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THE BOUNDEDNESS OF SOME BILINEAR SINGULAR INTEGRAL OPERATORS ON BESOV SPACES

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ABSTRACT. In this paper we weaken the kernel conditions of bilinear Calderón-Zygmund operators and prove boundedness on Besov spaces.

1. Introduction

Related to multilinear integral operators, some earlier works can be found in R. Coifman and Y. Meyer's papers ([1]-[3]). Later in 90's, some nice works about multilinear singular integral operators have been done by M. Lacey and C. Thiele about the L^p boundedness of bilinear Hilbert transform ([13]-[14]), L. Grafakos and R. H. Torres about the boundedness of multilinear Calderón-Zygmund operators ([6]-[9]). In 1984, G. David and J. L. Journé gave the T(1) theorem which was a general law to verify the L^2 boundedness for classical C-Z singular integral operators ([4]). L. Grafakos and R. H. Torres proved the T(1) theorem in [7] for the multilinear C-Z singular integral operators. In [6], L. Grafakos and R. H. Torres have given several open questions about the boundedness of multilinear Calderón-Zygmund operators. One may consider whether we can weaken the kernel conditions of multilinear Calderón-Zygmund operators and also obtain the $L^p(1 < p < \infty)$ boundedness of multilinear Calderón-Zygmund operators. In this paper we try to give some results about it. For convenience the bilinear operators are considered only. Before we give our main results, let's recall some notations about bilinear Calderón-Zygmund operators first.

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DEFINITION 1.1. Consider a bilinear operator $T : \mathcal{D}(\mathbb{R}^d) \times \mathcal{D}(\mathbb{R}^d) \to \mathcal{D}'(\mathbb{R}^d)$, for $f, g \in \mathcal{D}(\mathbb{R}^d), x \notin \operatorname{supp} f \cap \operatorname{supp} g$,

$$T(f,g)(x) = \int_{\mathbb{R}^d \times \mathbb{R}^d} K(x,y,z) f(y) g(z) dy dz, \tag{1.1}$$

where $K(x,y,z)$ is a distributional kernel on $(\mathbb{R}^d)^3$, suppose that it satisfies C-Z kernel conditions, if

$$|K(y_0,y_1,y_2)| \le \frac{A}{\left(\sum\limits_{k,l=0}^{2} |y_k - y_l|\right)^{2d}}, \tag{1.2}$$

$$\begin{aligned} &|K(y_0,y_1,y_2) - K(y_0',y_1,y_2)| + |K(y_1,y_0,y_2) - K(y_1,y_0',y_2)| \\ &+|K(y_2,y_1,y_0) - K(y_2,y_1,y_0')| \le \frac{A|y_0 - y_0'|^\epsilon}{\left(\sum\limits_{k,l=0}^{2} |y_k - y_l|\right)^{2d+\epsilon}}, \end{aligned} \tag{1.3}$$

$$\text{for} \quad |y_0 - y_0'| \le \frac{1}{2} \max_{0 \le j \le 2} |y_j - y_k|,$$

where $0 < \epsilon \le 1$.

Next some kind of Besov spaces is introduced ([12]).

DEFINITION 1.2. A test function space $\mathcal{C}(\mathbb{R}^d)$ is defined as the set of functions f which satisfy the following conditions

(1) $|f(x)| \le C\frac{r^\gamma}{(r+|x|)^{d+\gamma}}$, $r > 0$, $0 < \gamma \le 1$;

(2) $|f(x) - f(y)| \le C\frac{|x-y|}{r+|x|}\frac{r^\gamma}{(r+|x|)^{d+\gamma}}$ for $|x-y| \le \frac{1}{2}(r+|x|)$.

(3) The norm of f can be defined as $\|f\|_{\mathcal{C}(\mathbb{R}^d)} = \inf\{C : (1) \text{ and } (2) \text{ hold}\}$.

DEFINITION 1.3. A sequence $\{\Phi_k\}_{k\ge 0}$ of linear operators is said to be an approximation to the identity if $\phi_k(x,y)$, the kernel of Φ_k, satisfies the following conditions

(1) $|\phi_k(x,y)| \le C\frac{2^{-k}}{(2^{-k}+|x-y|)^{d+1}}$,

(2) $|\phi_k(x,y) - \phi_k(x',y)| \le C\left(\frac{|x-x'|}{2^{-k}+|x-y|}\right)\frac{2^{-k}}{(2^{-k}+|x-y|)^{d+1}}$,

(3) $|\phi_k(y,x) - \phi_k(y,x')| \le C\left(\frac{|x-x'|}{2^{-k}+|x-y|}\right)\frac{2^{-k}}{(2^{-k}+|x-y|)^{d+1}}$ for $|x-x'| \le \frac{1}{2}(2^{-k}+|x-y|)$,

(4) $|[\phi_k(x,y) - \phi_k(x,y')] - [\phi_k(x',y) - \phi_k(x',y')]| \le C\left(\frac{|x-x'|}{2^{-k}+|x-y|}\right)^{\epsilon'}$

$$\times\left(\frac{|y-y'|}{2^{-k}+|x-y|}\right)^{\epsilon'}\frac{2^{-k(1-\epsilon')}}{(2^{-k}+|x-y|)^{d+1-\epsilon'}} \text{ for } |x-x'|\le\tfrac{1}{2}(2^{-k}+|x-y|)$$

and $|y-y'|\le\frac{1}{2}(2^{-k}+|x-y|)$,

(5) $\int_{\mathbb{R}^d}\phi_k(x,y)dx=\int_{\mathbb{R}^d}\phi_k(x,y)dy=1$,

where $0<\epsilon'<1$.

DEFINITION 1.4. Let $\{\Phi_k\}_{k\ge 0}$ be an approximation to identity , $\Psi_k=\Phi_k-\Phi_{k-1}(k>0)$ and $\Psi_0=\Phi_0$, $s\in(-1,1)$ and $1\le p,q\le\infty$. Then (1.4)

$$B_p^{s,q}(\mathbb{R}^d)=\left\{f\in C':\|f\|_{B_p^{s,q}(\mathbb{R}^d)}\sim\left(\sum_{k\ge 0}2^{ksq}\|\Psi_k(f)\|^q_{L^p(\mathbb{R}^d)}\right)^{1/q}<\infty\right\}.$$

DEFINITION 1.5. Let a bilinear operator $T:\mathcal{D}(\mathbb{R}^d)\times\mathcal{D}(\mathbb{R}^d)\to\mathcal{D}'(\mathbb{R}^d)$ satisfy (1.1) with distributional kernel $K(x,y,z)$, and we say that $K(x,y,z)$ is a weakened C-Z kernel if it satisfies the following conditions

$$\begin{aligned}\sup_{r,r'>0}\Bigg\{&\int_{\{y_1:r\le|y_0-y_1|\le 2r\}}\int_{\{y_2:r'\le|y_0-y_2|\le 2r'\}}\Big(|K(y_0,y_1,y_2)|\\&+|K(y_1,y_0,y_2)|+|K(y_2,y_1,y_1)|\Big)dy_1dy_2\\&+\int_{\{y_1:|y_0-y_1|\ge 2r\}}\int_{\{y_2:|y_0-y_2|\le 2r\}}\Big(|K(y_0,y_1,y_2)|\\&+|K(y_1,y_0,y_2)|+|K(y_2,y_1,y_1)|\Big)dy_1dy_2\Bigg\}<\infty,\end{aligned}\tag{1.5}$$

$$\begin{aligned}\sup_{y_0'}\sup_{\substack{r>0\\|y_0-y_0'|\le r}}&\int_{\{y_2:|y_2-y_0|\ge 2^jr\}}\int_{\{y_1:|y_1-y_0|\ge 2^{j+k}r\}}\Big(|K(y_0,y_1,y_2)\\&-K(y_0',y_1,y_2)|+|K(y_1,y_0,y_2)-K(y_1,y_0',y_2)|\\&+|K(y_2,y_1,y_0)-K(y_2,y_1,y_0')|\Big)dy_1dy_2=\gamma(j,k),\end{aligned}\tag{1.6}$$

$$\begin{aligned}\sup_{y_0'}\sup_{\substack{r>0\\|y_0-y_0'|<r}}\Bigg\{&\int_{\{y_2:|y_2-y_0|\le 2^{j+k}r\}}\int_{\{y_1:|y_1-y_0|\ge 2^{j+k}r\}}\Big(|K(y_0,y_1,y_2)\\&-K(y_0',y_1,y_2)|+|K(y_1,y_0,y_2)-K(y_1,y_0',y_2)|\\&+|K(y_2,y_1,y_0)-K(y_2,y_1,y_0')|\Big)dy_1dy_2\end{aligned}\tag{1.7}$$

$$+ \int_{\{y_0:|y_0-y_0'|\le r\}} \int_{\{y_1:|y_1-y_0|\ge 2^{j+k}r\}} \Big(|K(y_0,y_1,y_2) - K(y_0',y_1,y_2)|$$
$$+ |K(y_1,y_0,y_2) - K(y_1,y_0',y_2)| + |K(y_2,y_1,y_0)$$
$$- K(y_2,y_1,y_0')| \Big) dy_1 dy_0$$
$$+ \int_{\{y_0:|y_0-y_0'|\le r\}} \int_{\{y_2:|y_2-y_0|\ge 2^{j+k}r\}} \Big(|K(y_0,y_1,y_2)$$
$$- K(y_0',y_1,y_2)| + |K(y_1,y_0,y_2) - K(y_1,y_0',y_2)| + |K(y_2,y_1,y_0)$$
$$- K(y_2,y_1,y_0')| \Big) dy_2 dy_0 \Big\} = \gamma_1(j,k),$$

$$\sum_{j,k=1}^{\infty} (\gamma(j,k) + \gamma_1(j,k)) + \sum_{j=1}^{\infty} (\gamma(0,j) + \gamma_1(0,j)) < +\infty. \tag{1.8}$$

REMARK 1.1. The ideas of above definitions of the weakened kernel come from S. Hofmann and Y. S. Han [11]. Obviously the conditions are weaker than the standard bilinear C-Z kernel (Definition 1.1).

DEFINITION 1.6. Let $T : \mathcal{D}(\mathbb{R}^d) \times \mathcal{D}(\mathbb{R}^d) \to \mathcal{D}'(\mathbb{R}^d)$, and we say that T satisfies bilinear weak boundedness property(BWBP), if for any $f, g, h \in \mathcal{D}(\mathbb{R}^d)$, $\operatorname{supp} f, g, h \in B$ which is any ball in $\mathbb{R}^d$ with radius R ,

$$|\langle T(f,g), h\rangle| \le CR^d \|f\|_\infty \|g\|_\infty \|h\|_\infty. \tag{1.9}$$

The purpose of the paper is to prove the following theorem

THEOREM 1.1. *Let a bilinear operator $T : \mathcal{D}(\mathbb{R}^d) \times \mathcal{D}(\mathbb{R}^d) \to \mathcal{D}'(\mathbb{R}^d)$ satisfy* (1.1) *with distributional kernel $K(x,y,z)$ satisfying weakened C-Z kernel conditions. Suppose that for any $f, g, h \in \mathcal{D}_0$, $\langle T(1,g),h\rangle = \langle T(f,1),h\rangle = \langle T(f,g),1\rangle = \langle T(1,1),h\rangle = \langle T(1,g),1\rangle = \langle T(f,1),\ 1\rangle = 0$ and T also satisfies bilinear weak boundedness property. Then T can be extended to be a bounded operator from $B^{0,q_1}_{p_1}(\mathbb{R}^d) \times B^{0,q_2}_{p_2}(\mathbb{R}^d)$ into $B^{0,q}_{p}(\mathbb{R}^d)$, where $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}$, $\frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q}(1 < p, q < \infty)$.*

REMARK 1.2. In the above theorem, $f, g, h \in \mathcal{D}_0$ means that $f, g, h \in \mathcal{D}$ and $\int f(x)dx = \int g(x)dx = \int h(x)dx = 0$, by using bilinear weak boundedness property and conditions (1.6), (1.7), and (1.8), it's easy to obtain that $\langle T(1,g),h\rangle = \langle T(f,1),h\rangle = \langle T(f,g),1\rangle = \langle T(1,1),\ h\rangle = \langle T(1,g),1\rangle = \langle T(f,1),1\rangle = 0$ converges.

Through out the paper, the constant C is not essential and maybe different somewhere.

2. Proof of the main theorem

In this paper, one of the most important things is to choose Calderón representation formula, here we take one inhomogeneous version and this following fact can be found in [10], [12].

LEMMA 2.1. (Inhomogeneous Calderón formula) *Suppose that* $\{\Phi_k\}_{k\geq 0}$ *is an approximation to identity,* $\Psi_k = \Phi_k - \Phi_{k-1} (k > 0)$ *and* $\Psi_0 = \Phi_0$. *Set*

$$D_N(f) = \sum_{0\leq k\leq N} \Psi_k \Psi_k^N(f) + \sum_{k>N} \Psi_k \Psi_k^N(f), \tag{2.1}$$

where $\Psi_k^N = \sum_{|j|\leq N} \Psi_{k+j}$. *Then there exist a fixed large enough integer* $N > 0$ *such that for* $f \in B_p^{0,q}(\mathbb{R}^d)(1 < p,q < \infty)$, $\|D_N(f)\|_{B_p^{0,q}} \leq C_N \|f\|_{B_p^{0,q}} (C_N < 1)$. D_N *has its reverse operator* $(D_N)^{-1}$ *and* $\|(D_N)^{-1}(f)\|_{B_p^{0,q}} \leq C\|f\|_{B_p^{0,q}}$.

In the proof of the main theorem we choose $\phi_k(x-y) = 2^{kd}\phi(2^k(x-y))$ to be the kernel of Φ_k, and $\phi \in \mathcal{D}(\mathbb{R}^d)$. With Lemma 2.1, actually we can use following formula as a substitute for the inhomogeneous Calderón formula

$$D_N(f) = \sum_{k=0}^{\infty} \Psi_k \Psi_k^N(f) \text{ for } f \in B_p^{0,q}(\mathbb{R}^d).$$

Using the Calderón formula, for any $f, g, h \in \mathcal{D}$, one has

$$\begin{aligned} &\langle T(D_N(f), D_N(g)), D_N(h)\rangle \\ = &\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \langle \Psi_l T(\Psi_j, \Psi_k)(\Psi_j^N(f), \Psi_k^N(g)), \Psi_l^N(h)\rangle, \end{aligned} \tag{2.2}$$

where we set the kernel of $\Psi_l T(\Psi_j, \Psi_k)$ to be

$$\begin{aligned} &\Psi_l T(\Psi_j, \Psi_k)(y_1, y_2, y_3) \\ = &\int\!\!\int\!\!\int \psi_l(x_1 - y_1) K(x_1, x_2, x_3) \psi_j(x_2 - y_2) \psi_k(x_3 - y_3) dx_1 dx_2 dx_3. \end{aligned}$$

We'll prove the following lemma.

LEMMA 2.2. *Let T be defined as in Theorem 1.1. Then for $l \le j \le k$, $j = l + m$, $k = j + n$, there exists $C > 0$ such that*

$$\int\int (|\Psi_l T(\Psi_j, \Psi_k)(y_1, y_2, y_3)|dy_1dy_2 + |\Psi_l T(\Psi_j, \Psi_k)(y_1, y_2, y_3)|dy_2dy_3 + |\Psi_l T(\Psi_j, \Psi_k)(y_1, y_2, y_3)|dy_1dy_3) < C\tilde{\gamma}(m, n),$$

where $\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \tilde{\gamma}(m, n) < +\infty$ *and* $\tilde{\gamma}(0, 0) = 1$.

Proof. Note that $\psi_j(\cdot)$ is the kernel of operator $\Psi_j (j \ge 0)$. Without loss of generality, we set $\operatorname{supp} \psi_0(x) \subset [0, 1]$.

i) The first case: $l, m, n > 0$. We will prove

$$\int\int |\Psi_l T(\Psi_j, \Psi_k)(y_1, y_2, y_3)|dy_1dy_2 < C\tilde{\gamma}(m, n).$$

Observe

$$\begin{aligned}
&\int\int |\Psi_l T(\Psi_j, \Psi_k)(y_1, y_2, y_3)|dy_1dy_2 \\
&\le \Bigg(\int_{\{y_1:|y_3-y_1|\ge 4\cdot 2^{-l}\}} \int_{\{y_2:|y_3-y_2|\ge 4\cdot 2^{-(l+m)}\}} \\
&\quad + \int_{\{y_1:|y_3-y_1|\ge 4\cdot 2^{-l}\}} \int_{\{y_2:|y_3-y_2|\le 4\cdot 2^{-(l+m)}\}} \\
&\quad + \int_{\{y_1:|y_3-y_1|\le 4\cdot 2^{-l}\}} \int_{\{y_2:|y_3-y_2|\ge 4\cdot 2^{-(l+m)}\}} \\
&\quad + \int_{\{y_1:|y_3-y_1|\le 4\cdot 2^{-l}\}} \int_{\{y_2:|y_3-y_2|\le 4\cdot 2^{-(l+m)}\}} \Bigg) \\
&\qquad |\Psi_l T(\Psi_j, \Psi_k)(y_1, y_2, y_3)|dy_1dy_2 = I + II + III + IV.
\end{aligned}$$

Firstly we deal with I, using condition (1.6) and (1.8),

$$\begin{aligned}
I = &\int_{\{y_1:|y_3-y_1|\ge 4\cdot 2^{-l}\}} \int_{\{y_2:|y_3-y_2|\ge 4\cdot 2^{-(l+m)}\}} \\
&\Bigg| \int\int\int (K(x_1, x_2, x_3) - K(x_1, x_2, y_3)) \\
&\psi_l(x_1 - y_1)\psi_j(x_2 - y_2)\psi_k(x_3 - y_3)dx_1dx_2dx_3 \Bigg| dy_1dy_2 \le \gamma(m, n).
\end{aligned}$$

To deal with II, from (1.7) and (1.8) we can deduce

$$II = \int_{\{y_1:|y_3-y_1|\geq 4\cdot 2^{-l}\}} \int_{\{y_2:|y_3-y_2|\leq 4\cdot 2^{-(l+m)}\}} \left| \int\int\int (K(x_1,x_2,x_3) - K(x_1,x_2,y_3)) \right.$$
$$\left. \psi_l(x_1-y_1)\psi_j(x_2-y_2)\psi_k(x_3-y_3)dx_1dx_2dx_3 \right| dy_1dy_2 \leq \gamma_1(m,n).$$

To deal with III, we have

$$III = \Bigg(\int_{\{y_1:|y_3-y_1|\leq 4\cdot 2^{-l}\}} \int_{\{y_2:|y_3-y_2|\geq 4\cdot 2^{-l}\}}$$
$$+ \int_{\{y_1:|y_3-y_1|\leq 4\cdot 2^{-l}\}} \int_{\{y_2:4\cdot 2^{-(l+m)}\leq|y_3-y_2|\leq 4\cdot 2^{-l}\}} \Bigg) \cdots = III_1 + III_2.$$

Using (1.7) and (1.8) directly, we can obtain $III_1 \leq \gamma_1(m,n)$. Because for any $f,g,h \in \mathcal{D}_0$, $\langle T(1,g),h\rangle = \langle T(f,1),h\rangle = \langle T(f,g),1\rangle = 0$, we set $\chi(x) = 1$ if $|x| \leq 1$ and $\chi(x) = 0$ if $|x| > 1$, then

$$III_2 = \int_{\{y_1:|y_3-y_1|\leq 4\cdot 2^{-l}\}} \int_{\{y_2:4\cdot 2^{-(l+m)}\leq|y_3-y_2|\leq 4\cdot 2^{-l}\}} \left| \int\int\int (K(x_1,x_2,x_3) - K(x_1,x_2,y_3))(\psi_l(x_1-y_1) \right.$$
$$- \psi_l(y_3-y_1))[\chi(2^{l+m+n}|x_1-y_3|) + (1-\chi(2^{l+m+n}|x_1-y_3|)]$$
$$\left. \psi_j(x_2-y_2)\psi_k(x_3-y_3)dx_1dx_2dx_3 \right| dy_1dy_2 = III_{2,1} + III_{2,2}.$$

Using $|\psi_l(x_1-y_1) - \psi_l(y_3-y_1)| \leq 2^{l(d+1)}|x_1-y_3|$, we can obtain $III_{2,1} \leq 2^{-(m+n)}\gamma_1(0,n)$ by (1.7), (1.8) directly. To deal with $III_{2,2}$, note that $2^{-(l+m+n)} \leq |x_1-y_3| \leq C2^{-l}$, ($C$ large enough), from (1.7) and (1.8), we can obtain

$$III_{2,2} \leq \gamma(m,n) + \sum_{i=1}^{m+n} 2^{i-(m+n)}\gamma(i,n).$$

Next

$$
\begin{aligned}
IV \le & \Big(\int\limits_{\{y_1:|y_1-y_3|\le 4\cdot 2^{-(l+m+n)}\}} \int\limits_{\{y_2:|y_2-y_3|\le 4\cdot 2^{-(l+m)}\}} \\
& + \int\limits_{\{y_1:4\cdot 2^{-(l+m+n)}\le |y_1-y_3|\le 4\cdot 2^{-l}\}} \int\limits_{\{y_2:|y_2-y_3|\le 4\cdot 2^{-(l+m)}\}} \Big) \cdots \\
= & IV_1 + IV_2.
\end{aligned}
$$

Then to deal with IV_1, set $\eta(x) \in \mathcal{D}$ and if $|x| \le 4$, $\eta(x) = 1$; if $|x| \ge 6$, $\eta(x) = 0$, we have

$$
\begin{aligned}
IV_1 = & \int\int \Big| \int\int\int K(x_1,x_2,x_3)(\psi_l(x_1-y_1) - \psi_l(y_3-y_1)) \\
& [\eta(2^{(l+m+n)}|x_1-y_3|) + (1-\eta(2^{(l+m+n)}|x_1-y_3|))]\psi_j(x_2-y_2) \\
& \psi_k(x_3-y_3)dx_1dx_2dx_3 \Big| dy_1dy_2 = IV_{1,1} + IV_{1,2}.
\end{aligned}
$$

Notice that $|\psi_l(x_1-y_1) - \psi_l(y_3-y_1)| < 2^{ld}2^l|x_1-y_3|$, then using BWBP property to $IV_{1,1}$, we have

$$
IV_{1,1} \le C2^{-(l+m)d} \cdot 2^{-(l+m+n)d} \cdot 2^{ld}2^{-(m+n)} \cdot 2^{(l+m+n)d} \le C2^{-m(d+1)} \cdot 2^{-n}.
$$

To deal with $IV_{1,2}$, we have

$$
\begin{aligned}
& IV_{1,2} \\
\le & \int\limits_{\{y_1:|y_1-y_3|\le 4\cdot 2^{-(l+m+n)}\}} \int\limits_{\{y_2:|y_2-y_3|\le 4\cdot 2^{-(l+m)}\}} \\
& \Big| \Big(\int\limits_{\{x_1:6\cdot 2^{-(l+m+n)}|x_1-y_3|\le 4\cdot 2^{-(l+m)}\}} + \int\limits_{\{x_1:4\cdot 2^{-(l+m)}|x_1-y_3|\le 4\cdot 2^{-l}\}} \Big) \\
& \int\int (K(x_1,x_2,x_3) - K(x_1,x_2,x_3))(\psi_l(x_1-y_1) - \psi_l(y_3-y_1)) \\
& (1-\eta(2^{(l+m+n)}|x_1-y_3|))\psi_j(x_2-y_2)\psi_k(x_3-y_3)dx_1dx_2dx_3 \Big| dy_1dy_2 \\
\le & C(2^{-m}\sum_{i=1}^{n} 2^{i-n}\gamma(i,0) + \sum_{i=1}^{m} 2^{i-m}\gamma_1(i,n)).
\end{aligned}
$$

Using $\langle T(1,1),h\rangle=0$ for $h\in\mathcal{D}_0$, then

$$\begin{aligned}
IV_2 \leq& \int\int\Big|\int\int\int K(x_1,x_2,x_3)(\psi_l(x_1-y_1)-\psi_l(y_3-y_1))\\
&[\eta(2^{l+m+n}|x_1-y_3|)+(1-\eta(2^{l+m+n}|x_1-y_3|))]\\
&(\psi_j(x_2-y_2)-\psi_j(y_3-y_2))\\
&[\eta(2^{l+m+n}|x_2-y_3|)+(1-\eta(2^{l+m+n}|x_2-y_3|))]\\
&\psi_j(x_3-y_3)dx_1dx_2dx_3\Big|dy_1dy_2\\
\leq& \int\int\Big|\int\int\int K(x_1,x_2,x_3)(\psi_l(x_1-y_1)-\psi_l(y_3-y_1))\cdot\\
&(\psi_j(x_2-y_2)-\psi_j(y_3-y_2))\{\eta(2^{l+m+n}|x_1-y_3|)\\
&\times\eta(2^{l+m+n}|x_2-y_3|)+\eta(2^{l+m+n}|x_1-y_3|)\\
&\times(1-\eta(2^{l+m+n}|x_2-y_3|))\\
&+(1-\eta(2^{l+m+n}|x_1-y_3|))\times\eta(2^{l+m+n}|x_2-y_3|)\\
&+(1-\eta(2^{l+m+n}|x_1-y_3|))\\
&\times(1-\eta(2^{l+m+n}|x_2-y_3|))\}\psi_j(x_3-y_3)dx_1dx_2dx_3\Big|dy_1dy_2\\
\leq& IV_{2,1}+IV_{2,2}+IV_{2,3}+IV_{2,4}.
\end{aligned}$$

Next using BWBP property and regular conditions of ψ_l and ψ_j, it's easy to obtain $IV_{2,1}\leq 2^{-2n-m}$. To deal with $IV_{2,2}$, we have

$$\begin{aligned}
IV_{2,2}=& \int\int\Big|\int\int\int(K(x_1,x_2,x_3)-K(x_1,x_2,y_3))(\psi_l(x_1-y_1)\\
&-\psi_l(y_3-y_1))(\psi_j(x_2-y_2)-\psi_j(y_3-y_2))\eta(2^{l+m+n}|x_1-y_3|)\\
&\times(1-\eta(2^{l+m+n}|x_2-y_3|))\psi_j(x_3-y_3)dx_1dx_2dx_3\Big|dy_1dy_2\\
\leq& C2^{-m-n}.
\end{aligned}$$

Then

$$\begin{aligned}
IV_{2,3}=& \int\int\Big|\int\int\int(K(x_1,x_2,x_3)-K(x_1,x_2,y_3))(\psi_l(x_1-y_1)\\
&-\psi_l(y_3-y_1))(\psi_j(x_2-y_2)-\psi_j(y_3-y_2))\\
&(1-\eta(2^{l+m+n}|x_1-y_3|))
\end{aligned}$$

$$\times\, \eta(2^{l+m+n}|x_2-y_3|)\psi_j(x_3-y_3)dx_1dx_2dx_3\Big|dy_1dy_2$$

$$\leq \int\int\Big(\Big|\int_{\{x_1:6\cdot 2^{-(l+m+n)}\leq|x_1-y_3|\leq 4\cdot 2^{-(l+m)}\}}\int\int\Big|$$

$$+\int_{\{x_1:4\cdot 2^{-(l+m)}\leq|x_1-y_3|\leq 4\cdot 2^{-l}\}}\int\int$$

$$+\int_{\{x_1:|x_1-y_3|\geq 4\cdot 2^{-l}\}}\int\int\cdots\Big|\Big)dy_1dy_2$$

$$\leq C(2^{-m}\sum_{i=1}^{n}2^{i-n}\gamma_1(0,i)+2^{-n}\sum_{i=1}^{m}2^{i-m}\gamma_1(0,i)+\gamma_1(m,n)).$$

Finally to $IV_{2,4}$, we have

$$IV_{2,4}=\int\int\Big|\int\int\int(K(x_1,x_2,x_3)-K(x_1,x_2,y_3))(\psi_l(x_1-y_1)$$

$$-\,\psi_l(y_3-y_1))(\psi_j(x_2-y_2)-\psi_j(y_3-y_2))$$

$$(1-\eta(2^{l+m+n}|x_1-y_3|))$$

$$\times\,(1-\eta(2^{l+m+n}|x_2-y_3|))\psi_j(x_3-y_3)dx_1dx_2dx_3\Big|dy_1dy_2$$

$$\leq C\int\int\Big|\int_{\{x_1:|x_1-y_3|\geq 4\cdot 2^{-l}\}}\int\int+\int\int_{\{x_2:|x_2-y_3|\geq 4\cdot 2^{-l}\}}\int$$

$$+\int_{\{x_1:6\cdot 2^{-(l+m+n)}\leq|x_1-y_3|\leq 4\cdot 2^{-l}\}}\int_{\{x_2:6\cdot 2^{-(l+m+n)}\leq|x_2-y_3|\leq 4\cdot 2^{-(l+m)}\}}\int$$

$$+\int_{\{x_1:6\cdot 2^{-(l+m+n)}\leq|x_1-y_3|\leq 4\cdot 2^{-l}\}}\int_{\{x_2:4\cdot 2^{-(l+m)}\leq|x_2-y_3|\leq 4\cdot 2^{-l}\}}\int\Big|\cdots$$

$$\leq IV_{2,4,1}+IV_{2,4,2}+IV_{2,4,3}+IV_{2,4,4}.$$

Using (1.7) and (1.8), it's easy to obtain $IV_{2,4,1}+IV_{2,4,2}\leq C\gamma_1(m,n)$. Next we have

$$IV_{2,4,3}\leq\int\int\Big|\int_{\{x_1:6\cdot 2^{-(l+m+n)}\leq|x_1-y_3|\leq 4\cdot 2^{-(l+m)}\}}$$

$$\int_{\{x_2:6\cdot 2^{-(l+m+n)}\le|x_2-y_3|\le 4\cdot 2^{-(l+m)}\}}\int$$

$$+\int_{\{x_1:4\cdot 2^{-(l+m)}\le|x_1-y_3|\le 4\cdot 2^{-l}\}}\int_{\{x_2:6\cdot 2^{-(l+m+n)}\le|x_2-y_3|\le 4\cdot 2^{-(l+m)}\}}\int\Big|$$

$\dots$

$$\le C(2^{-m}\sum_{i=1}^{n}2^{i-n}\gamma(i,0)+2^{-n}\sum_{i=1}^{m}2^{i-m}\gamma(i,n)).$$

We can obtain $IV_{2,4,4}\le C(2^{-m}\sum\limits_{i=1}^{n}2^{i-n}\gamma(i,0)+2^{-n}\sum\limits_{i=1}^{m}2^{i-m}\gamma(i,n))$ more simply.

ii) We will prove that for $l=j=k=0$,

$$\int\int|\Psi_0T(\Psi_0,\Psi_0)(y_1,y_2,y_3)|dy_1dy_2<C.$$

We write

$$\int\int|\Psi_0T(\Psi_0,\Psi_0)(y_1,y_2,y_3)|dy_1dy_2$$

$$\le\int_{\{y_1:|y_1-y_3|\le 8C_1\}}\int_{\{y_2:|y_2-y_3|\le 8C_1\}}|\Psi_0T(\Psi_0,\Psi_0)(y_1,y_2,y_3)|dy_1dy_2$$

$$+\int_{\{y_1:|y_1-y_3|\ge 8C_1\}}\int_{\{y_2:|y_2-y_3|\le 8C_1\}}|\Psi_0T(\Psi_0,\Psi_0)(y_1,y_2,y_3)|dy_1dy_2$$

$$+\int_{\{y_1:|y_1-y_3|\ge 8C_1\}}\int_{\{y_2:|y_2-y_3|\le 8C_1\}}|\Psi_0T(\Psi_0,\Psi_0)(y_1,y_2,y_3)|dy_1dy_2$$

$$+\int_{\{y_1:|y_1-y_3|\ge 8C_1\}}\int_{\{y_2:|y_2-y_3|\ge 8C_1\}}|\Psi_0T(\Psi_0,\Psi_0)(y_1,y_2,y_3)|dy_1dy_2$$

$$=A_1+A_2+A_3+A_4,$$

where C_1 connected with N. To deal with A_1, using BWBP property, it's easy to obtain $A_1\le C$. A_2 and A_3 are similar, we only need to deal with A_2. Using (1.5), for fixed $z\in\operatorname{supp}h$ and $x\in\operatorname{supp}f$, we have

$$A_2=\int_{\{y_1:|y_1-y_3|\ge 8C_1\wedge|y_1-z|\le 8C_1\}}\int_{\{y_2:|y_2-y_3|\le 8C_1\wedge|y_2-x|\le 8C_1\}}$$

$$\left|\int\int\int K(x_1,x_2,x_3)\psi_0(x_1-y_1)\psi_0(x_2-y_2)\right.$$
$$\left.\psi_0(x_3-y_3)dx_1dx_2dx_3\right|dy_1dy_2$$
$$\leq C\int_{\{y_1:|y_1-y_3|\geq 8C_1\wedge|y_1-z|\leq 8C_1\}}\int_{\{y_2:|y_2-y_3|\leq 8C_1\wedge|y_2-x|\leq 8C_1\}}\int_{\{x_3:|x_3-y_3|\leq 4\}}$$
$$\left(\int_{\{x_1:|x_1-x_3|\geq 4\}}\int_{\{x_1:|x_2-x_3|\leq 4\}}|K(x_1,x_2,x_3)|dx_1dx_2\right)dx_3\leq C.$$

Also using (1.5) directly, we can prove $A_4\leq C$.

iii) Thirdly the cases $0<l=j=k$, $0<l=j<k$ and $0=l\leq j\leq k$ compound the two previous cases, which should make some minor modifications, here we omit its proof. From the three cases, we have actually proved that for $0\leq l\leq j\leq k$, $j=l+m$, $k=l+m+n$,

$$\int\int|\Psi_lT(\Psi_j,\Psi_k)(y_1,y_2,y_3)|dy_1dy_2<C\tilde{\gamma}(m,n),$$

where $\tilde{\gamma}(m,n)$ have relations with $\gamma(m,n)$ and $\gamma_1(m,n)$ satisfying

$$\sum_{m,n=0}^{\infty}\tilde{\gamma}(m,n)<\infty.$$

iv) To prove that

$$\int\int|\Psi_lT(\Psi_j,\Psi_k)(y_1,y_2,y_3)|dy_1dy_3<C\tilde{\gamma}(m,n), \tag{2.3}$$

or

$$\int\int|\Psi_lT(\Psi_j,\Psi_k)(y_1,y_2,y_3)|dy_2dy_3<C\tilde{\gamma}(m,n), \tag{2.4}$$

note that for fixed $z\in\operatorname{supp}h,\ y\in\operatorname{supp}g$,

$$\int\int|\Psi_lT(\Psi_j,\Psi_k)(y_1,y_2,y_3)|dy_1dy_3$$
$$=\int_{\{y_1:|y_1-z|\leq C_1\cdot 2^{-l}\}}\int_{\{y_1:|y_1-y|\leq C_1\cdot 2^{-(l+m+n)}\}}$$
$$|\Psi_lT(\Psi_j,\Psi_k)(y_1,y_2,y_3)|dy_1dy_3.$$

Then using the similar method in three previous cases, we also can prove (2.3) and (2.4). This completes the proof. □

PROPOSITION 2.1. *Let $\{\tilde{\Phi}_k\}_{k\geq 0}$ be an approximation to identity, $\tilde{\Psi}_k = \tilde{\Phi}_k - \tilde{\Phi}_{k-1}(k > 0)$, $\tilde{\Psi}_0 = \tilde{\Phi}_0$, and $1 \leq p, q \leq \infty$. Then*

(2.5)

$$B_p^{0,q}(\mathbb{R}^d) = \left\{ f \in \mathcal{C}' : \|f\|_{B_p^{0,q}(\mathbb{R}^d)} \sim \left(\sum_{j\geq 0} \|\tilde{\Psi}_j(f)\|^q_{L^p(\mathbb{R}^d)} \right)^{1/q} < \infty \right\}.$$

This proposition can refer to [5] or [10]. Now we give the proof of the main theorem.

Proof of Theorem 1.1. Using Höder's inequality and Lemma 2.2, for $1/p+1/p' = 1/p'+1/p_1+1/p_2 = 1$ and $1/q+1/q' = 1/q'+1/q_1+1/q_2 = 1$,

$$\begin{aligned}
&|\langle T(D_N(f), D_N(g)), D_N(h)\rangle| \\
&\leq \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} |\langle \Psi_l T(\Psi_j, \Psi_k)(\Psi_j^N(f), \Psi_k^N(g)), \Psi_l^N(h)\rangle| \\
&\leq 6 \sum_{l,m,n} \tilde{\gamma}(m,n) \|\Psi_l^N(h)\|_{p'} \|\Psi_{l+m}^N(f)\|_{p_1} \|\Psi_{l+m+n}^N(g)\|_{p_2} \\
&\leq C\{\sum_{l,m,n} \tilde{\gamma}(m,n) \|\Psi_l^N(h)\|_{p'}^{q'}\}^{1/q'} \\
&\quad \times \{\sum_{l,m,n} \tilde{\gamma}(m,n) \|\Psi_{l+m}^N(f)\|_{p_1}^{q} \|\Psi_{l+m+n}^N(g)\|_{p_2}^{q}\}^{1/q} \\
&\leq CN^{1/q'} \|h\|_{B_{p'}^{0,q'}} \{\sum_{l,m,n} \tilde{\gamma}(m,n) \|\Psi_{l+m}^N(f)\|_{p_1}^{q_1}\}^{1/q_1} \\
&\quad \times \{\sum_{l,m,n} \tilde{\gamma}(m,n) \|\Psi_{l+m+n}^N(g)\|_{p_2}^{q_2}\}^{1/q_2} \\
&\leq CN \|f\|_{B_{p_1}^{0,q_1}} \|g\|_{B_{p_2}^{0,q_2}} \|h\|_{B_{p'}^{0,q'}}.
\end{aligned}$$

With Proposition 2.1 and Lemma 2.1, we finish the proof of the main theorem.

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References

[1] R. Coifman and Y. Meyer,*On commutators of singular integrals and bilinear singular integrals*, Trans. Amer. Math. Soc. **212** (1975), 315–331.

[2] ———, *Commutateurs d'intégrales singulièrs et opérateurs multilinéaires*, Ann. Inst. Fourier. Grenoble. **28** (1978), no. 3, xi, 177-202.

[3] ———, *Au delà des Opérateurs pseudo-différentiels*, Astérisque **57** (1978).

[4] G. David and J. L. Journé, *A boundedness criterion for generalized Calderón-Zygmund operators*, Ann. of Math. **120** (1984), no. 2, 371–397.

[5] M. Frazier and B. Jawerth, *A discrete transform and decompositions of distribution spaces*, J. Func. Anal. **93** (1990), 34–170.

[6] L. Grafakos and R. H. Torres, *On multilinear singular integrals of Caderón-Zygmund type*, Publ. Mat. Vol. **extra** (2002), 57–91.

[7] ———,*Multilinear singular integrals of Caderón-Zygmund theory*, Advances. in Mathematics **165** (2002), 124–164.

[8] ———, *Discrete decompositions for bilinear operators and almost diagonal conditions*, Trans. Amer. Math. Soc. **354** (2002), no. 3, 1153–1176.

[9] ———, *Maximal operators and weighted norm inequalities for multilinear singular integrals*, Indiana. Univ. Math. J. **51** (2002), no. 5, 1261–1276.

[10] Y. S. Han, *Inhomogeneous Calderón reproducing formula on spaces of homogeneous type*, J. Geom. Anal. **7** (1997), no. 2, 259–284.

[11] Y. S. Han and S. Hofmann, *T1 theorems for Besov and Triebel-Lizorkin spaces*, Trans. Amer. Math. Soc. **337** (1993), 839–853.

[12] Y. S. Han, S. Z. Lu, and D. C. Yang, *Inhomogeneous Besov and Triebel-Lizorkin spaces on spaces of homogeneous type*, Approx. Theory. Appl. **15** (1999), no. 3, 37–65.

[13] M. Lacey and C. Thiele, *L^p estimates on the bilinear Hilbert transform, $2 < p < \infty$*, Ann. of Math. **146** (1997), no. 3, 693–724.

[14] ———, *On Calderón's conjecture*, Ann. of Math. **149** (1999), no. 2, 475–496.

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