace{0,\ldots,0}_{q \text{ times}}, \underbrace{0,\ldots,0}_{m-q \text{ times}}, \underbrace{0,\ldots,0}_{mn-2ms \text{ times}})
\end{aligned} \tag{3.2}$$

for all $x \in X$. It follows from (3.1) and (3.2) that

$$\begin{aligned}
&\left\| 2ms\ {}_{mn-2}C_{k-1} f\left(\frac{qx}{m}\right) - ms\ {}_{mn-2}C_{k-1} f\left(\frac{2qx}{m}\right)\right\| \\
&\le \varphi(\underbrace{0,\ldots,0}_{m-2q \text{ times}}, \underbrace{x,\ldots,x}_{q \text{ times}}, \underbrace{0,\ldots,0}_{q \text{ times}}, \underbrace{x,\ldots,x}_{q \text{ times}}, \underbrace{0,\ldots,0}_{m-q \text{ times}}, \ldots, \\
&\quad \underbrace{0,\ldots,0}_{m-2q \text{ times}}, \underbrace{x,\ldots,x}_{q \text{ times}}, \underbrace{0,\ldots,0}_{q \text{ times}}, \underbrace{x,\ldots,x}_{q \text{ times}}, \underbrace{0,\ldots,0}_{m-q \text{ times}}, \underbrace{0,\ldots,0}_{mn-2ms \text{ times}})
\end{aligned}$$

$$+\varphi(\underbrace{0,\dots,0}_{m-2q\text{ times}},\underbrace{x,\dots,x}_{q\text{ times}},\underbrace{x,\dots,x}_{q\text{ times}},\underbrace{0,\dots,0}_{q\text{ times}},\underbrace{0,\dots,0}_{m-q\text{ times}},\dots,$$
$$\underbrace{0,\dots,0}_{m-2q\text{ times}},\underbrace{x,\dots,x}_{q\text{ times}},\underbrace{x,\dots,x}_{q\text{ times}},\underbrace{0,\dots,0}_{q\text{ times}},\underbrace{0,\dots,0}_{m-q\text{ times}},\underbrace{0,\dots,0}_{mn-2ms\text{ times}})$$

for all $x \in X$. So

$$\begin{aligned}
&\left\|f(x)-\frac{1}{2}f(2x)\right\| \\
&\leq \frac{1}{2ms\ {}_{mn-2}C_{k-1}}\varphi(\underbrace{0,\dots,0}_{m-2q\text{ times}},\underbrace{\frac{mx}{q},\dots,\frac{mx}{q}}_{q\text{ times}},\underbrace{0,\dots,0}_{q\text{ times}}, \\
&\underbrace{\frac{mx}{q},\dots,\frac{mx}{q}}_{q\text{ times}},\underbrace{0,\dots,0}_{m-q\text{ times}},\dots,\underbrace{0,\dots,0}_{m-2q\text{ times}},\underbrace{\frac{mx}{q},\dots,\frac{mx}{q}}_{q\text{ times}}, \\
&\underbrace{0,\dots,0}_{q\text{ times}},\underbrace{\frac{mx}{q},\dots,\frac{mx}{q}}_{q\text{ times}},\underbrace{0,\dots,0}_{m-q\text{ times}},\underbrace{0,\dots,0}_{mn-2ms\text{ times}}) \\
&+\frac{1}{2ms\ {}_{mn-2}C_{k-1}}\varphi(\underbrace{0,\dots,0}_{m-2q\text{ times}},\underbrace{\frac{mx}{q},\dots,\frac{mx}{q}}_{q\text{ times}},\underbrace{\frac{mx}{q},\dots,\frac{mx}{q}}_{q\text{ times}} \\
&\underbrace{0,\dots,0}_{q\text{ times}},\underbrace{0,\dots,0}_{m-q\text{ times}},\dots,\underbrace{0,\dots,0}_{m-2q\text{ times}},\underbrace{\frac{mx}{q},\dots,\frac{mx}{q}}_{q\text{ times}}, \\
&\underbrace{\frac{mx}{q},\dots,\frac{mx}{q}}_{q\text{ times}},\underbrace{0,\dots,0}_{q\text{ times}},\underbrace{0,\dots,0}_{m-q\text{ times}},\underbrace{0,\dots,0}_{mn-2ms\text{ times}})
\end{aligned} \tag{3.‡}$$

for all $x \in X$. Hence

$$\begin{aligned}
&\left\|\frac{1}{2^d}f(2^dx)-\frac{1}{2^{d+1}}f(2^{d+1}x)\right\| \\
&=\frac{1}{2^d}\left\|f(2^dx)-\frac{1}{2}f(2\cdot 2^dx)\right\| \\
&\leq \frac{1}{2^{d+1}ms\ {}_{mn-2}C_{k-1}}\varphi(\underbrace{0,\dots,0}_{m-2q\text{ times}},\underbrace{\frac{2^dmx}{q},\dots,\frac{2^dmx}{q}}_{q\text{ times}},\underbrace{0,\dots,0}_{q\text{ times}}
\end{aligned} \tag{3.3}$$

$$\underbrace{\frac{2^d mx}{q},\ldots,\frac{2^d mx}{q}}_{q\text{ times}},\underbrace{0,\ldots,0}_{m-q\text{ times}},\ldots,\underbrace{0,\ldots,0}_{m-2q\text{ times}},\underbrace{\frac{2^d mx}{q},\ldots,\frac{2^d mx}{q}}_{q\text{ times}},$$

$$\underbrace{0,\ldots,0}_{q\text{ times}},\underbrace{\frac{2^d mx}{q},\ldots,\frac{2^d mx}{q}}_{q\text{ times}},\underbrace{0,\ldots,0}_{m-q\text{ times}},\underbrace{0,\ldots,0}_{mn-2ms\text{ times}})$$

$$+\frac{1}{2^{d+1}ms\ {}_{mn-2}C_{k-1}}\varphi(\underbrace{0,\ldots,0}_{m-2q\text{ times}},\underbrace{\frac{2^d mx}{q},\ldots,\frac{2^d mx}{q}}_{q\text{ times}},$$

$$\underbrace{\frac{2^d mx}{q},\ldots,\frac{2^d mx}{q}}_{q\text{ times}},\underbrace{0,\ldots,0}_{q\text{ times}},\underbrace{0,\ldots,0}_{m-q\text{ times}},\ldots,\underbrace{0,\ldots,0}_{m-2q\text{ times}},\underbrace{\frac{2^d mx}{q},\ldots,\frac{2^d mx}{q}}_{q\text{ times}},$$

$$\underbrace{\frac{2^d mx}{q},\ldots,\frac{2^d mx}{q}}_{q\text{ times}},\underbrace{0,\ldots,0}_{q\text{ times}},\underbrace{0,\ldots,0}_{m-q\text{ times}},\underbrace{0,\ldots,0}_{mn-2ms\text{ times}})$$

for all $x \in X$ and all positive integers d. By (3.3), we have

$$\left\|\frac{1}{2^l}f(2^l x)-\frac{1}{2^d}f\left(2^d x\right)\right\|$$

$$\leq \sum_{j=l}^{d-1}\frac{1}{2^{j+1}ms\ {}_{mn-2}C_{k-1}}\varphi(\underbrace{0,\ldots,0}_{m-2q\text{ times}},\underbrace{\frac{2^j mx}{q},\ldots,\frac{2^j mx}{q}}_{q\text{ times}},$$

$$\underbrace{0,\ldots,0}_{q\text{ times}},\underbrace{\frac{2^j mx}{q},\ldots,\frac{2^j mx}{q}}_{q\text{ times}},$$

$$\underbrace{0,\ldots,0}_{m-q\text{ times}},\ldots,\underbrace{0,\ldots,0}_{m-2q\text{ times}},\underbrace{\frac{2^j x}{q},\ldots,\frac{2^j mx}{q}}_{q\text{ times}}, \tag{3.4}$$

$$\underbrace{0,\ldots,0}_{q\text{ times}},\underbrace{\frac{2^j mx}{q},\ldots,\frac{2^j mx}{q}}_{q\text{ times}},\underbrace{0,\ldots,0}_{m-q\text{ times}},\underbrace{0,\ldots,0}_{mn-2ms\text{ times}})$$

$$+\sum_{j=l}^{d-1}\frac{1}{2^{j+1}ms\ {}_{mn-2}C_{k-1}}\varphi(\underbrace{0,\ldots,0}_{m-2q\text{ times}},\underbrace{\frac{2^j mx}{q},\ldots,\frac{2^j mx}{q}}_{q\text{ times}},$$

$$\underbrace{\frac{2^j mx}{q}, \dots, \frac{2^j mx}{q}}_{q \text{ times}}, \underbrace{0, \dots, 0}_{q \text{ times}}, \underbrace{0, \dots, 0}_{m-q \text{ times}}, \dots, \underbrace{0, \dots, 0}_{m-2q \text{ times}}, \underbrace{\frac{2^j mx}{q}, \dots, \frac{2^j mx}{q}}_{q \text{ times}},$$

$$\underbrace{\frac{2^j mx}{q}, \dots, \frac{2^j mx}{q}}_{q \text{ times}}, \underbrace{0, \dots, 0}_{q \text{ times}}, \underbrace{0, \dots, 0}_{m-q \text{ times}}, \underbrace{0, \dots, 0}_{mn-2ms \text{ times}})$$

for all $x \in X$ and all positive integers l and d with $l < d$. This shows that the sequence $\{\frac{1}{2^d} f(2^d x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{2^d} f(2^d x)\}$ converges for all $x \in X$. So we can define a mapping $T : X \to Y$ by

$$T(x) := \lim_{d \to \infty} \frac{1}{2^d} f(2^d x)$$

for all $x \in X$. Also, we get

$$\begin{aligned} \|D_1 T(x_1, \dots, x_{mn})\| &= \lim_{d \to \infty} \frac{1}{2^d} \|D_1 f(2^d x_1, \dots, 2^d x_{mn})\| \\ &\le \lim_{d \to \infty} \frac{1}{2^d} \varphi(2^d x_1, \dots, 2^d x_{mn}) \\ &= 0 \end{aligned}$$

for all $x_1, \dots, x_{mn} \in X$. Thus T is a generalized Trif's mapping. By Lemma 2.1, T is additive. Putting $l = 0$ and letting $d \to \infty$ in (3.4), we get (3.iii).

Now, let $L : X \to Y$ be another generalized Trif's mapping satisfying (3.iii). Then we have

$$\begin{aligned} &\|T(x) - L(x)\| \\ &= \frac{1}{2^d} \|T(2^d x) - L(2^d x)\| \\ &\le \frac{1}{2^d} (\|T(2^d x) - f(2^d x)\| + \|L(2^d x) - f(2^d x)\|) \\ &\le \frac{2}{2^{d+1} ms \; {}_{mn-2}C_{k-1}} \widetilde{\varphi}(\underbrace{0, \dots, 0}_{m-2q \text{ times}}, \underbrace{\frac{2^d mx}{q}, \dots, \frac{2^d mx}{q}}_{q \text{ times}}, \underbrace{0, \dots, 0}_{q \text{ times}}, \end{aligned}$$

$$\underbrace{\frac{2^d mx}{q}, \dots, \frac{2^d mx}{q}}_{q \text{ times}}, \underbrace{0, \dots, 0}_{m-q \text{ times}}, \dots, \underbrace{0, \dots, 0}_{m-2q \text{ times}}, \underbrace{\frac{2^d mx}{q}, \dots, \frac{2^d mx}{q}}_{q \text{ times}},$$

$$\underbrace{0,\dots,0}_{q\text{ times}},\underbrace{\frac{2^d mx}{q},\dots,\frac{2^d mx}{q}}_{q\text{ times}},\underbrace{0,\dots,0}_{m-q\text{ times}},\underbrace{0,\dots,0}_{mn-2ms\text{ times}})$$

$$+\frac{2}{2^{d+1}ms\ {}_{mn-2}C_{k-1}}\widetilde{\varphi}(\underbrace{0,\dots,0}_{m-2q\text{ times}},\underbrace{\frac{2^d mx}{q},\dots,\frac{2^d mx}{q}}_{q\text{ times}},$$

$$\underbrace{\frac{2^d mx}{q},\dots,\frac{2^d mx}{q}}_{q\text{ times}},\underbrace{0,\dots,0}_{q\text{ times}},\underbrace{0,\dots,0}_{m-q\text{ times}},\dots,\underbrace{0,\dots,0}_{m-2q\text{ times}},\underbrace{\frac{2^d mx}{q},\dots,\frac{2^d mx}{q}}_{q\text{ times}},$$

$$\underbrace{\frac{2^d mx}{q},\dots,\frac{2^d mx}{q}}_{q\text{ times}},\underbrace{0,\dots,0}_{q\text{ times}},\underbrace{0,\dots,0}_{m-q\text{ times}},\underbrace{0,\dots,0}_{mn-2ms\text{ times}}),$$

which tends to zero as $d \to \infty$ for all $x \in X$. So we can conclude that $T(x) = L(x)$ for all $x \in X$. This proves the uniqueness of T.

By the assumption, for each $u \in \mathcal{U}(\mathcal{A})$, we get

$$\begin{aligned}\|D_u T(x,\underbrace{0,\dots,0}_{mn-1\text{ times}})\| &= \lim_{d\to\infty}\frac{1}{2^d}\|D_u f(2^d x,\underbrace{0,\dots,0}_{mn-1\text{ times}})\| \\ &\le \lim_{d\to\infty}\frac{1}{2^d}\varphi(2^d x,\underbrace{0,\dots,0}_{mn-1\text{ times}}) = 0\end{aligned}$$

for all $x \in X$. So

$$mn\ {}_{mn-2}C_{k-2}T(\frac{ux}{mn}) + m\ {}_{mn-2}C_{k-1}T(\frac{ux}{m}) = k\ {}_{mn-1}C_{k-1}uT(\frac{x}{k})$$

for all $u \in \mathcal{U}(\mathcal{A})$ and all $x \in X$. Since ${}_{mn-2}C_{k-2} + {}_{mn-2}C_{k-1} = {}_{mn-1}C_{k-1}$ and T is additive,

$$T(ux) = uT(x) \tag{3.5}$$

for all $u \in \mathcal{U}(\mathcal{A})$ and all $x \in X$.

Now let $a \in \mathcal{A}$ ($a \neq 0$) and M an integer greater than $4|a|$. Then $|\frac{a}{M}| < \frac{1}{4} < 1 - \frac{2}{3} = \frac{1}{3}$. By [6, Theorem 1], there exist three elements $u_1, u_2, u_3 \in \mathcal{U}(\mathcal{A})$ such that $3\frac{a}{M} = u_1 + u_2 + u_3$. So by (3.5)

$$\begin{aligned}T(ax) = T(\frac{M}{3}\cdot 3\frac{a}{M}x) &= M\cdot T(\frac{1}{3}\cdot 3\frac{a}{M}x) \\ &= \frac{M}{3}T(3\frac{a}{M}x)\end{aligned}$$

$$
\begin{aligned}
&= \frac{M}{3} T(u_1 x + u_2 x + u_3 x) \\
&= \frac{M}{3} (T(u_1 x) + T(u_2 x) + T(u_3 x)) \\
&= \frac{M}{3} (u_1 + u_2 + u_3) T(x) \\
&= \frac{M}{3} \cdot 3 \frac{a}{M} T(x) = aT(x)
\end{aligned}
$$

for all $a \in \mathcal{A}$ and all $x \in X$. Hence

$$T(ax + by) = T(ax) + T(by) = aT(x) + bT(y)$$

for all $a, b \in \mathcal{A}(a, b \neq 0)$ and all $x, y \in X$. And $T(0x) = 0 = 0T(x)$ for all $x \in X$. So the generalized Trif's mapping $T : \mathcal{A} \to \mathcal{B}$ is an $\mathcal{A}$-linear mapping, as desired. □

Corollary 3.2. *Let θ and $p < 1$ be positive real numbers. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ such that*

$$\|D_u f(x_1, \dots, x_{mn})\| \leq \theta \sum_{j=1}^{mn} ||x_j||^p$$

for all $u \in \mathcal{U}(\mathcal{A})$ and all $x_1, \dots, x_{mn} \in X$. Then there exists a unique $\mathcal{A}$-linear generalized Trif's mapping $T : X \to Y$ such that

$$\|f(x) - T(x)\| \leq \frac{4m^{p-1}q^{1-p}\theta}{(2 - 2^p)\ {}_{mn-2}C_{k-1}} ||x||^p$$

for all $x \in X$.

Proof. Define $\varphi(x_1, \dots, x_{mn}) = \theta \sum_{j=1}^{mn} ||x_j||^p$, and apply Theorem 3.1. □

Theorem 3.3. *Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ for which there is a function $\varphi : X^{mn} \to [0, \infty)$ such that*

$$\widetilde{\varphi}(x_1, \dots, x_{mn}) := \sum_{j=1}^{\infty} 2^j \varphi\left(\frac{x_1}{2^j}, \dots, \frac{x_{mn}}{2^j}\right) < \infty, \tag{3.iv}$$

$$\|D_u f(x_1, \dots, x_{mn})\| \leq \varphi(x_1, \dots, x_{mn}) \tag{3.v}$$

for all $u \in \mathcal{U}(\mathcal{A})$ and all $x_1, \ldots, x_{mn} \in X$. Then there exists a unique $\mathcal{A}$-linear generalized Trif's mapping $T : X \to Y$ such that

(3.vi)

$$\begin{aligned}
&\|f(x) - T(x)\| \\
&\le \frac{1}{2ms\ {}_{mn-2}C_{k-1}} \widetilde{\varphi}(\underbrace{0, \ldots, 0}_{m-2q \text{ times}}, \underbrace{\frac{mx}{q}, \ldots, \frac{mx}{q}}_{q \text{ times}}, \underbrace{0, \ldots, 0}_{q \text{ times}}, \\
&\underbrace{\frac{mx}{q}, \ldots, \frac{mx}{q}}_{q \text{ times}}, \underbrace{0, \ldots, 0}_{m-q \text{ times}}, \ldots, \underbrace{0, \ldots, 0}_{m-2q \text{ times}}, \underbrace{\frac{mx}{q}, \ldots, \frac{mx}{q}}_{q \text{ times}}, \underbrace{0, \ldots, 0}_{q \text{ times}}, \\
&\underbrace{\frac{mx}{q}, \ldots, \frac{mx}{q}}_{q \text{ times}}, \underbrace{0, \ldots, 0}_{m-q \text{ times}}, \underbrace{0, \ldots, 0}_{mn-2ms \text{ times}}) \\
&+ \frac{1}{2ms\ {}_{mn-2}C_{k-1}} \widetilde{\varphi}(\underbrace{0, \ldots, 0}_{m-2q \text{ times}}, \underbrace{\frac{mx}{q}, \ldots, \frac{mx}{q}}_{q \text{ times}}, \underbrace{\frac{mx}{q}, \ldots, \frac{mx}{q}}_{q \text{ times}}, \\
&\underbrace{0, \ldots, 0}_{q \text{ times}}, \underbrace{0, \ldots, 0}_{m-q \text{ times}}, \ldots, \underbrace{0, \ldots, 0}_{m-2q \text{ times}}, \underbrace{\frac{mx}{q}, \ldots, \frac{mx}{q}}_{q \text{ times}}, \underbrace{\frac{mx}{q}, \ldots, \frac{mx}{q}}_{q \text{ times}}, \\
&\underbrace{0, \ldots, 0}_{q \text{ times}}, \underbrace{0, \ldots, 0}_{m-q \text{ times}}, \underbrace{0, \ldots, 0}_{mn-2ms \text{ times}})
\end{aligned}$$

for all $x \in X$.

Proof. Replacing x by $\frac{x}{2}$ in (3.‡), we have

$$\begin{aligned}
&\|f(x) - 2f(\frac{x}{2})\| \\
&\le \frac{1}{ms\ {}_{mn-2}C_{k-1}} \varphi(\underbrace{0, \ldots, 0}_{m-2q \text{ times}}, \underbrace{\frac{mx}{2q}, \ldots, \frac{mx}{2q}}_{q \text{ times}}, \underbrace{0, \ldots, 0}_{q \text{ times}}, \\
&\underbrace{\frac{mx}{2q}, \ldots, \frac{mx}{2q}}_{q \text{ times}}, \underbrace{0, \ldots, 0}_{m-q \text{ times}}, \ldots, \underbrace{0, \ldots, 0}_{m-2q \text{ times}}, \underbrace{\frac{mx}{2q}, \ldots, \frac{mx}{2q}}_{q \text{ times}}, \underbrace{0, \ldots, 0}_{q \text{ times}}, \\
&\underbrace{\frac{mx}{2q}, \ldots, \frac{mx}{2q}}_{q \text{ times}}, \underbrace{0, \ldots, 0}_{m-q \text{ times}}, \underbrace{0, \ldots, 0}_{mn-2ms \text{ times}})
\end{aligned}$$

$$+\frac{1}{ms\ {}_{mn-2}C_{k-1}}\varphi(\underbrace{0,\ldots,0}_{m-2q\text{ times}},\underbrace{\frac{mx}{2q},\ldots,\frac{mx}{2q}}_{q\text{ times}},\underbrace{\frac{mx}{2q},\ldots,\frac{mx}{2q}}_{q\text{ times}},$$

$$\underbrace{0,\ldots,0}_{q\text{ times}},\underbrace{0,\ldots,0}_{m-q\text{ times}},\ldots,\underbrace{0,\ldots,0}_{m-2q\text{ times}},\underbrace{\frac{mx}{2q},\ldots,\frac{mx}{2q}}_{q\text{ times}},\underbrace{\frac{mx}{2q},\ldots,\frac{mx}{2q}}_{q\text{ times}},$$

$$\underbrace{0,\ldots,0}_{q\text{ times}},\underbrace{0,\ldots,0}_{m-q\text{ times}},\underbrace{0,\ldots,0}_{mn-2ms\text{ times}})$$

for all $x \in X$. So

(3.6)

$$\left\|2^d f\left(\frac{x}{2^d}\right)-2^{d+1}f\left(\frac{x}{2^{d+1}}\right)\right\|$$

$$=2^d\left\|f\left(\frac{x}{2^d}\right)-2f\left(\frac{x}{2\cdot 2^d}\right)\right\|$$

$$\leq\frac{2^d}{ms\ {}_{mn-2}C_{k-1}}\varphi(\underbrace{0,\ldots,0}_{m-2q\text{ times}},\underbrace{\frac{mx}{2^{d+1}q},\ldots,\frac{mx}{2^{d+1}q}}_{q\text{ times}},\underbrace{0,\ldots,0}_{q\text{ times}},$$

$$\underbrace{\frac{mx}{2^{d+1}q},\ldots,\frac{mx}{2^{d+1}q}}_{q\text{ times}},\underbrace{0,\ldots,0}_{m-q\text{ times}},\ldots,\underbrace{0,\ldots,0}_{m-2q\text{ times}},\underbrace{\frac{mx}{2^{d+1}q},\ldots,\frac{mx}{2^{d+1}q}}_{q\text{ times}}$$

$$\underbrace{0,\ldots,0}_{q\text{ times}},\underbrace{\frac{mx}{2^{d+1}q},\ldots,\frac{mx}{2^{d+1}q}}_{q\text{ times}},\underbrace{0,\ldots,0}_{m-q\text{ times}},\underbrace{0,\ldots,0}_{mn-2ms\text{ times}})$$

$$+\frac{2^d}{ms\ {}_{mn-2}C_{k-1}}\varphi(\underbrace{0,\ldots,0}_{m-2q\text{ times}},\underbrace{\frac{mx}{2^{d+1}q},\ldots,\frac{mx}{2^{d+1}q}}_{q\text{ times}},$$

$$\underbrace{\frac{mx}{2^{d+1}q},\ldots,\frac{mx}{2^{d+1}q}}_{q\text{ times}},$$

$$\underbrace{0,\ldots,0}_{q\text{ times}},\underbrace{0,\ldots,0}_{m-q\text{ times}},\ldots,\underbrace{0,\ldots,0}_{m-2q\text{ times}},\underbrace{\frac{mx}{2^{d+1}q},\ldots,\frac{mx}{2^{d+1}q}}_{q\text{ times}},$$

$$\underbrace{\frac{mx}{2^{d+1}q},\ldots,\frac{mx}{2^{d+1}q}}_{q\text{ times}},\underbrace{0,\ldots,0}_{q\text{ times}},\underbrace{0,\ldots,0}_{m-q\text{ times}},\underbrace{0,\ldots,0}_{mn-2ms\text{ times}})$$

for all $x \in X$ and all positive integers d. By (3.6), we have

(3.7)

$$\begin{aligned}
&\left\|2^l f(\frac{x}{2^l}) - 2^d f(\frac{x}{2^d})\right\| \\
&\leq \sum_{j=l}^{d-1} \frac{2^j}{ms\ {}_{mn-2}C_{k-1}} \varphi(\underbrace{0,\dots,0}_{m-2q \text{ times}}, \underbrace{\frac{mx}{2^{j+1}q},\dots,\frac{mx}{2^{j+1}q}}_{q \text{ times}}, \underbrace{0,\dots,0}_{q \text{ times}}, \\
&\underbrace{\frac{mx}{2^{j+1}q},\dots,\frac{mx}{2^{j+1}q}}_{q \text{ times}}, \underbrace{0,\dots,0}_{m-q \text{ times}},\dots, \underbrace{0,\dots,0}_{m-2q \text{ times}}, \underbrace{\frac{mx}{2^{j+1}q},\dots,\frac{mx}{2^{j+1}q}}_{q \text{ times}}, \\
&\underbrace{0,\dots,0}_{q \text{ times}}, \underbrace{\frac{mx}{2^{j+1}q},\dots,\frac{mx}{2^{j+1}q}}_{q \text{ times}}, \underbrace{0,\dots,0}_{m-q \text{ times}}, \underbrace{0,\dots,0}_{mn-2ms \text{ times}}) \\
&+ \sum_{j=l}^{d-1} \frac{2^j}{ms\ {}_{mn-2}C_{k-1}} \varphi(\underbrace{0,\dots,0}_{m-2q \text{ times}}, \underbrace{\frac{mx}{2^{j+1}q},\dots,\frac{mx}{2^{j+1}q}}_{q \text{ times}}, \\
&\underbrace{\frac{mx}{2^{j+1}q},\dots,\frac{mx}{2^{j+1}q}}_{q \text{ times}}, \underbrace{0,\dots,0}_{q \text{ times}}, \underbrace{0,\dots,0}_{m-q \text{ times}},\dots, \underbrace{0,\dots,0}_{m-2q \text{ times}}, \\
&\underbrace{\frac{mx}{2^{j+1}q},\dots,\frac{mx}{2^{j+1}q}}_{q \text{ times}}, \underbrace{\frac{mx}{2^{j+1}q},\dots,\frac{mx}{2^{j+1}q}}_{q \text{ times}}, \underbrace{0,\dots,0}_{q \text{ times}}, \underbrace{0,\dots,0}_{m-q \text{ times}}, \\
&\underbrace{0,\dots,0}_{mn-2ms \text{ times}})
\end{aligned}$$

for all $x \in X$ and all positive integers l and d with $l < d$. This shows that the sequence $\{2^d f(\frac{x}{2^d})\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{2^d f(\frac{x}{2^d})\}$ converges for all $x \in X$. So we can define a mapping $T : X \to Y$ by

$$T(x) := \lim_{d\to\infty} 2^d f\left(\frac{x}{2^d}\right)$$

for all $x \in X$. Also, we get

$$\begin{aligned}
\|D_1 T(x_1,\dots,x_{mn})\| &= \lim_{d\to\infty} 2^d \left\|D_1 f\left(\frac{x_1}{2^d},\dots,\frac{x_{mn}}{2^d}\right)\right\| \\
&\leq \lim_{d\to\infty} 2^d \varphi\left(\frac{x_1}{2^d},\dots,\frac{x_{mn}}{2^d}\right) = 0
\end{aligned}$$

for all $x_1, \ldots, x_{mn} \in X$. Thus T is a generalized Trif's mapping. By Lemma 2.1, T is additive. Putting $l = 0$ and letting $d \to \infty$ in (3.7), we get (3.vi).

The rest of the proof is similar to the proof of Theorem 3.1. □

COROLLARY 3.4. *Let θ and $p > 1$ be positive real numbers. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ such that*

$$\|D_u f(x_1, \ldots, x_{mn})\| \leq \theta \sum_{j=1}^{mn} ||x_j||^p$$

for all $u \in \mathcal{U}(\mathcal{A})$ and all $x_1, \ldots, x_{mn} \in X$. Then there exists a unique $\mathcal{A}$-linear generalized Trif's mapping $T : X \to Y$ such that

$$\|f(x) - T(x)\| \leq \frac{4m^{p-1}q^{1-p}\theta}{(2^p - 2)\ {}_{mn-2}C_{k-1}} ||x||^p$$

for all $x \in X$.

Proof. Define $\varphi(x_1, \ldots, x_{mn}) = \theta \sum_{j=1}^{mn} ||x_j||^p$, and apply Theorem 3.3. □

4. Isomorphisms between unital C^*-algebras

Throughout this section, assume that $\mathcal{A}$ is a unital C^*-algebra with norm $|| \cdot ||$, unit e and unitary group $\mathcal{U}(\mathcal{A})$, and that $\mathcal{B}$ is a unital C^*-algebra with norm $\| \cdot \|$.

We are going to investigate C^*-algebra isomorphisms between unital C^*-algebras.

THEOREM 4.1. *Let $h : \mathcal{A} \to \mathcal{B}$ be a bijective mapping satisfying $h(0) = 0$ and $h(2^d uy) = h(2^d u)h(y)$ for all $u \in \mathcal{U}(\mathcal{A})$, all $y \in \mathcal{A}$, and $d = 0, 1, 2, \ldots$, for which there is a function $\varphi : \mathcal{A}^{mn} \to [0, \infty)$ satisfying (3.i) for all $x_1, \ldots, x_{mn} \in \mathcal{A}$ such that*

$$\Bigg\| mn\ {}_{mn-2}C_{k-2} h\left(\frac{\mu x_1 + \cdots + \mu x_{mn}}{mn}\right) + m\ {}_{mn-2}C_{k-1} \sum_{i=1}^{n} h\left(\frac{\mu x_{mi-m+1} + \cdots + \mu x_{mi}}{m}\right) - k \sum_{1 \leq i_1 < \cdots < i_k \leq mn} \mu h\left(\frac{x_{i_1} + \cdots + x_{i_k}}{k}\right) \Bigg\|$$

$$\leq \varphi\Big(x_1, \dots, x_{mn}\Big), \tag{4.i}$$

$$\Big\|h(2^d u^*) - h(2^d u)^*\Big\| \leq \varphi(\underbrace{2^d u, \dots, 2^d u}_{mn \text{ times}}) \tag{4.ii}$$

for all $u \in \mathcal{U}(\mathcal{A})$, all $x_1, \dots, x_{mn} \in \mathcal{A}$, all $\mu \in \mathbb{T}^1 := \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$ and $d = 0, 1, 2, \dots$. Assume that (4.iii) $\lim_{d\to\infty} \frac{h(2^d e)}{2^d}$ is invertible. Then the bijective mapping $h : \mathcal{A} \to \mathcal{B}$ is a C^*-algebra isomorphism.

Proof. We can consider a C^*-algebra as a Banach module over a unital C^*-algebra $\mathbb{C}$. So by Theorem 3.1, there exists a unique $\mathbb{C}$-linear mapping $H : \mathcal{A} \to \mathcal{B}$ such that

(4.iv)

$$\begin{aligned}
&\|h(x) - H(x)\| \\
&\leq \frac{1}{2ms\ {}_{mn-2}C_{k-1}} \widetilde{\varphi}(\underbrace{0, \dots, 0}_{m-2q \text{ times}}, \underbrace{\frac{mx}{q}, \dots, \frac{mx}{q}}_{q \text{ times}}, \underbrace{0, \dots, 0}_{q \text{ times}}, \\
&\underbrace{\frac{mx}{q}, \dots, \frac{mx}{q}}_{q \text{ times}}, \underbrace{0, \dots, 0}_{m-q \text{ times}}, \dots, \underbrace{0, \dots, 0}_{m-2q \text{ times}}, \underbrace{\frac{mx}{q}, \dots, \frac{mx}{q}}_{q \text{ times}}, \underbrace{0, \dots, 0}_{q \text{ times}}, \\
&\underbrace{\frac{mx}{q}, \dots, \frac{mx}{q}}_{q \text{ times}}, \underbrace{0, \dots, 0}_{m-q \text{ times}}, \underbrace{0, \dots, 0}_{mn-2ms \text{ times}}) \\
&+ \frac{1}{2ms\ {}_{mn-2}C_{k-1}} \widetilde{\varphi}(\underbrace{0, \dots, 0}_{m-2q \text{ times}}, \underbrace{\frac{mx}{q}, \dots, \frac{mx}{q}}_{q \text{ times}}, \underbrace{\frac{mx}{q}, \dots, \frac{mx}{q}}_{q \text{ times}}, \\
&\underbrace{0, \dots, 0}_{q \text{ times}}, \underbrace{0, \dots, 0}_{m-q \text{ times}}, \dots, \underbrace{0, \dots, 0}_{m-2q \text{ times}}, \underbrace{\frac{mx}{q}, \dots, \frac{mx}{q}}_{q \text{ times}}, \underbrace{\frac{mx}{q}, \dots, \frac{mx}{q}}_{q \text{ times}}, \\
&\underbrace{0, \dots, 0}_{q \text{ times}}, \underbrace{0, \dots, 0}_{m-q \text{ times}}, \underbrace{0, \dots, 0}_{mn-2ms \text{ times}})
\end{aligned}$$

for all $x \in \mathcal{A}$. The mapping $H : \mathcal{A} \to \mathcal{B}$ is given by

$$H(x) = \lim_{d\to\infty} \frac{1}{2^d} h(2^d x) \tag{4.1}$$

for all $x \in \mathcal{A}$.

By (3.i) and (4.ii), we get

$$H(u^*) = \lim_{d\to\infty} \frac{h(2^d u^*)}{2^d} = \lim_{d\to\infty} \frac{h(2^d u)^*}{2^d} = \left(\lim_{d\to\infty} \frac{h(2^d u)}{2^d}\right)^* = H(u)^*$$

for all $u \in \mathcal{U}(\mathcal{A})$. Since H is $\mathbb{C}$-linear and each $x \in \mathcal{A}$ is a finite linear combination of unitary elements (see [7, Theorem 4.1.7]), i.e., $x = \sum_{j=1}^d \lambda_j u_j$ $(\lambda_j \in \mathbb{C}, u_j \in \mathcal{U}(\mathcal{A}))$,

$$\begin{aligned} H(x^*) &= H\Big(\sum_{j=1}^d \overline{\lambda_j} u_j^*\Big) = \sum_{j=1}^d \overline{\lambda_j} H(u_j^*) = \sum_{j=1}^d \overline{\lambda_j} H(u_j)^* \\ &= \Big(\sum_{j=1}^d \lambda_j H(u_j)\Big)^* \\ &= H\Big(\sum_{j=1}^d \lambda_j u_j\Big)^* \\ &= H(x)^* \end{aligned}$$

for all $x \in \mathcal{A}$.

Since $h(2^d uy) = h(2^d u)h(y)$ for all $u \in \mathcal{U}(\mathcal{A})$, all $y \in \mathcal{A}$, and all $d = 0, 1, 2, \ldots,$

$$H(uy) = \lim_{d\to\infty} \frac{1}{2^d} h(2^d uy) = \lim_{d\to\infty} \frac{1}{2^d} h(2^d u)h(y) = H(u)h(y) \tag{4.2}$$

for all $u \in \mathcal{U}(\mathcal{A})$ and all $y \in \mathcal{A}$. By the additivity of H and (4.2),

$$2^d H(uy) = H(2^d uy) = H(u(2^d y)) = H(u)h(2^d y)$$

for all $u \in \mathcal{U}(\mathcal{A})$ and all $y \in \mathcal{A}$. Hence

$$H(uy) = \frac{1}{2^d} H(u)h(2^d y) = H(u)\frac{1}{2^d} h(2^d y) \tag{4.3}$$

for all $u \in \mathcal{U}(\mathcal{A})$ and all $y \in \mathcal{A}$. Taking the limit in (4.3) as $d \to \infty$, we obtain

$$H(uy) = H(u)H(y) \tag{4.4}$$

for all $u \in \mathcal{U}(\mathcal{A})$ and all $y \in \mathcal{A}$. Since H is $\mathbb{C}$-linear and each $x \in \mathcal{A}$ is a finite linear combination of unitary elements, i.e., $x = \sum_{j=1}^{d} \lambda_j u_j$ ($\lambda_j \in \mathbb{C}, u_j \in \mathcal{U}(\mathcal{A})$), it follows from (4.4) that

$$\begin{aligned} H(xy) &= H\Big(\sum_{j=1}^{d} \lambda_j u_j y\Big) = \sum_{j=1}^{d} \lambda_j H(u_j y) = \sum_{j=1}^{d} \lambda_j H(u_j) H(y) \\ &= H\Big(\sum_{j=1}^{d} \lambda_j u_j\Big) H(y) \\ &= H(x)H(y) \end{aligned}$$

for all $x, y \in \mathcal{A}$.

By (4.2) and (4.4),

$$H(e)H(y) = H(ey) = H(e)h(y)$$

for all $y \in \mathcal{A}$. Since $\lim_{d\to\infty} \frac{h(2^d e)}{2^d} = H(e)$ is invertible,

$$H(y) = h(y)$$

for all $y \in \mathcal{A}$.

Therefore, the bijective mapping $h : \mathcal{A} \to \mathcal{B}$ is a C^*-algebra isomorphism. □

COROLLARY 4.2. *Let $h : \mathcal{A} \to \mathcal{B}$ be a bijective mapping satisfying $h(0) = 0$ and $h(2^d uy) = h(2^d u)h(y)$ for all $u \in \mathcal{U}(\mathcal{A})$, all $y \in \mathcal{A}$, and all $d = 0, 1, 2, \ldots$, for which there exist constants $\theta \geq 0$ and $p \in [0, 1)$ such that*

$$\begin{aligned} &\Big\| mn\ {}_{mn-2}C_{k-2}\, h\Big(\frac{\mu x_1 + \cdots + \mu x_{mn}}{mn}\Big) \\ &+ m\ {}_{mn-2}C_{k-1} \sum_{i=1}^{n} h\Big(\frac{\mu x_{mi-m+1} + \cdots + \mu x_{mi}}{m}\Big) \\ &- k \sum_{1 \leq i_1 < \cdots < i_k \leq mn} \mu h\Big(\frac{x_{i_1} + \cdots + x_{i_k}}{k}\Big) \Big\| \leq \theta \sum_{j=1}^{mn} ||x_j||^p, \end{aligned}$$

$$\Big\| h(2^d u^*) - h(2^d u)^* \Big\| \leq mn 2^{dp} \theta$$

for all $\mu \in \mathbb{T}^1$, all $u \in \mathcal{U}(\mathcal{A})$, $d = 0, 1, 2, \ldots$, and all $x_1, \cdots, x_{mn} \in \mathcal{A}$. Assume that $\lim_{d\to\infty} \frac{h(2^d e)}{2^d}$ is invertible. Then the bijective mapping $h : \mathcal{A} \to \mathcal{B}$ is a C^*-algebra isomorphism.

Proof. Define $\varphi(x_1, \cdots, x_{mn}) = \theta \sum_{j=1}^{mn} ||x_j||^p$, and apply Theorem 4.1. □

THEOREM 4.3. *Let $h : \mathcal{A} \to \mathcal{B}$ be a bijective mapping satisfying $h(0) = 0$ and $h(2^d uy) = h(2^d u)h(y)$ for all $u \in \mathcal{U}(\mathcal{A})$, all $y \in \mathcal{A}$, and $d = 0, 1, 2, \ldots$, for which there is a function $\varphi : \mathcal{A}^{mn} \to [0, \infty)$ satisfying* (3.i), (4.ii), *and* (4.iii) *such that*

$$\begin{aligned} &\Big\| mn\ {}_{mn-2}C_{k-2} h\Big(\frac{\mu x_1 + \cdots + \mu x_{mn}}{mn}\Big) \\ &+ m\ {}_{mn-2}C_{k-1} \sum_{i=1}^{n} h\Big(\frac{\mu x_{mi-m+1} + \cdots + \mu x_{mi}}{m}\Big) \\ &- k \sum_{1 \le i_1 < \cdots < i_k \le mn} \mu h\Big(\frac{x_{i_1} + \cdots + x_{i_k}}{k}\Big)\Big\| \\ &\le \varphi(x_1, \cdots, x_{mn}) \end{aligned} \tag{4.v}$$

for all $x_1, \cdots, x_{mn} \in \mathcal{A}$ and $\mu = 1, i$. If $h(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in \mathcal{A}$, then the bijective mapping $h : \mathcal{A} \to \mathcal{B}$ is a C^*-algebra isomorphism.

Proof. Put $\mu = 1$ in (4.v). By the same reasoning as in the proof of Theorem 4.1, there exists a unique generalized Trif's mapping $H : \mathcal{A} \to \mathcal{B}$ satisfying (4.iv). By the same reasoning as in the proof of [12, Theorem], the additive mapping $H : \mathcal{A} \to \mathcal{B}$ is $\mathbb{R}$-linear.

Put $\mu = i$ in (4.v). By the same method as in the proof of Theorem 4.1, one can obtain that

$$H(ix) = \lim_{d\to\infty} \frac{h(2^d ix)}{2^d} = \lim_{d\to\infty} \frac{ih(2^d x)}{2^d} = iH(x)$$

for all $x \in \mathcal{A}$.

For each element $\lambda \in \mathbb{C}$, $\lambda = r + it$, where $r, t \in \mathbb{R}$. So

$$\begin{aligned} H(\lambda x) &= H(rx + itx) = rH(x) + tH(ix) = rH(x) + itH(x) \\ &= (r + it)H(x) = \lambda H(x) \end{aligned}$$

for all $\lambda \in \mathbb{C}$ and all $x \in \mathcal{A}$. So

$$H(\zeta x + \eta y) = H(\zeta x) + H(\eta y) = \zeta H(x) + \eta H(y)$$

for all $\zeta, \eta \in \mathbb{C}$, and all $x, y \in \mathcal{A}$. Hence the additive mapping $H : \mathcal{A} \to \mathcal{B}$ is $\mathbb{C}$-linear.

The rest of the proof is the same as in the proof of Theorem 4.1. □

Now we prove the Cauchy-Rassias stability of C^*-algebra homomorphisms in unital C^*-algebras.

THEOREM 4.4. *Let $h : \mathcal{A} \to \mathcal{B}$ be a mapping satisfying $h(0) = 0$ for which there exists a function $\varphi : \mathcal{A}^{mn} \to [0, \infty)$ satisfying* (3.i), (4.i), *and* (4.ii) *such that*

$$\|h(2^d u \cdot 2^d v) - h(2^d u)h(2^d v)\| \le \varphi(2^d u, 2^d v, \underbrace{0, \ldots, 0}_{mn-2 \text{ times}}) \tag{4.vi}$$

for all $u, v \in \mathcal{U}(\mathcal{A})$ and $d = 0, 1, 2, \ldots$. Then there exists a unique C^*-algebra homomorphism $H : \mathcal{A} \to \mathcal{B}$ satisfying (4.iv).

Proof. By the same reasoning as in the proof of Theorem 4.1, there exists a unique $\mathbb{C}$-linear involutive mapping $H : \mathcal{A} \to \mathcal{B}$ satisfying (4.iv).

By (4.vi),

$$\begin{aligned}\frac{1}{2^{2d}}\|h(2^d u \cdot 2^d v) - h(2^d u)h(2^d v)\| &\le \frac{1}{2^{2d}}\varphi(2^d u, 2^d v, \underbrace{0, \ldots, 0}_{mn-2 \text{ times}}) \\ &\le \frac{1}{2^{d}}\varphi(2^d u, 2^d v, \underbrace{0, \ldots, 0}_{mn-2 \text{ times}}),\end{aligned}$$

which tends to zero by (3.i) as $d \to \infty$. By (4.1),

$$\begin{aligned}H(uv) &= \lim_{d\to\infty} \frac{h(2^d u \cdot 2^d v)}{2^{2d}} = \lim_{d\to\infty} \frac{h(2^d u)h(2^d v)}{2^{2d}} \\ &= \lim_{d\to\infty} \frac{h(2^d u)}{2^d}\frac{h(2^d v)}{2^d} \\ &= H(u)H(v)\end{aligned}$$

for all $u, v \in \mathcal{U}(\mathcal{A})$. Since H is $\mathbb{C}$-linear and each $x \in \mathcal{A}$ is a finite linear combination of unitary elements, i.e., $x = \sum_{j=1}^{d} \lambda_j u_j$ $(\lambda_j \in \mathbb{C}, u_j \in$

$\mathcal{U}(\mathcal{A})$),

$$\begin{aligned} H(xv) = H\Big(\sum_{j=1}^{d} \lambda_j u_j v\Big) = \sum_{j=1}^{d} \lambda_j H(u_j v) &= \sum_{j=1}^{d} \lambda_j H(u_j) H(v) \\ &= H\Big(\sum_{j=1}^{d} \lambda_j u_j\Big) H(v) \\ &= H(x)H(v) \end{aligned}$$

for all $x \in \mathcal{A}$ and all $v \in \mathcal{U}(\mathcal{A})$. By the same method as given above, one can obtain that

$$H(xy) = H(x)H(y)$$

for all $x, y \in \mathcal{A}$. So the mapping $H : \mathcal{A} \to \mathcal{B}$ is a C^*-algebra homomorphism, as desired. □

5. Homomorphisms between Lie JC^*-algebras

The original motivation to introduce the class of nonassociative algebras known as Jordan algebras came from quantum mechanics (see [21]). Let $\mathcal{L}(\mathcal{H})$ be the real vector space of all bounded self-adjoint linear operators on $\mathcal{H}$, interpreted as the (bounded) *observables* of the system. In 1932, Jordan observed that $\mathcal{L}(\mathcal{H})$ is a (nonassociative) algebra via the *anticommutator product* $x \circ y := \frac{xy+yx}{2}$. A commutative algebra X with product $x \circ y$ is called a *Jordan algebra*. A unital Jordan C^*-subalgebra of a C^*-algebra, endowed with the anticommutator product, is called a *JC^*-algebra*.

A unital C^*-algebra $\mathcal{C}$, endowed with the Lie product $[x, y] = \frac{xy-yx}{2}$ on $\mathcal{C}$, is called a *Lie C^*-algebra*. A unital C^*-algebra $\mathcal{C}$, endowed with the Lie product $[\cdot, \cdot]$ and the anticommutator product $\circ$, is called a *Lie JC^*-algebra* if $(\mathcal{C}, \circ)$ is a JC^*-algebra and $(\mathcal{C}, [\cdot, \cdot])$ is a Lie C^*-algebra.

Throughout this paper, let $\mathcal{A}$ be a unital Lie JC^*-algebra with norm $||\cdot||$, unit e and unitary group $\mathcal{U}(\mathcal{A}) = \{u \in \mathcal{A} \mid uu^* = u^*u = e\}$, and $\mathcal{B}$ a unital Lie JC^*-algebra with norm $||\cdot||$ and unit e'.

DEFINITION 5.1. A $\mathbb{C}$-linear mapping $H : \mathcal{A} \to \mathcal{B}$ is called a *Lie JC^*-algebra homomorphism* if $H : \mathcal{A} \to \mathcal{B}$ satisfies

$$\begin{aligned} H(x \circ y) &= H(x) \circ H(y), \\ H([x, y]) &= [H(x), H(y)], \\ H(x^*) &= H(x)^* \end{aligned}$$

for all $x, y \in \mathcal{A}$.

REMARK 5.1. A $\mathbb{C}$-linear mapping $H : \mathcal{A} \to \mathcal{B}$ is a C^*-algebra homomorphism if and only if the mapping $H : \mathcal{A} \to \mathcal{B}$ is a Lie JC^*-algebra homomorphism.

Assume that H is a Lie JC^*-algebra homomorphism. Then

$$\begin{aligned} H(xy) = H([x,y] + x \circ y) &= H([x,y]) + H(x \circ y) \\ &= [H(x), H(y)] + H(x) \circ H(y) \\ &= H(x)H(y) \end{aligned}$$

for all $x, y \in \mathcal{A}$. So H is a C^*-algebra homomorphism.

Assume that H is a C^*-algebra homomorphism. Then

$$\begin{aligned} H([x,y] = H(\frac{xy - yx}{2}) &= \frac{H(x)H(y) - H(y)H(x)}{2} = [H(x), H(y)], \\ H(x \circ y) = H(\frac{xy + yx}{2}) &= \frac{H(x)H(y) + H(y)H(x)}{2} = H(x) \circ H(y) \end{aligned}$$

for all $x, y \in \mathcal{A}$. So H is a Lie JC^*-algebra homomorphism.

We are going to investigate Lie JC^*-algebra homomorphisms between Lie JC^*-algebras.

THEOREM 5.1. *Let $h : \mathcal{A} \to \mathcal{B}$ be a mapping satisfying $h(0) = 0$ and $h(2^d u \circ y) = h(2^d u) \circ h(y)$ for all $y \in \mathcal{A}$, all $u \in \mathcal{U}(\mathcal{A})$ and $d = 0, 1, 2, \ldots$, for which there exists a function $\varphi : \mathcal{A}^{mn+2} \to [0, \infty)$ such that*

$$\tag{5.i} \begin{aligned} \widetilde{\varphi}(x_1, \ldots, x_{mn}, z, w) &:= \sum_{j=0}^{\infty} \frac{1}{2^j} \varphi(2^j x_1, \ldots, 2^j x_{mn}, 2^j z, 2^j w) \\ &< \infty, \end{aligned}$$

$$\tag{5.ii} \begin{aligned} &\Bigg\| mn\ {}_{mn-2}C_{k-2} h\left(\frac{\mu x_1 + \cdots + \mu x_{mn} + \frac{[z,w]}{{}_{mn-2}C_{k-2}}}{mn} \right) \\ &+ m\ {}_{mn-2}C_{k-1} \sum_{i=1}^{n} h\left(\frac{\mu x_{mi-m+1} + \cdots + \mu x_{mi}}{m} \right) \\ &- k \sum_{1 \le i_1 < \cdots < i_k \le mn} \mu h\left(\frac{x_{i_1} + \cdots + x_{i_k}}{k} \right) - [h(z), h(w)] \Bigg\| \\ &\le \varphi(x_1, \ldots, x_{mn}, z, w), \end{aligned}$$

$$\|h(2^d u^*) - h(2^d u)^*\| \le \varphi(\underbrace{2^d u, \ldots, 2^d u}_{mn \text{ times}}, 0, 0) \tag{5.iii}$$

for all $\mu \in \mathbb{T}^1$*, all* $u \in \mathcal{U}(\mathcal{A})$*, all* $x_1, \ldots, x_{mn}, z, w \in \mathcal{A}$ *and* $d = 0, 1, 2, \ldots$. *Assume*

$$\lim_{d \to \infty} \frac{h(2^d e)}{2^d} = e'. \tag{5.iv}$$

Then the mapping $h : \mathcal{A} \to \mathcal{B}$ *is a Lie* JC^**-algebra homomorphism.*

Proof. By the same reasoning as in the proof of Theorem 4.1, there exists a unique $\mathbb{C}$-linear involutive mapping $H : \mathcal{A} \to \mathcal{B}$ such that

(5.v)

$$\begin{aligned}
&\|h(x) - H(x)\| \\
&\le \frac{1}{2ms\ {}_{mn-2}C_{k-1}} \widetilde{\varphi}(\underbrace{0, \ldots, 0}_{m-2q \text{ times}}, \underbrace{\frac{mx}{q}, \ldots, \frac{mx}{q}}_{q \text{ times}}, \underbrace{0, \ldots, 0}_{q \text{ times}}, \\
&\quad \underbrace{\frac{mx}{q}, \ldots, \frac{mx}{q}}_{q \text{ times}}, \underbrace{0, \ldots, 0}_{m-q \text{ times}}, \ldots, \underbrace{0, \ldots, 0}_{m-2q \text{ times}}, \underbrace{\frac{mx}{q}, \ldots, \frac{mx}{q}}_{q \text{ times}}, \underbrace{0, \ldots, 0}_{q \text{ times}}, \\
&\quad \underbrace{\frac{mx}{q}, \ldots, \frac{mx}{q}}_{q \text{ times}}, \underbrace{0, \ldots, 0}_{m-q \text{ times}}, \underbrace{0, \ldots, 0}_{mn-2ms \text{ times}}, 0, 0) \\
&\quad + \frac{1}{2ms\ {}_{mn-2}C_{k-1}} \widetilde{\varphi}(\underbrace{0, \ldots, 0}_{m-2q \text{ times}}, \underbrace{\frac{mx}{q}, \ldots, \frac{mx}{q}}_{q \text{ times}}, \underbrace{\frac{mx}{q}, \ldots, \frac{mx}{q}}_{q \text{ times}}, \underbrace{0, \ldots, 0}_{q \text{ times}}, \\
&\quad \underbrace{0, \ldots, 0}_{m-q \text{ times}}, \ldots, \underbrace{0, \ldots, 0}_{m-2q \text{ times}}, \underbrace{\frac{mx}{q}, \ldots, \frac{mx}{q}}_{q \text{ times}}, \underbrace{\frac{mx}{q}, \ldots, \frac{mx}{q}}_{q \text{ times}}, \\
&\quad \underbrace{0, \ldots, 0}_{q \text{ times}}, \underbrace{0, \ldots, 0}_{m-q \text{ times}}, \underbrace{0, \ldots, 0}_{mn-2ms \text{ times}}, 0, 0)
\end{aligned}$$

for all $x \in \mathcal{A}$.

It follows from (4.1) that

$$H(x) = \lim_{d \to \infty} \frac{h(2^{2d} x)}{2^{2d}} \tag{5.1}$$

for all $x \in \mathcal{A}$. Let $x_1 = \cdots = x_{mn} = 0$ in (5.ii). Then we get

$$\left\| mn\ {}_{mn-2}C_{k-2} h\left(\frac{[z,w]}{mn\ {}_{mn-2}C_{k-2}}\right) - [h(z), h(w)]\right\| \le \varphi(\underbrace{0, \dots, 0}_{mn \text{ times}}, z, w)$$

for all $z, w \in \mathcal{A}$. So

$$\begin{aligned} &\frac{1}{2^{2d}}\left\| mn\ {}_{mn-2}C_{k-2} h\left(\frac{[2^d z, 2^d w]}{mn\ {}_{mn-2}C_{k-2}}\right) - [h(2^d z), h(2^d w)]\right\| \\ &\le \frac{1}{2^{2d}}\varphi(\underbrace{0, \dots, 0}_{mn \text{ times}}, 2^d z, 2^d w) \\ &\le \frac{1}{2^{d}}\varphi(\underbrace{0, \dots, 0}_{mn \text{ times}}, 2^d z, 2^d w) \end{aligned} \tag{5.2}$$

for all $z, w \in \mathcal{A}$. By (5.i), (5.1), and (5.2),

$$\begin{aligned} &mn\ {}_{mn-2}C_{k-2} H\left(\frac{[z,w]}{mn\ {}_{mn-2}C_{k-2}}\right) \\ &= \lim_{d\to\infty} \frac{mn\ {}_{mn-2}C_{k-2} h\left((2^{2d}\frac{[z,w]}{mn\ {}_{mn-2}C_{k-2}}\right)}{2^{2d}} \\ &= \lim_{d\to\infty} \frac{mn\ {}_{mn-2}C_{k-2} h\left(\frac{[2^d z, 2^d w]}{mn\ {}_{mn-2}C_{k-2}}\right)}{2^{2d}} \\ &= \lim_{d\to\infty} \frac{1}{2^{2d}}[h(2^d z), h(2^d w)] \\ &= \lim_{d\to\infty} [\frac{h(2^d z)}{2^d}, \frac{h(2^d w)}{2^d}] \\ &= [H(z), H(w)] \end{aligned}$$

for all $z, w \in \mathcal{A}$. So

$$H([z,w]) = mn\ {}_{mn-2}C_{k-2} H\left(\frac{[z,w]}{mn\ {}_{mn-2}C_{k-2}}\right) = [H(z), H(w)]$$

for all $z, w \in \mathcal{A}$.

Since $h(2^d u \circ y) = h(2^d u) \circ h(y)$ for all $y \in \mathcal{A}$, all $u \in \mathcal{U}(\mathcal{A})$ and $d = 0, 1, 2, \ldots,$

$$\begin{aligned} H(u \circ y) &= \lim_{d\to\infty} \frac{1}{2^d} h(2^d u \circ y) = \lim_{d\to\infty} \frac{1}{2^d} h(2^d u) \circ h(y) \\ &= H(u) \circ h(y) \end{aligned} \tag{5.3}$$

for all $y \in \mathcal{A}$ and all $u \in \mathcal{U}(\mathcal{A})$. By the additivity of H and (5.3),

$$2^d H(u \circ y) = H(2^d u \circ y) = H(u \circ (2^d y)) = H(u) \circ h(2^d y)$$

for all $y \in \mathcal{A}$ and all $u \in \mathcal{U}(\mathcal{A})$. Hence

$$H(u \circ y) = \frac{1}{2^d} H(u) \circ h(2^d y) = H(u) \circ \frac{1}{2^d} h(2^d y) \tag{5.4}$$

for all $y \in \mathcal{A}$ and all $u \in \mathcal{U}(\mathcal{A})$. Taking the limit in (5.4) as $d \to \infty$, we obtain

$$H(u \circ y) = H(u) \circ H(y) \tag{5.5}$$

for all $y \in \mathcal{A}$ and all $u \in \mathcal{U}(\mathcal{A})$. Since H is $\mathbb{C}$-linear and each $x \in \mathcal{A}$ is a finite linear combination of unitary elements i.e., $x = \sum_{j=1}^{d} \lambda_j u_j$ ($\lambda_j \in \mathbb{C}, u_j \in \mathcal{U}(\mathcal{A})$),

$$\begin{aligned} H(x \circ y) &= H\Big(\sum_{j=1}^{d} \lambda_j u_j \circ y\Big) = \sum_{j=1}^{d} \lambda_j H(u_j \circ y) \\ &= \sum_{j=1}^{d} \lambda_j H(u_j) \circ H(y) \\ &= H\Big(\sum_{j=1}^{d} \lambda_j u_j\Big) \circ H(y) \\ &= H(x) \circ H(y) \end{aligned}$$

for all $x, y \in \mathcal{A}$.

By (5.iv), (5.3), and (5.5),

$$H(y) = H(e \circ y) = H(e) \circ h(y) = e' \circ h(y) = h(y)$$

for all $y \in \mathcal{A}$. So

$$H(y) = h(y)$$

for all $y \in \mathcal{A}$.

Therefore, the mapping $h : \mathcal{A} \to \mathcal{B}$ is a Lie JC^*-algebra homomorphism. □

THEOREM 5.2. *Let $h : \mathcal{A} \to \mathcal{B}$ be a mapping satisfying $h(2x) = 2h(x)$ for all $x \in \mathcal{A}$ for which there exists a function $\varphi : \mathcal{A}^{mn+2} \to [0, \infty)$ satisfying* (5.i), (5.ii), (5.iii), *and* (5.iv) *such that*

$$\|h(2^d u \circ y) - h(2^d u) \circ h(y)\| \le \varphi(u, y, \underbrace{0, \dots, 0}_{mn \text{ times}}) \tag{5.vi}$$

for all $y \in \mathcal{A}$, all $u \in \mathcal{U}(\mathcal{A})$ and $d = 0, 1, 2, \dots$. Then the mapping $h : \mathcal{A} \to \mathcal{B}$ is a Lie JC^*-algebra homomorphism.

Proof. By the same reasoning as in the proof of Theorem 4.1, there exists a unique $\mathbb{C}$-linear mapping $H : \mathcal{A} \to \mathcal{B}$ satisfying (5.v).

By (5.vi) and the assumption that $h(2x) = 2h(x)$ for all $x \in \mathcal{A}$,

$$\begin{aligned}
&\|h(2^d u \circ y) - h(2^d u) \circ h(y)\| \\
&= \frac{1}{2^{2l}} \|h(2^l \cdot 2^d u \circ 2^l y) - h(2^l \cdot 2^d u) \circ h(2^l y)\| \\
&\le \frac{1}{2^{2l}} \varphi(2^l u, 2^l y, \underbrace{0, \dots, 0}_{mn \text{ times}}) \\
&\le \frac{1}{2^{l}} \varphi(2^l u, 2^l y, \underbrace{0, \dots, 0}_{mn \text{ times}}),
\end{aligned}$$

which tends to zero as $l \to \infty$ by (5.i). So

$$h(2^d u \circ y) = h(2^d u) \circ h(y)$$

for all $y \in \mathcal{A}$, all $u \in \mathcal{U}(\mathcal{A})$ and $d = 0, 1, 2, \dots$. But by (4.1),

$$H(x) = \lim_{d \to \infty} \frac{1}{2^d} h(2^d x) = h(x)$$

for all $x \in \mathcal{A}$.

The rest of the proof is similar to the proof of Theorem 5.1. □

We are going to show the Cauchy-Rassias stability of Lie JC^*-algebra homomorphisms in Lie JC^*-algebras.

THEOREM 5.3. *Let $h : \mathcal{A} \to \mathcal{B}$ be a mapping satisfying $h(0) = 0$ for which there exists a function $\varphi : \mathcal{A}^{mn+4} \to [0, \infty)$ such that*

$$\begin{aligned}
&\widetilde{\varphi}(x_1, \dots, x_{mn}, z, w, a, b) \\
&:= \sum_{j=0}^{\infty} \frac{1}{2^j} \varphi(2^j x_1, \dots, 2^j x_{mn}, 2^j z, 2^j w, 2^j a, 2^j b) < \infty,
\end{aligned} \tag{5.vii}$$

$$
\begin{aligned}
(5.\mathrm{viii})\quad & \Big\| mn\ {}_{mn-2}C_{k-2} h\Big(\frac{\mu x_1 + \cdots + \mu x_{mn} + \frac{[z,w]+a\circ b}{{}_{mn-2}C_{k-2}}}{mn}\Big) \\
&+ m\ {}_{mn-2}C_{k-1} \sum_{i=1}^{n} h\Big(\frac{\mu x_{mi-m+1} + \cdots + \mu x_{mi}}{m}\Big) \\
&- k \sum_{1\le i_1 < \cdots < i_k \le mn} \mu h\Big(\frac{x_{i_1} + \cdots + x_{i_k}}{k}\Big) - [h(z), h(w)] \\
&- h(a)\circ h(b)\Big\| \le \varphi(x_1, \dots, x_{mn}, z, w, a, b)
\end{aligned}
$$

$$
(5.\mathrm{ix})\qquad \|h(2^d u^*) - h(2^d u)^*\| \le \varphi(\underbrace{2^d u, \dots, 2^d u}_{mn \text{ times}}, 0, 0, 0, 0)
$$

for all $\mu \in \mathbb{T}^1$, all $u \in \mathcal{U}(\mathcal{A})$, $d = 0, 1, 2, \dots$, and all $x_1, \dots, x_{mn}, z, w, a, b \in \mathcal{A}$. Then there exists a unique Lie JC^*-algebra homomorphism $H : \mathcal{A} \to \mathcal{B}$ such that

(5.x)

$$
\begin{aligned}
&\|h(x) - H(x)\| \\
&\le \frac{1}{2ms\ {}_{mn-2}C_{k-1}} \widetilde{\varphi}(\underbrace{0, \dots, 0}_{m-2q \text{ times}}, \underbrace{\frac{mx}{q}, \dots, \frac{mx}{q}}_{q \text{ times}}, \underbrace{0, \dots, 0}_{q \text{ times}}, \\
&\quad \underbrace{\frac{mx}{q}, \dots, \frac{mx}{q}}_{q \text{ times}}, \underbrace{0, \dots, 0}_{m-q \text{ times}}, \dots, \underbrace{0, \dots, 0}_{m-2q \text{ times}}, \underbrace{\frac{mx}{q}, \dots, \frac{mx}{q}}_{q \text{ times}}, \underbrace{0, \dots, 0}_{q \text{ times}}, \\
&\quad \underbrace{\frac{mx}{q}, \dots, \frac{mx}{q}}_{q \text{ times}}, \underbrace{0, \dots, 0}_{m-q \text{ times}}, \underbrace{0, \dots, 0}_{mn-2ms \text{ times}}, 0, 0, 0, 0) \\
&\quad + \frac{1}{2ms\ {}_{mn-2}C_{k-1}} \widetilde{\varphi}(\underbrace{0, \dots, 0}_{m-2q \text{ times}}, \underbrace{\frac{mx}{q}, \dots, \frac{mx}{q}}_{q \text{ times}}, \underbrace{\frac{mx}{q}, \dots, \frac{mx}{q}}_{q \text{ times}}, \\
&\quad \underbrace{0, \dots, 0}_{q \text{ times}}, \underbrace{0, \dots, 0}_{m-q \text{ times}}, \dots, \underbrace{0, \dots, 0}_{m-2q \text{ times}}, \underbrace{\frac{mx}{q}, \dots, \frac{mx}{q}}_{q \text{ times}}, \underbrace{\frac{mx}{q}, \dots, \frac{mx}{q}}_{q \text{ times}}, \\
&\quad \underbrace{0, \dots, 0}_{q \text{ times}}, \underbrace{0, \dots, 0}_{m-q \text{ times}}, \underbrace{0, \dots, 0}_{mn-2ms \text{ times}}, 0, 0, 0, 0)
\end{aligned}
$$

for all $x \in \mathcal{A}$.

Proof. By the same reasoning as in the proof of Theorem 4.1, there exists a unique $\mathbb{C}$-linear involutive mapping $H : \mathcal{A} \to \mathcal{B}$ satisfying (5.x). The rest of the proof is similar to the proof of Theorem 5.1. □

6. Cauchy-Rassias stability of Lie JC^*-algebra derivations in Lie JC^*-algebras

DEFINITION 6.1. A $\mathbb{C}$-linear mapping $D : \mathcal{A} \to \mathcal{A}$ is called a *Lie JC^*-algebra derivation* if $D : \mathcal{A} \to \mathcal{A}$ satisfies

$$\begin{aligned} D(x \circ y) &= (Dx) \circ y + x \circ (Dy), \\ D([x, y]) &= [Dx, y] + [x, Dy], \\ D(x^*) &= D(x)^* \end{aligned}$$

for all $x, y \in \mathcal{A}$.

REMARK 6.1. A $\mathbb{C}$-linear mapping $D : \mathcal{A} \to \mathcal{A}$ is a C^*-algebra derivation if and only if the mapping $D : \mathcal{A} \to \mathcal{A}$ is a Lie JC^*-algebra derivation.

Assume that D is a Lie JC^*-algebra derivation. Then

$$\begin{aligned} D(xy) = D([x, y] + x \circ y) &= D([x, y]) + D(x \circ y) \\ &= [Dx, y] + [x, Dy] + (Dx) \circ y + x \circ (Dy) \\ &= (Dx)y + x(Dy) \end{aligned}$$

for all $x, y \in \mathcal{A}$. So D is a C^*-algebra derivation.

Assume that D is a C^*-algebra derivation. Then

$$\begin{aligned} D([x, y]) = D\left(\frac{xy - yx}{2}\right) &= \frac{(Dx)y + x(Dy) - (Dy)x - y(Dx)}{2} \\ &= [Dx, y] + [x, Dy], \\ D(x \circ y) = D\left(\frac{xy + yx}{2}\right) &= \frac{(Dx)y + x(Dy) + (Dy)x + y(Dx)}{2} \\ &= (Dx) \circ y + x \circ (Dy) \end{aligned}$$

for all $x, y \in \mathcal{A}$. So H is a Lie JC^*-algebra derivation.

We prove the Cauchy-Rassias stability of Lie JC^*-algebra derivations in Lie JC^*-algebras.

THEOREM 6.1. *Let $h : \mathcal{A} \to \mathcal{A}$ be a mapping satisfying $h(0) = 0$ for which there exists a function $\varphi : \mathcal{A}^{mn+4} \to [0, \infty)$ satisfying* (5.vii) *and* (5.ix) *such that*

(6.i)

$$\begin{aligned}&\Big\| mn\ {}_{mn-2}C_{k-2} h\Big(\frac{\mu x_1 + \cdots + \mu x_{mn} + \frac{[z,w]+a\circ b}{{}_{mn-2}C_{k-2}}}{mn}\Big) \\ &+ m\ {}_{mn-2}C_{k-1} \sum_{i=1}^{n} h\Big(\frac{\mu x_{mi-m+1} + \cdots + \mu x_{mi}}{m}\Big) \\ &- k \sum_{1\le i_1<\cdots<i_k\le mn} \mu h\Big(\frac{x_{i_1} + \cdots + x_{i_k}}{k}\Big) - [h(z), w] - [z, h(w)] \\ &- h(a)\circ b - a\circ h(b)\Big\| \\ &\le \varphi(x_1, \ldots, x_{mn}, z, w, a, b)\end{aligned}$$

for all $\mu \in \mathbb{T}^1$ and all $x_1, \ldots, x_{mn}, z, w, a, b \in \mathcal{A}$. Then there exists a unique Lie JC^*-algebra derivation $D : \mathcal{A} \to \mathcal{A}$ such that

(6.ii)

$$\begin{aligned}&\|h(x) - D(x)\| \\ &\le \frac{1}{2ms\ {}_{mn-2}C_{k-1}} \widetilde{\varphi}(\underbrace{0,\ldots,0}_{m-2q \text{ times}}, \underbrace{\frac{mx}{q},\ldots,\frac{mx}{q}}_{q \text{ times}}, \underbrace{0,\ldots,0,}_{q \text{ times}} \\ &\underbrace{\frac{mx}{q},\ldots,\frac{mx}{q}}_{q \text{ times}}, \underbrace{0,\ldots,0}_{m-q \text{ times}}, \ldots, \underbrace{0,\ldots,0}_{m-2q \text{ times}}, \underbrace{\frac{mx}{q},\ldots,\frac{mx}{q}}_{q \text{ times}}, \underbrace{0,\ldots,0,}_{q \text{ times}} \\ &\underbrace{\frac{mx}{q},\ldots,\frac{mx}{q}}_{q \text{ times}}, \underbrace{0,\ldots,0}_{m-q \text{ times}}, \underbrace{0,\ldots,0}_{mn-2ms \text{ times}}, 0,0,0,0) \\ &+ \frac{1}{2ms\ {}_{mn-2}C_{k-1}} \widetilde{\varphi}(\underbrace{0,\ldots,0}_{m-2q \text{ times}}, \underbrace{\frac{mx}{q},\ldots,\frac{mx}{q}}_{q \text{ times}}, \underbrace{\frac{mx}{q},\ldots,\frac{mx}{q}}_{q \text{ times}}, \\ &\underbrace{0,\ldots,0,}_{q \text{ times}} \underbrace{0,\ldots,0}_{m-q \text{ times}}, \ldots, \underbrace{0,\ldots,0}_{m-2q \text{ times}}, \underbrace{\frac{mx}{q},\ldots,\frac{mx}{q}}_{q \text{ times}}, \underbrace{\frac{mx}{q},\ldots,\frac{mx}{q}}_{q \text{ times}}, \\ &\underbrace{0,\ldots,0,}_{q \text{ times}} \underbrace{0,\ldots,0}_{m-q \text{ times}}, \underbrace{0,\ldots,0}_{mn-2ms \text{ times}}, 0,0,0,0)\end{aligned}$$

for all $x \in \mathcal{A}$.

Proof. Put $z = w = a = b = 0$ in (6.i). By the same reasoning as in the proof of Theorem 4.1, there exists a unique $\mathbb{C}$-linear involutive mapping $D : \mathcal{A} \to \mathcal{A}$ satisfying (6.ii). The $\mathbb{C}$-linear mapping $D : \mathcal{A} \to \mathcal{A}$ is given by

$$D(x) = \lim_{d\to\infty} \frac{1}{2^d} h(2^d x) \tag{6.1}$$

for all $x \in \mathcal{A}$.

It follows from (6.1) that

$$D(x) = \lim_{d\to\infty} \frac{h(2^{2d}x)}{2^{2d}} \tag{6.2}$$

for all $x \in \mathcal{A}$. Let $x_1 = \cdots = x_{mn} = a = b = 0$ in (6.i). Then we get

$$\left\| mn \ {}_{mn-2}C_{k-2} h\left(\frac{[z,w]}{mn \ {}_{mn-2}C_{k-2}}\right) - [h(z),w] - [z,h(w)] \right\| \\ \leq \varphi(\underbrace{0,\dots,0}_{mn \text{ times}}, z, w, 0, 0)$$

for all $z, w \in \mathcal{A}$. So

$$\begin{aligned} &\frac{1}{2^{2d}} \left\| mn \ {}_{mn-2}C_{k-2} h\left(\frac{[2^d z, 2^d w]}{mn \ {}_{mn-2}C_{k-2}}\right) - [h(2^d z), 2^d w] \right. \\ &\left. - [2^d z, h(2^d w)] \right\| \\ &\leq \frac{1}{2^{2d}} \varphi(\underbrace{0,\dots,0}_{mn \text{ times}}, 2^d z, 2^d w, 0, 0) \\ &\leq \frac{1}{2^{d}} \varphi(\underbrace{0,\dots,0}_{mn \text{ times}}, 2^d z, 2^d w, 0, 0) \end{aligned} \tag{6.3}$$

for all $z, w \in \mathcal{A}$. By (5.vii), (6.2), and (6.3),

$$\begin{aligned} &mn \ {}_{mn-2}C_{k-2} D\left(\frac{[z,w]}{mn \ {}_{mn-2}C_{k-2}}\right) \\ &= \lim_{d\to\infty} \frac{mn \ {}_{mn-2}C_{k-2} h\left(2^{2d} \frac{[z,w]}{mn \ {}_{mn-2}C_{k-2}}\right)}{2^{2d}} \end{aligned}$$

$$= \lim_{d\to\infty} \frac{mn\ {}_{mn-2}C_{k-2} h\Big(\frac{[2^d z, 2^d w]}{mn\ {}_{mn-2}C_{k-2}}\Big)}{2^{2d}}$$
$$= \lim_{d\to\infty} \left(\left[\frac{h(2^d z)}{2^d}, \frac{2^d w}{2^d}\right] + \left[\frac{2^d z}{2^d}, \frac{h(2^d w)}{2^d}\right] \right)$$
$$= [D(z), w] + [z, D(w)]$$

for all $z, w \in \mathcal{A}$. So

$$D([z,w]) = mn\ {}_{mn-2}C_{k-2} D\Big(\frac{[z,w]}{mn\ {}_{mn-2}C_{k-2}}\Big) = [D(z), w] + [z, D(w)]$$

for all $z, w \in \mathcal{A}$.

Similarly, one can obtain that

$$mn\ {}_{mn-2}C_{k-2} D\Big(\frac{a \circ b}{mn\ {}_{mn-2}C_{k-2}}\Big)$$
$$= \lim_{d\to\infty} \frac{mn\ {}_{mn-2}C_{k-2} h\Big(2^{2d}\frac{a\circ b}{mn\ {}_{mn-2}C_{k-2}}\Big)}{2^{2d}}$$
$$= \lim_{d\to\infty} \frac{mn\ {}_{mn-2}C_{k-2} h\Big(\frac{(2^d a)\circ(2^d b)}{mn\ {}_{mn-2}C_{k-2}}\Big)}{2^{2d}}$$
$$= \lim_{d\to\infty} \left(\frac{h(2^d a)}{2^d} \circ \frac{2^d b}{2^d} + \frac{2^d a}{2^d} \circ \frac{h(2^d b)}{2^d} \right)$$
$$= D(a) \circ b + a \circ D(b)$$

for all $a, b \in \mathcal{A}$. So

$$D(a \circ b) = mn\ {}_{mn-2}C_{k-2} D\Big(\frac{a \circ b}{mn\ {}_{mn-2}C_{k-2}}\Big) = D(a) \circ b + a \circ D(b)]$$

for all $a, b \in \mathcal{A}$. Hence the $\mathbb{C}$-linear mapping $D : \mathcal{A} \to \mathcal{A}$ is a Lie JC^*-algebra derivation satisfying (6.ii), as desired. □

COROLLARY 6.2. *Let $h : \mathcal{A} \to \mathcal{A}$ be a mapping satisfying $h(0) = 0$*

for which there exist constants $\theta \geq 0$ and $p \in [0,1)$ such that

$$\Big\| mn\ {}_{mn-2}C_{k-2} h\Big(\frac{\mu x_1 + \cdots + \mu x_{mn} + \frac{[z,w]+a\circ b}{{}_{mn-2}C_{k-2}}}{mn}\Big) + m\ {}_{mn-2}C_{k-1} \sum_{i=1}^{n} h\Big(\frac{\mu x_{mi-m+1} + \cdots + \mu x_{mi}}{m}\Big) - k \sum_{1\leq i_1<\cdots<i_k\leq mn} \mu h\Big(\frac{x_{i_1} + \cdots + x_{i_k}}{k}\Big) - [h(z),w] - [z,h(w)] - h(a)\circ b - a\circ h(b)\Big\| \leq \theta \sum_{j=1}^{mn} \|x_j\|^p + \|z\|^p + \|w\|^p + \|a\|^p + \|b\|^p,$$

$$\|h(2^d u^*) - h(2^d u)^*\| \leq mn2^{dp}\theta$$

for all $\mu \in \mathbb{T}^1$, all $u \in \mathcal{U}(\mathcal{A})$, $d = 0,1,2,\ldots$, and all $x_1, \ldots, x_{mn}, z, w, a, b \in \mathcal{A}$. Then there exists a unique Lie JC^*-algebra derivation $D : \mathcal{A} \to \mathcal{A}$ such that

$$\|h(x) - D(x)\| \leq \frac{4m^{p-1}q^{1-p}\theta}{(2-2^p)_{mn-2}C_{k-1}} \|x\|^p$$

for all $x \in \mathcal{A}$.

Proof. Define $\varphi(x_1, \ldots, x_{mn}, z, w, a, b) = \theta(\sum_{j=1}^{mn} \|x_j\|^p + \|z\|^p + \|w\|^p + \|a\|^p + \|b\|^p)$, and apply Theorem 6.1. □

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