

BALANCEDNESS OF INTEGER DOMINATION GAMES

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ABSTRACT. In this paper, we consider cooperative games arising from integer domination problem on graphs. We introduce two games, $\{k\}$ -domination game and its monotonic relaxed game, and focus on their cores. We first give characterizations of the cores and the relationship between them. Furthermore, a common necessary and sufficient condition for the balancedness of both games is obtained by making use of the technique of linear programming and its duality.

1. Introduction

In this paper we investigate cooperative cost games that arise from integer domination problems on graphs, which are widely studied in graph theory. Given a graph $G = (V, E; \omega)$ with vertex weight function $\omega : V \rightarrow R_+$ and a given positive integer k , a function $g : V \rightarrow \{0, 1, 2, \dots, k\}$ is a $\{k\}$ -dominating function of G if for every vertex $v \in V$, $\sum_{u \in N[v]} g(u) \geq k$, where $N[v] = \{v\} \cup \{u \in V : (u, v) \in E\}$ is the closed neighborhood of v in graph G . The $\{k\}$ -domination problem is to find a $\{k\}$ -dominating function g which minimizes the total weight $\sum_{v \in V} g(v)\omega(v)$. When $k = 1$, this problem is just the weighted minimum dominating set problem.

The $\{k\}$ -domination problem has some practical applications. For example, let $G = (V, E)$ be a graph in which vertices represent cities and edges represent pairs of cities that are neighbors. Assume there is a need to build service stations such that each city can receive help from at least k service stations in its own city or in its neighboring cities. Assume there is also a fixed cost for building a service station in a each city. The problem is determining the number of service stations

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built in each city such that the total building cost is minimal among the participating cities. This problem is equivalent to the problem of finding a minimum weight $\{k\}$ -dominating function on G .

A natural question that arises from above example is how to distribute the total cost of building service stations among all the participating cities. In this paper, we introduce two closely related cooperative cost games, the $\{k\}$ -domination game and its monotonic relaxed game, that are modelled on the cost distribution problem. Especially, we focus on characterizing the cores for both game models.

The combinatorial optimization techniques have been often utilized in much cooperative games. Especially, integer linear programming and its duality theory have proven itself a very powerful tool in the study of cores. Shapley and Shubik [10] formulated the assignment game as a two-sided market, and showed that the core is exactly the set of optimal solutions of a dual linear programming on the assignment game problem. This approach is further exploited in the study of linear production game [2, 8], partition game [4], packing and covering games [3]. Recently, Velzen [11] introduced three kinds of cooperative games that arise from the weighted minimum dominating set problem on a graph. It was shown that the core of each game is non-empty if and only if the corresponding linear programming relaxation of the weighted minimum dominating set problem has an integer optimal solution, and in this case, an element in the core can be found in polynomial time. The integer domination games presented in the paper extend Velzen's with more practical application.

In Section 2, we introduce some notions from cooperative game theory and present two cooperative games that are modelled on the cost allocation problems arising from integer domination problems on graphs. In Section 3, we study the characterizations of cores of the two games. Section 4 is dedicated to the balancedness of the two games. We give a relationship between the cores of the two games, and derive a common necessary and sufficient condition for non-emptiness of the cores for both games. That is to say, if one of the two games possesses a core element, then the other possesses a core element as well.

2. Definition of $\{k\}$ -domination games

In this section, we introduce two cooperative cost games that are modelled on the cost allocation problem arising from integer domination problems on graphs. We begin with some concepts and notions in cooperative game theory.

2.1. Cooperative game

A cooperative game (in characteristic function form) $\Gamma = (V, c)$ consists of a player set $V = \{1, 2, \dots, n\}$ and a characteristic function $c : 2^V \rightarrow R$ with $c(\emptyset) = 0$. For each coalition $S \subseteq V$, $c(S)$ represents the revenue or cost achieved by the players in S together. The main issue is how to fairly distribute the total revenue or cost $c(V)$ among all the players. We define terms only for cost games, as symmetric statement holds for revenue games.

A vector $z = (z_1, z_2, \dots, z_n)$ is called an *imputation* if and only if $\sum_{i \in V} z_i = c(V)$ and $z_i \leq c(i)$ for each i . $z(S)$ is defined to be $\sum_{i \in S} z_i$ for each $S \subseteq V$. Now, the *core* of a game $\Gamma = (V, c)$ is defined by

$$\text{Core}(\Gamma) = \{z \in R^n : z(V) = c(V) \text{ and } z(S) \leq c(S), \forall S \subseteq V\},$$

where $z(S) = \sum_{i \in S} z_i$ for $S \subseteq V$. The set of constraints imposed on $\text{Core}(\Gamma)$, which is called group rationality, ensures that no coalition would have an incentive to split from the grand coalition V , and do better on its own.

The study of the core is closely associated with another important concept, the balanced set. The collection \mathcal{B} of subsets of N is balanced if there exists a set of positive numbers β_S ($S \in \mathcal{B}$), such that for each $i \in V$, we have $\sum_{i \in S \in \mathcal{B}} \beta_S = 1$. A game (V, c) is called *balanced* if $\sum_{S \in \mathcal{B}} \beta_S c(S) \leq c(V)$ holds for every balanced collection \mathcal{B} with weights $\{\beta_S : S \in \mathcal{B}\}$. With techniques essentially the same as linear programming duality, Bondareva [1] and Shapley [9] proved that a game has non-empty core if and only if it is balanced.

A game $\Gamma = (V, c)$ is called a *monotonic* game if it satisfies $c(S) \leq c(T)$ for every $S \subseteq T \subseteq V$. Given a balanced monotonic game $\Gamma = (V, c)$ and $z \in \text{Core}(\Gamma)$, it holds that $z_i = c(V) - \sum_{j \in V \setminus \{i\}} z_j \geq c(V) - c(V \setminus \{i\}) \geq 0$ for every $i \in V$. That is, each core element of a monotonic balanced game is non-negative.

2.2. $\{k\}$ -domination games

Let $G = (V, E)$ be an undirected graph with vertex set V and edge set E . Two distinct vertices $u, v \in V$ are called adjacent if $\{u, v\} \in E$. For any non-empty set $V' \subseteq V$, the induced subgraph by V' , denoted by $G[V']$, is a subgraph of G whose vertex set is V' and whose edge set is the set of edges having both endpoints in V' . For any subset $S \subseteq V$, we define the closed neighboring set of S to be the union of the closed neighborhoods of all vertices in S , denoted by $N[S] = \bigcup_{v \in S} N[v]$.

Given a graph $G = (V, E; \omega)$ with vertex weight function $\omega : V \rightarrow R_+$ and a positive integer k , a function $g : V \rightarrow \{0, 1, 2, \dots, k\}$ is a $\{k\}$ -dominating function of G if for every vertex $v \in V$, $\sum_{u \in N[v]} g(u) \geq k$. Thus, if S is a dominating set of graph G and we define the function g where $g(v) = 1$ if $v \in S$ and $g(v) = 0$ if $v \notin S$; then g is a 1-dominating function of G . The $\{k\}$ -domination problem is to find a $\{k\}$ -dominating function g which minimizes the total weight $\sum_{v \in V} g(v)\omega(v)$.

A function $g : V \rightarrow \{0, 1, 2, \dots, k\}$ is said to $\{k\}$ -dominate a set $S \subseteq V$, if for each vertex $v \in S$, $\sum_{u \in N[v]} g(u) \geq k$. In the rest of this paper, for convenience, we denote $\sum_{u \in S} g(u)$ and $g(i)$ by $g(S)$ and g_i , respectively.

Given a graph $G = (V, E; \omega)$ with vertex weight function $\omega : V \rightarrow R_+$, the $\{k\}$ -domination game $\Gamma = (V, c)$ corresponding to G is defined as follows:

- (1) The player set is $V = \{1, 2, \dots, n\}$;
- (2) For each coalition $S \subseteq V$,

$$c(S) = \min \left\{ \sum_{i \in S} g_i \omega_i \mid g : S \rightarrow \{0, 1, 2, \dots, k\} \text{ and } \forall j \in S, \sum_{i \in N[j] \cap S} g_i \geq k \right\}.$$

That is, the cost $c(S)$ is the minimum weight of $\{k\}$ -dominating function in the induced subgraph $G[S]$. In this game, each coalition cannot use the cities not belonging to itself. However, in some situation, it sometimes makes more sense to build service stations in any cities not restricted the ones in the coalition as long as each city in the coalition can receive help from k service stations in its own or neighboring cities. Motivated by this observation, we define another related game, the monotonic $\{k\}$ -domination game $\tilde{\Gamma} = (V, \tilde{c})$, by dropping the requirement that coalitions are only allowed to use their own cities. Formally, the monotonic $\{k\}$ -domination game $\tilde{\Gamma} = (V, \tilde{c})$ corresponding to G is defined as follows:

- (1) The player set is $V = \{1, 2, \dots, n\}$;
- (2) For each coalition $S \subseteq V$,

$$\tilde{c}(S) = \min \left\{ \sum_{i \in V} g_i \omega_i \mid g : V \rightarrow \{0, 1, 2, \dots, k\} \text{ and } \forall j \in S, \sum_{i \in N[j]} g_i \geq k \right\}.$$

That is, the value $\tilde{c}(S)$ is the minimum weight of a function which $\{k\}$ -dominates the set S . Obviously, this game is monotonic, i.e., for any subset S and T with $S \subseteq T$, $\tilde{c}(S) \leq \tilde{c}(T)$.

Since, for each coalition, there are more $\{k\}$ -dominating functions to consider in the monotonic $\{k\}$ -domination game than in $\{k\}$ -domination game, it holds that $c(S) \geq \tilde{c}(S)$ for all $S \subset V$. The grand coalition V has the same possibilities in both games, $c(V) = \tilde{c}(V)$.

3. Characterization of the cores

In this section, we present characterizations of the cores for both $\{k\}$ -domination games. For convenience, we introduce the following terms. Let $G = (V, E)$ be a graph. The closed neighborhood of a vertex $v \in V$ in graph G is called v -star. If $T \subseteq N[v]$ contains v , then T is called a v -substar. The set of all v -substars in graph G is denoted by \mathcal{T}_v , i.e., $\mathcal{T}_v = \{T \subseteq V : T \text{ is a } v\text{-substar}\}$.

For a dominating set D of graph $G = (V, E)$, it is easy to see that the vertex set V can be covered by disjoint v -substars with $v \in D$, i.e. $V = \bigcup_{v \in D} T_v$, where $T_v \in \mathcal{T}_v$ and $T_u \cap T_v = \emptyset$ if $u \neq v$. In the following lemma, we extend this observation to $\{k\}$ -dominating function g . We will show that every vertex is contained in exactly k substars in some suitably chosen collection of substars induced by the function g . In fact, the proof of the following lemma shows a way to construct such a collection of substars.

LEMMA 3.1. *Let $g : V \rightarrow \{0, 1, 2, \dots, k\}$ be a $\{k\}$ -dominating function of graph $G = (V, E)$. Then for each $v \in V$, there exists a multiset \mathcal{T}_v^g of v -substars (may be repeated) such that $|\mathcal{T}_v^g| = g(v)$, such that every vertex $u \in V$ is contained in exactly k substars in the collection $\mathcal{T}^g = \{\mathcal{T}_v^g : v \in V\}$.*

Proof. Since $g : V \rightarrow \{0, 1, 2, \dots, k\}$ is a $\{k\}$ -dominating function of graph G , $\forall u \in V$, we have $\sum_{v \in N[u]} g(v) \geq k$. Then for each vertex $v \in V$, we take a set of integers $\{F^v(u) : u \in V\}$ satisfying the following constraints:

$$(3.1) \quad \begin{cases} F^v(v) = g(v) \\ 0 \leq F^v(u) \leq g(v) & \text{for } u \in N[v] \setminus \{v\} \\ F^v(u) = 0 & \text{for } u \notin N[v] \end{cases}$$

and the set $\bigcup_{v \in V} \{F^v(u) : u \in V\}$ satisfies the constraint:

$$(3.2) \quad \forall u \in V : \sum_{v \in \{v:u \in N[v]\}} F^v(u) = k.$$

Intuitively, $F^v(u)$ can be considered as the number of times of vertex u being used by vertex v to ensure vertex v to be exactly $\{k\}$ -dominated. It is easy to see that a integer set $\bigcup_{v \in V} \{F^v(u) : u \in V\}$ satisfying constrains (3.1) and (3.2) exists.

For each vertex $v \in V$, we take the positive values in the set $\{F^v(u) : u \in V\}$, and arrange those in increasing order, say, $0 < l_1 < l_2 < \dots < l_s$, where $l_s = g(v)$. Then we construct a multiset of v -substars \mathcal{T}_v^g consisting v -substars defined as follows:

$$\left\{ \begin{array}{l} T_{v(l_1)} = \{u \in N[v] : F^v(u) \geq l_1\} \\ \qquad \qquad \qquad \rightarrow l_1 \text{ copies of } T_{v(l_1)} \text{ appearing in } \mathcal{T}_v^g. \\ T_{v(l_2)} = \{u \in N[v] : F^v(u) \geq l_2\} \\ \qquad \qquad \qquad \rightarrow l_2 - l_1 \text{ copies of } T_{v(l_2)} \text{ appearing in } \mathcal{T}_v^g. \\ \dots\dots\dots \\ T_{v(l_s)} = \{u \in N[v] : F^v(u) \geq l_s\} \\ \qquad \qquad \qquad \rightarrow l_s - l_{s-1} \text{ copies of } T_{v(l_s)} \text{ appearing in } \mathcal{T}_v^g. \end{array} \right.$$

Obviously, \mathcal{T}_v^g has $g(v)(= l_s)$ elements, and each vertex $u \in V$ appears in exactly $F^v(u)$ v -substars in \mathcal{T}_v^g . Now, let $\mathcal{T}^g = \{\mathcal{T}_v^g : v \in V\}$. Since $\sum_{v \in \{v:u \in N[v]\}} F^v(u) = k$, each vertex $u \in V$ is contained in exactly k substars in the collection \mathcal{T}^g . \square

Let $\Gamma = (V, c)$ and $\tilde{\Gamma} = (V, \tilde{c})$ be the corresponding $\{k\}$ -domination game and monotonic $\{k\}$ -domination game, respectively. Now we provide efficient core descriptions of both games in terms of coalitions corresponding to v -stars and v -substars.

THEOREM 3.2. *Let $G = (V, E; \omega)$ be a graph with vertex weight function $\omega : V \rightarrow R_+$, and $\tilde{\Gamma} = (V, \tilde{c})$ be the corresponding monotonic $\{k\}$ -domination game. It holds that $z \in \text{Core}(\tilde{\Gamma})$ if and only if*

- (1) $z \geq 0$, and $z(V) = \tilde{c}(V)$;
- (2) for each $j \in V$, $z(N[j]) \leq k\omega_j$.

Proof. Suppose that $z \in \text{Core}(\tilde{\Gamma})$. Then (1) follows from $\tilde{\Gamma} = (V, \tilde{c})$ being a monotonic game. Given $j \in V$, we define a function $f : V \rightarrow \{0, 1, 2, \dots, k\}$ where $f_j = k$ and $f_i = 0$ for any $i \neq j$. Clearly, this function $\{k\}$ -dominates the set $N[j]$, which implies that $N[j]$ is a coalition with cost at most $k\omega_j$. That is, $z(N[j]) \leq \tilde{c}(N[j]) \leq k\omega_j$.

To prove the converse, we show that $z(S) \leq \tilde{c}(S)$ holds for all $S \subseteq V$. Let $S \subseteq V$ be an arbitrary subset and $f^* : V \rightarrow \{0, 1, 2, \dots, k\}$ be an optimal weighted function which $\{k\}$ -dominates the subset S . That is, $\tilde{c}(S) = \sum_{j \in V} f_j^* \omega_j$. Then we have

$$z(S) \leq \frac{1}{k} \sum_{j \in V} f_j^* z(N[j]) \leq \sum_{j \in V} f_j^* \omega_j = \tilde{c}(S),$$

where the first inequality holds because $z \geq 0$ and f^* k -dominates the subset S , the second inequality holds because of our assumption (2). Hence $z \in \text{Core}(\tilde{\Gamma})$. \square

In the next theorem we characterize the core of the $\{k\}$ -domination game which is similar to that of the monotonic $\{k\}$ -domination game given above. Except that v -substars are used in stead of v -stars, that is, a vector is a core element of the $\{k\}$ -domination game if and only if no coalition corresponding to a v -substar has an incentive to leave the grand coalition.

THEOREM 3.3. *Let $G = (V, E; \omega)$ be a graph with vertex weight function $\omega : V \rightarrow R_+$, and $\Gamma = (V, c)$ be the corresponding $\{k\}$ -domination game. It holds that $z \in \text{Core}(\Gamma)$ if and only if*

- (1) $z(V) = c(V)$;
- (2) for each $j \in V$, and each j -substar $T_j \in \mathcal{T}_j$, we have $z(T_j) \leq k\omega_j$.

Proof. Suppose that $z \in \text{Core}(\Gamma)$. Then we have $z(V) = c(V)$. For each $j \in V$, and each subset j -substar $T_j \in \mathcal{T}_j$, the function $g : T_j \rightarrow \{0, 1, 2, \dots, k\}$ such that $g_j = k$ and $g_i = 0$ for any $i \neq j$ is a $\{k\}$ -dominating function in the induced graph $G[T_j]$, it implies that T_j is a coalition with cost at most $k\omega_j$. Hence $z(T_j) \leq c(T_j) \leq k\omega_j$.

Now we prove the converse. Let $S \subseteq V$ be an arbitrary coalition and $g^* : S \rightarrow \{0, 1, 2, \dots, k\}$ be a minimum weight $\{k\}$ -dominating function in the induced graph $G[S]$, that is, $\sum_{j \in S} g_j^* \omega_j = c(S)$.

By Lemma 3.1, for each vertex $j \in S$, there exist a multiset $\mathcal{T}_j^{g^*}$ with g_j^* j -substars, and each vertex $i \in S$ is contained in exactly k substars in $\mathcal{T}^{g^*} = \{\mathcal{T}_j^{g^*} : j \in S\}$. Therefore

$$z(S) = \frac{1}{k} \sum_{T \in \mathcal{T}^{g^*}} z(T) \leq \frac{1}{k} \sum_{j \in S} g_j^* k\omega_j = \sum_{j \in S} g_j^* \omega_j = c(S),$$

where the inequality follows from our assumption (2). \square

4. Balancedness of $\{k\}$ -domination games

In this section we focus on the relationship between the cores of the two $\{k\}$ -domination games and balancedness condition for both of them. We obtain a necessary and sufficient condition for balancedness of the two games making use of integer linear programming corresponding to the $\{k\}$ -domination problem and duality theory.

Let $G = (V, E; \omega)$ be a graph with vertex weight function $\omega : V \rightarrow R_+$, $\Gamma = (V, c)$ and $\tilde{\Gamma} = (V, \tilde{c})$ be the corresponding $\{k\}$ -domination game and monotonic $\{k\}$ -domination game, respectively. In Section 2, we have shown that $c(S) \geq \tilde{c}(S)$ for every $S \subset V$, and $c(V) = \tilde{c}(V)$. From this we derive that

$$(4.1) \quad \text{Core}(\tilde{\Gamma}) \subseteq \text{Core}(\Gamma).$$

Moreover, we will prove that $\text{Core}(\tilde{\Gamma})$ coincides with the nonnegative part of $\text{Core}(\Gamma)$.

THEOREM 4.1. *Let Γ and $\tilde{\Gamma}$ be the $\{k\}$ -domination game and monotonic $\{k\}$ -domination game corresponding to graph $G = (V, E; \omega)$, respectively. Then we have*

$$(4.2) \quad \text{Core}(\tilde{\Gamma}) = \text{Core}(\Gamma) \cap R_+^n.$$

Proof. Because $\tilde{\Gamma} = (V, \tilde{c})$ is a monotonic game, we have $z \geq 0$ for all $z \in \text{Core}(\tilde{\Gamma})$. In addition, by (4.1), it holds that $\text{Core}(\tilde{\Gamma}) \subseteq \text{Core}(\Gamma) \cap R_+^n$.

Now, we shall show that $\text{Core}(\Gamma) \cap R_+^n \subseteq \text{Core}(\tilde{\Gamma})$. Let $z \in \text{Core}(\Gamma) \cap R_+^n$. Clearly, $z(V) = c(V) = \tilde{c}(V)$. Let $S \subset V$ be an arbitrary subset. Then there exists a function $g^* : V \rightarrow \{0, 1, 2, \dots, k\}$ which $\{k\}$ -dominates the set S , such that $\tilde{c}(S) = \sum_{j \in V} g_j^* w_j$.

Since $z \geq 0$, we have

$$\sum_{j \in V} g_j^* z(N[j]) \geq kz(S),$$

and also since $z \in \text{Core}(\Gamma)$, by Theorem 3.3, we have $z(N[j]) \leq kw_j$ for any $j \in V$. Therefore,

$$z(S) \leq \frac{1}{k} \sum_{j \in V} g_j^* z(N[j]) \leq \frac{1}{k} \sum_{j \in V} g_j^* kw_j = \tilde{c}(S).$$

That is, $z \in \text{Core}(\tilde{\Gamma})$. □

In the remainder of this section we focus on the balancedness of both $\{k\}$ -domination game and monotonic $\{k\}$ -domination game. Let $G =$

$(V, E; \omega)$ be a weighted graph with vertex set $V = \{1, 2, \dots, n\}$. Let $A = (a_{ij})_{n \times n}$ be the closed neighborhood matrix of G , where $a_{ij} = 1$ if vertex i is in the closed neighborhood $N[j]$ (that is, $a_{ij} = 1$ if $\{i, j\} \in E(G)$, and $a_{ii} = 1$ for all $i \in V$), and $a_{ij} = 0$ otherwise. We describe the problem of minimum weight $\{k\}$ -dominating function using the following integer linear programming (IP):

$$(4.3) \quad (\text{IP}) : \begin{array}{l} \min \quad \sum_{j=1}^n \omega_j x_j \\ \text{s.t.} \quad \begin{cases} Ax \geq k \\ x = (x_1, x_2, \dots, x_n)^t \in \{0, 1, \dots, k\}^n. \end{cases} \end{array}$$

For this integer programming, we have the linear programming relaxation (LP) and its dual (DLP) as follows:

$$(4.4) \quad (\text{LP}) : \begin{array}{l} \min \quad \sum_{j=1}^n \omega_j x_j \\ \text{s.t.} \quad \begin{cases} Ax \geq k \\ x = (x_1, x_2, \dots, x_n)^t \geq 0 \end{cases} \end{array}$$

$$(4.5) \quad (\text{DLP}) : \begin{array}{l} \max \quad k \sum_{i=1}^n y_i \\ \text{s.t.} \quad \begin{cases} yA \leq \omega \\ y = (y_1, y_2, \dots, y_n) \geq 0, \end{cases} \end{array}$$

where $\omega = (\omega_1, \omega_2, \dots, \omega_n)$. We remark that in the linear programming relaxation (LP), we omit the constraint $x \leq k$ since it is redundant under minimizing the objective function.

In the following, we show that both games are balanced if and only if the above linear programming relaxation (LP) has integral optimal solution, i.e., the optimal value of (LP) equals the minimum weight of $\{k\}$ -dominating function of graph G . We first prove this result for monotonic $\{k\}$ -domination game using the similar technique in Deng et al., [3].

THEOREM 4.2. *Let $G = (V, E; \omega)$ be a graph with vertex weight function $\omega : V \rightarrow R_+$, and $\tilde{\Gamma} = (V, \tilde{c})$ be the corresponding monotonic $\{k\}$ -domination game. Then the core of $\tilde{\Gamma}$ is non-empty if and only if linear programming (LP) has an integer optimal solution. In such case, a vector $z = (z_1, z_2, \dots, z_n)$ is in the core if and only if $\frac{1}{k}z$ is an optimal solution to (DLP).*

Proof. Let $z \in \text{Core}(\tilde{\Gamma})$. Theorem 3.2 implies that $\frac{1}{k}z$ is a feasible solution to (DLP) and $z(V)$ equals the optimal value of (IP). Then we have

$$z(V) = \text{opt}(\text{IP}) \geq \text{opt}(\text{LP}) \geq \text{opt}(\text{DLP}) \geq z(V),$$

the second inequality holds because of the duality theorem of linear programming. (Here we use the notation $\text{opt}(Q)$ representing the optimum objective value of programming problem Q .)

On the other hand, if $\text{opt}(\text{IP}) = \text{opt}(\text{LP})$ and $y = (y_1, y_2, \dots, y_n)$ is an optimal solution of (DLP), then for the vertex $z = ky$

$$z(V) = ky(V) = \text{opt}(\text{DLP}) = \text{opt}(\text{LP}) = \text{opt}(\text{IP}).$$

Also since y is a feasible solution to (DLP), y satisfies that $y \geq 0$ and $y(N[j]) \leq \omega_j$, it follows that $z \geq 0$ and $z(N[j]) \leq k\omega_j$. By Theorem 3.2, $z \in \text{Core}(\tilde{\Gamma})$. \square

In the next theorem, we show that there is a necessary and sufficient condition for their balancedness common both games, that is, both games are balanced if and only if the corresponding (LP) in (4.4) has an integer optimal solution.

THEOREM 4.3. *Let $G = (V, E; \omega)$ be a graph with vertex weight function $\omega : V \rightarrow R_+$, $\Gamma = (V, c)$ and $\tilde{\Gamma} = (V, \tilde{c})$ be the corresponding $\{k\}$ -domination game and monotonic $\{k\}$ -domination game, respectively. The following statements are equivalent :*

- (1) $\text{opt}(\text{LP}) = \text{opt}(\text{ILP})$;
- (2) $\tilde{\Gamma} = (V, \tilde{c})$ is balanced;
- (3) $\Gamma = (V, c)$ is balanced.

Proof. The statement (1) \Leftrightarrow (2) follows from Theorem 4.2. The statement (2) \Rightarrow (3) follows from the observation (4.1) that $\text{Core}(\tilde{\Gamma}) \subseteq \text{Core}(\Gamma)$. So we only need to show the statement (3) \Rightarrow (2). According to Theorem 4.1, it is enough to show that if $\text{Core}(\Gamma) \neq \emptyset$, then there exists an $x \in \text{Core}(\Gamma)$ such that $x \geq 0$, which implies that $x \in \text{Core}(\tilde{\Gamma})$.

By contradiction, suppose that $x \not\geq 0$ for all $x \in \text{Core}(\Gamma)$. Let $z = (z_1, z_2, \dots, z_n)$ be an element of $\text{Core}(\Gamma)$ such that the number of zero components are as large as possible. Since $z \not\geq 0$, we assume that $z_{i_0} < 0$ for some $i_0 \in \{1, 2, \dots, n\}$, and let j_0 be the vertex nearest to vertex i_0 among all the vertices j with $z_j > 0$, that is, $j_0 = \text{argmin}\{d_G(i_0, j) : z_j > 0\}$. Denote $\varepsilon = \min\{z_{j_0}, -z_{i_0}\}$. We construct a new cost allocation

\tilde{z} as follows:

$$\tilde{z}_i = \begin{cases} z_{i_0} + \varepsilon & i = i_0, \\ z_{j_0} - \varepsilon & i = j_0, \\ z_i & i \in N \setminus \{i_0, j_0\}. \end{cases}$$

It is easy to see that $\tilde{z}(V) = z(V)$, and $\tilde{z}_{i_0} \leq 0$, $\tilde{z}_{j_0} \geq 0$ and at least one of \tilde{z}_{i_0} and \tilde{z}_{j_0} is zero. We shall show that \tilde{z} is also an element of $\text{Core}(\Gamma)$.

According to Theorem 3.3, it is sufficient to show that for all $j \in V$ and all j -substar $T_j \in \mathcal{T}_j$, it holds that $\tilde{z}(T_j) \leq k\omega_j$. Let $j \in V$ and T_j be a j -substar.

Case 1. $i_0 \notin T_j$ or $\{i_0, j_0\} \subseteq T_j$.

Then $\tilde{z}(T_j) \leq z(T_j)$. It follows, from $z \in \text{Core}(\Gamma)$, that $\tilde{z}(T_j) \leq z(T_j) \leq k\omega_j$.

Case 2. $i_0 \in T_j$ and $j_0 \notin T_j$.

In this case, the cost allocated to T_j at \tilde{z} is larger than the one allocated at z . We consider the following two subcases.

Subcase 2.1. $j_0 \in N[j]$.

Then $T_j \cup \{j_0\} \in \mathcal{T}_j$ and

$$\tilde{z}(T_j) \leq \tilde{z}(T_j) + \tilde{z}_{j_0} = \tilde{z}(T_j \cup \{j_0\}) = z(T_j \cup \{j_0\}) \leq k\omega_j.$$

The second equality $\tilde{z}(T_j \cup \{j_0\}) = z(T_j \cup \{j_0\})$ holds because i_0 and j_0 are both in $T_j \cup \{j_0\}$; the last inequality holds because $z \in \text{Core}(\Gamma)$ and $T_j \cup \{j_0\} \in \mathcal{T}_j$.

Subcase 2.2. $j_0 \notin N[j]$.

First we suppose that $j = i_0$. By the definition of j_0 , we have that $z_l \leq 0$ for all $l \in T_j \setminus \{i_0\}$. It follows that $\tilde{z}(T_j) \leq \tilde{z}_{i_0} \leq 0 \leq k\omega_{i_0} = k\omega_j$.

Secondly, we suppose that $j \neq i_0$. Then $T_j \setminus \{i_0\} \in \mathcal{T}_j$ and

$$\tilde{z}(T_j) = z(T_j \setminus \{i_0\}) + \tilde{z}_{i_0} \leq k\omega_j,$$

the last inequality holds because $z \in \text{Core}(\Gamma)$ and $\tilde{z}_{i_0} \leq 0$.

Therefore, we conclude that $\tilde{z} \in \text{Core}(\Gamma)$. Since \tilde{z} has 1 or 2 more zero components than z , which contradicts to our assumption on z . Therefore, there must be a non-negative vector in the $\text{Core}(\Gamma)$, and so $\text{Core}(\tilde{\Gamma}) \neq \emptyset$. \square

EXAMPLE. Let $G = (V, E; \omega)$ be the graph given in Figure 1, and let $\omega = (1, 1, 1, 1)$.

(1) When $k = 3m$ for some positive integer m , it holds that $z = (m, m, m, m)$ is an optimal solution to the corresponding (LP) given in (4.4). It follows, from Theorem 4.3, that both $\{k\}$ -domination game and monotonic $\{k\}$ -domination game corresponding to G are balanced.

(2) When $k \neq 3m$ for any positive integer m , it is easy to verify that $z = (\frac{k}{3}, \frac{k}{3}, \frac{k}{3}, \frac{k}{3})$ is a fractional optimal solution to the corresponding (LP), and the optimal objective value is $4k/3$. So the (LP) has no integer optimal solution, it follows that the cores of both games are empty.

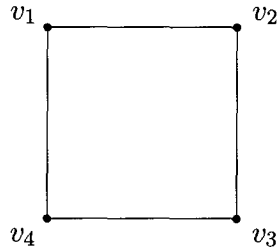


FIGURE 1. Graph G

From Theorems 4.2 and 4.3, both games are balanced when the linear programming (4.4) has an integer optimal solution, and we can obtain a core element for both games from an optimal solution to its dual programming given in (4.5). That is, a core element can be found in polynomial time when core is non-empty. However, the problem of testing whether the linear programming has an integer optimal solution is difficult in general. For $k = 1$, it was shown to be \mathcal{NP} -hard to determine the minimum weight of dominating set (see [5]). We conjecture that for any positive integer k , the problem of testing whether the linear programming (LP) given in (4.4) has an integer optimal solution. That is, testing the balancedness for both $\{k\}$ -domination games is \mathcal{NP} -hard solution.

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