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# STRONG DIFFERENTIAL SUBORDINATION AND APPLICATIONS TO UNIVALENCY CONDITIONS

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Abstract. For the Briot-Bouquet differential equations of the form given in [1]

$$u(z) + \frac{zu'(z)}{z\frac{f'(z)}{f(z)}[\alpha u(z)+\beta]} = g(z),$$

we can reduce them to

$$v(z) + F(z)\frac{v'(z)}{v(z)} = h(z),$$

where $v(z) = \alpha u(z) + \beta$, $h(z) = \alpha g(z) + \beta$ and $F(z) = f(z)/f'(z)$.

In this paper we are going to give conditions in order that if $u$ and $v$ satisfy, respectively, the equations

$$(1) \qquad u(z) + F(z)\frac{u'(z)}{u(z)} = h(z),$$

$$v(z) + G(z)\frac{v'(z)}{v(z)} = g(z)$$

with certain conditions on the functions $F$ and $G$ applying the concept of strong subordination $g \prec\prec h$ given in [2] by the author, implies that $v \prec u$, where $\prec$ indicates subordination.

## 1. Introduction

In 1935, Goluzin [3] considered the simple first order subordination $zp'(z) \prec h(z)$. He showed that if $h$ is convex then $p(z) \prec q(z)$, with $zq'(z) = h(z)$ $(q(0) = 0)$ and this is the best dominant.

Successive generalizations of this result have been done by Robinson [9] in 1947, Suffridge [10] in 1970, Hallembeck and Ruscheweyh [4] in

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1975, Miller and Mocanu [6] in 1985 and Antonino and Romaguera [2] in 1994.

In all these results were compared functions $p$ an $q$ that were verifying the same differential relation. This is, in the Goluzin´s result, for example, $zp'(z) = b(z)$ and $zq'(z) = h(z)$ and $b(z) \prec h(z)$, but in both cases it is verified the relation $zf'(z)$.

In this paper we are going to see the case in which the functions $p$ and $q$ satisfy different differential equations .

## 2. Preliminaries

Let $F$ and $G$ be analytic in the unit disk $U$. The function $F$ is subordinate to $G$, written $F \prec G$, if $G$ is univalent, $F(0) = G(0)$ and $F(U) \subset G(U)$.

Let $\mathbb{A}_n$ denote the set of functions $f(z) = z + a_{n+1}z^{n+1} + \cdots, n \geq 1$ that are analytic in the unit disk $U$, and let $\mathbb{A} = \mathbb{A}_1$. For $a \in C$, a complex number, and $n$ a positive integer, let $\mathbb{H}[a,n]$ denote the set of functions $f(z) = a + a_n z^n + \cdots, n \geq 1$ that are analytic in $U$.

Let $F$ be analytic and univalent in $U$, with $F(0) = 0$. The class of $F$-convex functions, denoted by $F\mathbb{K}$ are those of $f \in \mathbb{A}$ for which $\mathrm{Re}\left[F(z)\frac{f''(z)}{f'(z)} + 1\right] > 0$. The class of $F$-starlike functions, denoted by $F\mathbb{S}^*$, are those of $f \in A$ for which $\mathrm{Re}\left[F(z)\frac{f'(z)}{f(z)}\right] > 0$. The class of close-to-$F$-convex functions, denoted by $F\mathbb{C}$, are those of $f \in \mathbb{A}$ for which there is a $g \in F\mathbb{S}^*$ such that $\mathrm{Re}\left[F(z)\frac{f'(z)}{g(z)}\right] > 0$. If $F(z) = z$, then we have the convex, starlike and close-to-convex functions, respectively. In this case, these sets are denoted by $\mathbb{K}$, $\mathbb{S}^*$, $\mathbb{C}$ and $\mathbb{S}$ respectively. It is well known that $\mathbb{K} \subset \mathbb{S}^* \subset \mathbb{C} \subset \mathbb{S}$.

We will consider that in (1) $F$ and $G$ are analytic and univalent functions in $U$ ($G$ analytic in $\bar{U}$) with $F(0) = G(0) = 0$, $h(z)$ is an analytic convex function in $U$ with $h(0) \neq 0$, and we will designate for

$$g(z,\xi) \equiv v(z) + \frac{G(\xi)}{\xi} z \frac{v'(z)}{v(z)} \tag{2}$$

the analytic function in $U \times \bar{U}$ and for $g(z) = g(z,z)$, with $g'(0) \neq 0$.

We are going to enunciate two lemmas that will be used in this paper.

LEMMA 1. ([5], Lemma 1) *Let $p \in \mathbb{H}[1,n]$ and let $q$ be analytic and univalent in $\overline{U}$ with $p(0) = q(0)$. If $p$ is not subordinate to $q$,* *then there exist points* $z_0 \in U$ *and* $\xi_0 \in \partial U$, *and an* $m \geq n$ *for which* $p(|z| < |z_0|) \subset q(U)$,

a) $p(z_0) = q(\xi_0)$, and

b) $z_0 p'(z_0) = m\xi_0 q'(\xi_0)$.

The next lemma deals with the notion of a subordination chain. A function $L(z,t)$, $z \in U$, $t \geq 0$, is a subordination chain if $L(z,t)$ is an analytic and univalent function of $z$ for all $t > 0$ and is a continuously differentiable function of $t$ on $[0,\infty[$ for all $z \in U$, and $L(z,s) \prec L(z,t)$ when $0 \leq s \leq t$.

LEMMA 2. ([8], p.159) *The function* $L(z,t) = a_1(t)z + \cdots$, *with* $a_1(t) \neq 0$ *for all* $t \geq 0$ *and* $\lim_{t-\infty} |a_1(t)| = \infty$, *is a subordination chain if and only if* $\operatorname{Re}\left[ z \frac{\frac{\partial L}{\partial z}}{\frac{\partial L}{\partial t}} \right] > 0$ *for* $z \in U$ *and* $t \geq 0$.

## 3. Differential subordination

DEFINITION 1. A function $L(z,t,\xi,s), z \in U, \xi \in \bar{U}, t \geq 0, s > 0$, is a set of subordination chains if for each $\xi \in \bar{U}$ and some $s$, $L(z,t,\xi,s)$ is a subordination chain.

DEFINITION 2. Let $H(z,\xi)$ be analytic in $U \times \bar{U}$ and let $f(z)$ analytic and univalent in $U$. The function $H(z,\xi)$ is strongly subordinate to f(z), written $H(z,\xi) \prec\prec f(z)$, for $\xi \in \bar{U}$, if the function of z, $H(z,\xi)$, is subordinate to f(z).

In the following theorems we will suppose that the functions $u$ and $v$ that satisfy the equations (1) are different from zero in U. Conditions for this to happen can be seen in [1].

THEOREM 1. *Suppose that* $u(z)$ *is an analytic and univalent solution of the differential equation*

$$u(z) + F(z)\frac{u'(z)}{u(z)} = h(z) \tag{3}$$

*and that* $v(z)$ *is an analytic function that satisfies the equation*

$$v(z) + G(z)\frac{v'(z)}{v(z)} = g(z).$$

*If a)* $g(z,\xi) \prec\prec h(z)$ *and b)* $\operatorname{Re}\left[s\frac{G(z)}{z} - \frac{F(z)}{z}\right]\frac{u'(z)}{h'(z)u(z)} \geq 0 \ \ \forall z \in U$ *and* $s \geq 1$*, then* $v \prec u$.

*Proof.* Without loss of generality we can assume that the conditions of the theorem are satisfied on the closed disk $\bar{U}$ (or $\bar{U}\times\bar{U}$). In opposite case, we can replace $u(z)$ by $u_r(z) = u(rz), v(z)$ by $v_r(z) = v(rz), F(z)$ by $F_r(z) = F(rz)$, $g(z,\xi)$ by $g_r(z,\xi) = g(rz,\xi)$ and $h(z)$ by $h_r(z) = h(rz)$, where $0 < r < 1$. These new functions satisfy the conditions of the theorem on $\bar{U}$ (or $\bar{U}\times\bar{U}$). We would then prove that $p_r \prec v_r$ for all $0 < r < 1$. By letting $r \longrightarrow 1^-$, we would obtain $p(z) \prec v(z)$.

Suppose that a) and b) are satisfied, but $v$ is not subordinate to $u$. According to Lemma 1, there are points $z_0 \in U$ and $\xi_0 \in \partial U$, and $m \geqslant 1$ such that $v(z_0) = u(\xi_0)$ and $z_0 v'(z_0) = m\xi_0 u'(\xi_0)$. Using these results in (2) we obtain

$$(4) \qquad g(z_0,\xi_0) = v(z_0) + \frac{G(\xi_0)}{\xi_0}\frac{z_0 v'(z_0)}{v(z_0)} = u(\xi_0) + m\xi_0\frac{G(\xi_0)}{\xi_0}\frac{u'(\xi_0)}{u(\xi_0)}.$$

From (3), we have

$$u(\xi_0) = h(\xi_0) - F(\xi_0)\frac{u'(\xi_0)}{u(\xi_0)}$$

and if we use this equation in (4) we have

$$(5) \qquad g(z_0,\xi_0) = h(\xi_0) + \left[m\frac{G(\xi_0)}{\xi_0} - \frac{F(\xi_0)}{\xi_0}\right]\frac{u'(\xi_0)}{h'(\xi_0)u(\xi_0)}\,[\xi_0 h'(\xi_0]$$

and because

$$\operatorname{Re}\left[m\frac{G(\xi_0)}{\xi_0} - \frac{F(\xi_0)}{\xi_0}\right]\frac{u'(\xi_0)}{h'(\xi_0)u(\xi_0)} \geq 0$$

and $\xi_0 h'(\xi_0)$ is an outward normal to the boundary of the convex domain $h(U)$, we deduce that $g(z_0,\xi_0)$ of (5) represents a complex outside of $h(U)$. This contradicts $g \prec\prec h$, and we conclude that $v \prec u$. □

EXAMPLE 1. Let $F = (M^2 - R - 1)z$ with $R > 2$, $\frac{R}{R-2} < M < \sqrt{R+1}$, and let $h(z) = R + 1 + M\frac{Mz-1}{M-z}$. Since $\omega = M\frac{Mz-1}{M-z}$ mapping $U$ in $|\omega| < M$, $h(z)$ is a convex function. The differential equation $u(z) + F(z)\frac{u'(z)}{u(z)} = h(z)$ is satisfied by the univalent function $u(z) = \frac{MR}{M-z}$.

Let $g(z,\xi)$ be the function

$$g(z,\xi) = R + 1 + z + z\frac{1+\xi}{R+1+z},$$

where $G(\xi) = \xi + \xi^2$, is strongly subordinated to h(z) and the differential equation $v(z) + (1+z)z\dfrac{v'(z)}{v(z)} = g(z)$ is satisfied by $v(z) = R + 1 + z$.

Moreover,

$$\begin{aligned} & \operatorname{Re}\left[s\frac{G(z)}{z} - \frac{F(z)}{z}\right]\frac{u'(z)}{h'(z)u(z)} \\ = & \operatorname{Re}\left[s(1+z) - (M^2 - R - 1)\right]\frac{M-z}{M(M^2-1)} > 0, \quad s \geq 1. \end{aligned}$$

According to Theorem 1, $v \prec u$ since it is verified directly.

If the function $h(z)$ is univalent but is not convex, then the conditions of the previous theorem can be modified to have an analogous result.

THEOREM 2. *Suppose that $u(z)$ is an analytic and univalent solution of the differential equation*

$$u(z) + F(z)\frac{u'(z)}{u(z)} = h(z) \tag{6}$$

*and that $v(z)$ is an analytic function that satisfies the equation*

$$v(z) + G(z)\frac{v'(z)}{v(z)} = g(z).$$

*If*

a) $g(z,\xi) \prec\prec h(z)$,

b) $P(z) = z\dfrac{u'}{u}$ *is starlike, and*

c) $\operatorname{Re}\left[s\dfrac{G(\xi)}{\xi} - \dfrac{F(\xi)}{\xi}\right]\dfrac{u'(z)}{h'(z)u(z)} > 0$, $\forall (z,\xi) \in U \times \bar{U}$ *and* $\forall s \geq 1$,

*then* $v \prec u$.

*Proof.* Let

$$L(z,t,\xi,s) = h(z) + t\left[s\frac{G(\xi)}{\xi} - \frac{F(\xi)}{\xi}\right]P(z) \tag{7}$$

be an analytic function in $U$ for all $t \geqslant 0$, every $\xi \in \bar{U}$ and $s \geqslant 1$, and it is continuously differentiable on $]0,\infty[$ respect of t, for all $z \in U, \xi \in \bar{U}$

and $s \geqslant 1$. Routine calculations allow to obtain $\forall t \geqslant 0$, for each $\xi$ and $s$ fixed

$$\frac{\partial L}{\partial z} = h'(z) + t\left[s\frac{G(\xi)}{\xi} - \frac{F(\xi)}{\xi}\right]P'(z),$$

$$\frac{\partial L}{\partial t} = \left[s\frac{G(\xi)}{\xi} - \frac{F(\xi)}{\xi}\right]P(z),$$

and we can calculate

$$z\frac{\dfrac{\partial L}{\partial z}}{\dfrac{\partial L}{\partial t}} = \frac{1}{\left[s\dfrac{G(\xi)}{\xi} - \dfrac{F(\xi)}{\xi}\right]\dfrac{u'(z)}{h'(z)u(z)}} + tz\frac{P'(z)}{P(z)},$$

from b) and c), $\operatorname{Re} z\frac{\frac{\partial L}{\partial z}}{\frac{\partial L}{\partial t}} > 0$. Moreover,

$$\left.\frac{\partial L}{\partial z}\right|_{z=0} = a_1(t) = h'(0) + t\left[s\frac{G(\xi)}{\xi} - \frac{F(\xi)}{\xi}\right]\frac{u'(0)}{u(0)h'(0)}h'(0)$$

$$= h'(0)\left[1 + t\left[s\frac{G(\xi)}{\xi} - \frac{F(\xi)}{\xi}\right]\frac{u'(0)}{u(0)h'(0)}\right] \neq 0,$$

since $t \geq 0$ and from c),

$$\operatorname{Re}\left[s\frac{G(\xi)}{\xi} - \frac{F(\xi)}{\xi}\right]\frac{u'(0)}{u(0)h'(0)} > 0,$$

also $\lim_{t\to\infty} |a_1(t)| = \infty$ for every $\xi$ and s fixed. From Lemma 2, we conclude that $L(z,t,\xi,s)$ is a subordination chain for $\xi$ and $s$ fixed, that we have designated as set of subordination chains; we observe that $h(z) = L(z,0,\xi,s)$.

Suppose that $v$ is not subordinate to $u$. Then we can apply a similar argument to that of Theorem 1 to establish that

$$g(z_0,\xi_0) = h(\xi_0) + t\left[m\frac{G(\xi_0)}{\xi_0} - \frac{F(\xi_0)}{\xi_0}\right]P(\xi_0).$$

From (6), with $z = \xi = \xi_0$ and $s = m$, we have that $g(z_0,\xi_0) = L(\xi_0,t,\xi_0,m)$ and since $L(\xi_0,0,\xi_0,m) \in L(z,t,\xi_0,m)$ and $L(\xi_0,0,\xi_0,m) \notin h(U)$, $g(z,\xi)$ is not subordinated to $h(z)$ in opposition to what we have supposed. Hence, $v \prec u$. □

EXAMPLE 2. Let $F(z)$, $h(z)$ and $u(z)$ be as in Example 1. Obviously $h(z)$ is univalent. Let

$$G(z) = \frac{a-b}{b} z(R + bz)$$

and

$$g(z,\xi) = R + bz + (a - b)\frac{(R + b\xi)z}{R + bz},$$

where $0 < b < a$, $0 < b < R$ and $2 < M < \sqrt{R+1}$. Let the differential equation

$$v(z) + G(z)\frac{v'(z)}{v(z)} = g(z),$$

be satisfied by $v(z) = R + bz$. We are going to see that the conditions of Theorem 2 are satisfied.

1) $g(z,\xi) \prec\prec h(z)$.

It is sufficient to see that

$$\left| -1 + bz + (a - b)z\frac{R + b\xi}{R + bz} \right| < M.$$

But $\left| -1 + bz + (a - b)z\frac{R + b\xi}{R + bz} \right| \leq 1 + b + (a - b)\frac{R + b}{R - b}$ and fixing $b = R/10$ and taking $(a - b)$ sufficiently small for that $(a - b)\frac{R + b}{R - b} < \frac{1}{10}$, then $1 + R/10 + 1/10 < M < \sqrt{R+1}$, this is, $11 + R < 10M < 10\sqrt{R+1}$ which is compatible for $R < 9$.

2) $P(z)$ is starlike.

As $z\frac{P'(z)}{P(z)} = 1 + \frac{z}{M - z}$ is sufficient to see that $\left| \frac{z}{M - z} \right| \leq \frac{1}{M - 1} < 1$. But since $M > 2$ the inequality is satisfied.

3) $\operatorname{Re}\left[ s\frac{G(\xi)}{\xi} - \frac{F(\xi)}{\xi} \right] \frac{u'(z)}{h'(z)u(z)} > 0$.

For that $\operatorname{Re}\left[ s\frac{a-b}{b}(R + b\xi) + R + 1 - M^2 \right] \frac{M - z}{M(M^2 - 1)} > 0$ is sufficient to see that $\operatorname{Re}\left[ s\frac{a-b}{b}(R + b\xi) + R + 1 - M^2 \right](M - z) > 0$. In this expression let's put $\xi = re^{i\varphi}$ and $z = \rho e^{i\theta}, 0 < r < 1, 0 < \rho < 1$ and

$\varphi, \theta \in [0, 2\pi]$.

$$\begin{aligned}
&\operatorname{Re}\left[s\frac{a-b}{b}(R+bre^{i\varphi})+R+1-M^2\right](M-\rho e^{i\theta}) \\
&= \left[s\frac{a-b}{b}(R+br\cos\varphi)+R+1-M^2\right](M-\rho\cos\theta) \\
&\quad + s\frac{a-b}{b}b\rho r\sin\varphi\sin\theta \\
&\geq \left[s\frac{a-b}{b}(R-b)+R+1-M^2\right](M-1)-s(a-b) \\
&\quad + (R+1-M^2)(M-1) \\
&= s\frac{a-b}{b}\left[RM-R-Mb\right]+(R+1-M^2)(M-1).
\end{aligned}$$

As $a-b>0, R+1-M^2>0$ and $M-1>0$, in order that the latter expression is $>0$, is sufficient that $M(R-b)-R>0 \Longrightarrow M>\dfrac{R}{R-b}$. If $b=R/10$, for example, then $10/9<M<\sqrt{R+1}$ and it would be enough to take $3<R<9$.

The conditions of Theorem 2 can be established and, consequently $v \prec u$.

COROLLARY 2.1. *Let* $u(z)+F(z)\dfrac{u'(z)}{u(z)}=h(z)$ *and suppose that the conditions a) and b) of Theorem 1 or the conditions a), b) and c) of Theorem 2 are verified, with*

$$g(z,\xi)=G'(z)+\frac{G(\xi)}{\xi}z\frac{f''(z)}{f'(z)}, \tag{8}$$

*where* $f(z)$ *is an analytic function in* $U$ *such that* $G(z)\dfrac{f'(z)}{f(z)}\neq 0$ *in* $U$. *Then*

$$G(z)\frac{f'(z)}{f(z)} \prec u(z). \tag{9}$$

*Proof.* The equation $v(z)+G(z)\dfrac{v'(z)}{v(z)}=g(z)$, with $g(z)=G'(z)+G(z)\dfrac{f''(z)}{f'(z)}$ has the solution $v(z)=\dfrac{G(z)f'(z)}{f(z)}$ and using Theorem 1 or Theorem 2, as correspond, we obtain the conclusion. □

If $\operatorname{Re} h(z) > 0$ under certain conditions $\operatorname{Re} u(z) > 0$ (see [1]) and from (7) with $\xi = z$, $f(z)$ is $G$-convex and from (8) $G$-starlike. This is, in this context, a $G$-convex function is G-starlike. This concept has been introduced and used by the author in other papers.

If in the equations (1) we replace $F$ by $\gamma F$ and $G$ by $\gamma G$ $(\gamma > 0)$ the conclusions of Theorem 1 and Theorem 2 are not changed. In the following theorems we will apply this new version.

THEOREM 3. *Let* $u(z) + \gamma F(z)\dfrac{u'(z)}{u(z)} = h(z)$ *and suppose that the conditions a) and b) of Theorem 1 or the conditions a), b) and c) of Theorem 2 are verified, with*

$$g(z,\xi) = \frac{G(\xi)}{\xi} z \frac{H'(z)}{H(z)},$$

*where* $G(z) \in \mathbb{A}_n$, $\gamma > 0$ *and* $H(z) \in \mathbb{A}_n$. *If* $f(z)$ *is defined as*

$$f(z) = \left[\int_0^z \frac{H^{1/\gamma}(t)}{\gamma G(t)} dt\right]^\gamma \tag{10}$$

*which is different from zero in* $U - \{0\}$, *then* $f(z) \in \mathbb{A}_n$ *and*

$$G(z)\frac{f'(z)}{f(z)} \prec u(z). \tag{11}$$

*Proof.* From condition a) of the Theorem 1 or Theorem 2, we are assuming that $\dfrac{H(z)}{G(z)} \neq 0$ in $U$. If in the equation $\gamma G(z)\dfrac{v'(z)}{v(z)} + v(z) = G(z)\dfrac{H'(z)}{H(z)}$ we set $g(z) = G(z)\dfrac{H'(z)}{H(z)}$ then $g(z) \in \mathbb{H}[1,n]$ and we obtain

$$\gamma G(z)\frac{v'(z)}{v(z)} + v(z) = g(z).$$

The function

$$v(z) = \frac{H^{1/\gamma}}{\displaystyle\int_0^z \frac{H^{1/\gamma}(t)}{\gamma G(t)} dt}$$

is a nonzero analytic solution of this equation and satisfies the conditions of Theorem 1 or Theorem 2. Consequently, $v \prec u$. From (9) and from the latter expression, we have that $f(z) = H(z)/v^\gamma(z)$ and $f \in \mathbb{A}_n$. By logarithmically differentiating

$$G(z)\frac{f'(z)}{f(z)} = G(z)\frac{H'(z)}{H(z)} - \gamma G(z)\frac{v'(z)}{v(z)} = v(z) \prec u(z).$$

This completes the proof. □

A simple computation using (9) leads to

$$(1-\gamma)G(z)\frac{f'(z)}{f(z)} + \gamma\frac{zG(\xi)}{\xi G(z)}\left(G(z)\frac{f''(z)}{f'(z)} + G'(z)\right) = g(z,\xi)$$

and designating by $J(\gamma G, f; G)$ the term on the left, we have that

$$J(\gamma\frac{z}{\xi}G(\xi), f(z); G(z))$$
$$= \left(G(z) - \gamma\frac{z}{\xi}G(\xi)\right)\frac{f'(z)}{f(z)} + \gamma\frac{zG(\xi)}{\xi}\left(\frac{f''(z)}{f'(z)} + \frac{G'(z)}{G(z)}\right)$$

and, more generally,

$$J\left(\gamma\frac{z}{\xi}H(\xi), f(z), G(z)\right)$$
$$= \gamma\frac{z}{\xi}H(\xi)\left(\frac{f''(z)}{f'(z)} + \frac{G'(z)}{G(z)}\right) + \left(G(z) - \gamma\frac{z}{\xi}H(\xi)\right)\frac{f'(z)}{f(z)}.$$

Theorem 4 can be written now in the following form: If the condition b) of Theorem 1 is satisfied (or the conditions b) and c) of Theorem 2), then

$$J(\gamma\frac{z}{\xi}G(\xi), f(z); G(z) = g(z,\xi) \prec\prec h \Longrightarrow G(z)\frac{f'(z)}{f(z)} \prec u(z).$$

In [5] the authors have investigated a form of this result. Our following theorem contains this result.

THEOREM 4. *Let's suppose that the conditions b) of Theorem 1 or b) and c) of Theorem 2 are verified for the functions $u(z)$, $h(z)$, $G(z)$. Let $g(z,\xi)$ be defined of the following form: for all $f \in \mathbb{A}_n$ such that $G(z)f'(z)/f(z)$ is analytic and nonzero function in $U$*

$$g(z,\xi) = J(\gamma\frac{z}{\xi}G(\xi), f(z); G(z))$$

*and if*

$$K(z) = G(z)\exp\int_0^z \frac{u(t) - G'(t)}{G(t)}dt, \tag{12}$$

*then*

$$J(\gamma \frac{z}{\xi} G(\xi), f(z); G(z)) \prec\prec J(\gamma F, K; G) \Longrightarrow \tag{13}$$

$$G(z)\frac{f'(z)}{f(z)} \prec G(z)\frac{K'(z)}{K(z)} = u(z). \tag{14}$$

*Proof.* From (12), we obtain $u(z) = G(z)K'(z)/K(z)$ and substituting in $h(z) = u(z) + \gamma F(z)u'(z)/u(z)$ we have that

$$\begin{aligned} h(z) &= (G(z) - \gamma F(z))\frac{K'(z)}{K(z)} + \gamma F(z)\left(\frac{K''(z)}{K'(z)} + \frac{G'(z)}{G(z)}\right) \\ &= J(\gamma F, K; G). \end{aligned}$$

Since $u(z)$ is univalent, also is univalent $G(z)K'(z)/K(z)$, and the subordination (14) are well defined. If we set $g(z,z) = g(z) = J(\gamma G, f; G)$, $g(z) \in \mathbb{H}[1,n]$. Using this $g(z)$ in Theorem 1 or Theorem 2, the equation $\gamma G(z)\frac{v'}{v} + v = J(\gamma G, f; G)$, has a nonzero analytic solution $v(z) = G(z)f'(z)/f(z)$ that satisfies $G(z)f'(z)/f(z) \prec G(z)k'(z)/k(z)$. □

EXAMPLE 3. Let

$$\begin{aligned} u(z) + F(z)\frac{u'(z)}{u(z)} &= h(z), \\ v(z) + G(z)\frac{v'(z)}{v(z)} &= g(z), \end{aligned}$$

with $F(z) = az \ (a \in ]-1,0[), h(z) = \left[(1+az)^2 + a^2 z\right]/(1+az), G(z) = z + az^2$. The first one is satisfied by $u = 1/(1+az)$. If $|a| < 1/26$, then $h(z)$ is a convex function and the conditions a) and b) of Theorem 1 are satisfied. If $g(z) = 1 + b_n z^n$, with $|b_n| \le a^2$, then $b(z) \prec h(z)$ and we can apply Theorem 1.

We are going to give an application of the previous results to see how conditions of univalency can be obtained.

EXAMPLE 4. Let $h(z) = 1 + mz, u(z) = e^z, F(z) = 1 + mz - e^z$ $(m \neq 1)$ and $G(z) = z$. If

$$a = \min_{z \in \partial U} \operatorname{Re} \frac{e^z - 1}{z} > 0,$$

then for $0 < m < 1 + a$ the conditions of Theorem 1 are verified with $s \ge 1$. If $f \in \mathbb{A}$ and

$$1 + z\frac{f''(z)}{f'(z)} \prec h(z),$$

then by Corollary 1, $zf'(z)/f(z) \prec u(z)$. This means that

$$\left| z\frac{f''(z)}{f'(z)} \right| < m \Longrightarrow \operatorname{Re} z\frac{f'(z)}{f(z)} > 0.$$

That is, $f$ is starlike and therefore univalent.

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