J. Korean Math. Soc. **43** (2006), No. 2, pp. 271–281

ON THE COMMUTATIVE PRODUCT OF DISTRIBUTIONS

BRIAN FISHER AND KENAN TAŞ

ABSTRACT. The commutative products of the distributions $x^r \ln^p |x|$ and $x^{-r-1} \ln^q |x|$ and of $\operatorname{sgn} x\, x^r \ln^p |x|$ and $\operatorname{sgn} x\, x^{-r-1} \ln^q |x|$ are evaluated for $r = 0, \pm 1, \pm 2, \ldots$ and $p, q = 0, 1, 2, \ldots$.

In the following, we let $\mathcal{D}$ be the space of infinitely differentiable functions with compact support and let $\mathcal{D}'$ be the space of distributions defined on $\mathcal{D}$. The distributions $x_+^{-1} \ln^p x_+$ and $x_-^{-1} \ln^p x_-$ are defined by the equations

$$x_+^{-1} \ln^p x_+ = (p+1)^{-1}(\ln^{p+1} x_+)', \;\; x_-^{-1} \ln^p x_- = -(p+1)^{-1}(\ln^{p+1} x_-)'$$

for $p = 0, 1, 2, \ldots$, see Gel'fand and Shilov [6].

The distributions $x_+^{-r} \ln^p x_+$ and $x_-^{-r} \ln^p x_-$ are then defined inductively by the equations

$$\begin{aligned}(x_+^{-r+1} \ln^p x_+)' &= -(r-1)x_+^{-r} \ln^p x_+ + p x_+^{-r} \ln^{p-1} x_+,\\ (x_-^{-r+1} \ln^p x_-)' &= (r-1)x_-^{-r} \ln^p x_- - p x_-^{-r} \ln^{p-1} x_-\end{aligned}$$

for $r = 1, 2, \ldots$ and $p = 0, 1, 2, \ldots$. Note that this is not the same as Gel'fand and Shilov's definitions.

The distribution $x^{-r} \ln^p |x|$ is then defined by

$$x^{-r} \ln^p |x| = x_+^{-r} \ln^p x_+ + (-1)^r x_-^{-r} \ln^p x_-$$

for $r = 1, 2, \ldots$ and $p = 0, 1, 2, \ldots$, which is in agreement with Gel'fand and Shilov's definition. In particular, it is easily proved that if φ is a function in $\mathcal{D}$ with support contained in the interval $[-1, 1]$, then

$$\langle x^{-1} \ln^p |x|, \varphi(x) \rangle = \int_{-1}^{1} x^{-1} \ln^p |x| [\varphi(x) - \varphi(0)]\, dx \tag{1}$$

for $p = 0, 1, 2, \ldots$.

Received October 19, 2004. Revised October 25, 2005.
2000 Mathematics Subject Classification: 46F10.
Key words and phrases: distribution, delta-function, product of distributions.

Further, the distribution $\operatorname{sgn} x\, x^{-r} \ln^p |x|$ is defined by

$$\operatorname{sgn} x\, x^{-r} \ln^p |x| = x_+^{-r} \ln^p x_+ - (-1)^r x_-^{-r} \ln^p x_-$$

for $r = 0, \pm 1, \pm 2, \dots$ and $p = 0, 1, 2, \dots$.

It follows that

$$(x^{-r} \ln^p |x|)' = -\, r x^{-r-1} \ln^p |x| + p x^{-r-1} \ln^{p-1} |x|,$$

$$(\operatorname{sgn} x\, x^{-r} \ln^p |x|)' = -\, r \operatorname{sgn} x\, x^{-r-1} \ln^p |x| + p \operatorname{sgn} x\, x^{-r-1} \ln^{p-1} |x|$$

for $r = 0, \pm 1, \pm 2, \dots$ and $p = 0, 1, 2, \dots$.

The definition of the product of a distribution and an infinitely differentiable function is the following, see for example Gel'fand and Shilov [6] or Schwartz [7].

DEFINITION 1. Let f be a distribution in $\mathcal{D}'$ and let g be an infinitely differentiable function. The product fg is defined by

$$\langle fg, \varphi \rangle = \langle f, g\varphi \rangle$$

for all functions φ in $\mathcal{D}$.

Schwartz claimed that no suitable generalization of this definition could be defined but Gel'fand and Shilov pointed out that it was possible to define the product of a distribution and a sufficiently continuously differentiable function. More precisely, a first extension of the product of a distribution and an infinitely differentiable function is the following, see for example [2].

DEFINITION 2. Let f and g be distributions in $\mathcal{D}'$ for which on the interval (a, b), f is the k-th derivative of a locally summable function F in $L^p(a, b)$ and $g^{(k)}$ is a locally summable function in $L^q(a, b)$ with $1/p + 1/q = 1$. Then the product $fg = gf$ of f and g is defined on the interval (a, b) by

$$fg = \sum_{i=0}^{k} \binom{k}{i} (-1)^i [F g^{(i)}]^{(k-i)}.$$

It follows easily from Definition 2 that the following products hold:

$$(|x|^\lambda \ln^p |x|)(|x|^\mu \ln^q |x|) = |x|^{\lambda+\mu} \ln^{p+q} |x|, \tag{2}$$

$$(\operatorname{sgn} x |x|^\lambda \ln^p |x|)(|x|^\mu \ln^q |x|) = \operatorname{sgn} x |x|^{\lambda+\mu} \ln^{p+q} |x|, \tag{3}$$

$$(|x|^\lambda \ln^p |x|)(\operatorname{sgn} x |x|^\mu \ln^q |x|) = \operatorname{sgn} x |x|^{\lambda+\mu} \ln^{p+q} |x|, \tag{4}$$

$$(\operatorname{sgn} x |x|^\lambda \ln^p |x|)(\operatorname{sgn} x |x|^\mu \ln^q |x|) = |x|^{\lambda+\mu} \ln^{p+q} |x| \tag{5}$$

for $\lambda + \mu > -1$ and $p, q = 0, 1, 2, \dots$.

Now let $\rho(x)$ be a function in $\mathcal{D}$ having the following properties:
(i) $\rho(x) = 0$ for $|x| \geq 1$,
(ii) $\rho(x) \geq 0$,
(iii) $\rho(x) = \rho(-x)$,
(iv) $\int_{-1}^{1} \rho(x)\, dx = 1$.

Putting $\delta_n(x) = n\rho(nx)$ for $n = 1, 2, \ldots$, it follows that $\{\delta_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the Dirac delta-function $\delta(x)$.

If now f is an arbitrary distribution in $\mathcal{D}'$, we define

$$f_n(x) = (f * \delta_n)(x) = \langle f(t), \delta_n(x-t)\rangle$$

for $n = 1, 2, \ldots$. It follows that $\{f_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the distribution $f(x)$.

The following definition for the commutative product of two distributions was given in [2] and generalizes Definition 2.

DEFINITION 3. Let f and g be distributions in $\mathcal{D}'$ and let $f_n(x) = (f*\delta_n)(x)$ and $g_n(x) = (g*\delta_n)(x)$. We say that the commutative product $f.g$ of f and g exists and is equal to the distribution h on the interval (a, b) if

$$\lim_{n\to\infty} \langle f_n(x)g_n(x), \varphi(x)\rangle = \langle h(x), \varphi(x)\rangle$$

for all functions φ in $\mathcal{D}$ with support contained in the interval $[a, b]$.

A number of results on the commutative product of distributions were obtained in [2], [3], [4] and [5].

It was proved in [2] that if the product fg exists by Definition 2, it exists by Definition 3 and $fg = f.g$.

The following theorem is easily proved.

THEOREM 1. *Let f and g be distributions in $\mathcal{D}'$ and suppose that the commutative products $f.g$ and $f.g'$ (or $f'.g$) exists. Then the commutative product $f'.g$ (or $f.g'$) exists and*

$$(f.g)' = f'.g + f.g'. \tag{6}$$

In [1], Colombeau gave a more general definition for the product of distributions. He considered quotients of the space $\mathcal{E}(\mathcal{D}(\Omega))$ of all infinitely differentiable functions on the space $\mathcal{D}(\Omega)$, where Ω denotes any open subset of the reals. An algebra $\mathcal{G}(\Omega)$ is then defined which contains the space $\mathcal{D}'(\Omega)$ of all distributions on Ω and such that the algebra $\mathcal{E}(\Omega)$ of all infinitely differentiable functions on Ω is a subalgebra of $\mathcal{G}(\Omega)$. The resulting product of two distributions in $\mathcal{D}(\Omega)$ is then an element in $\mathcal{G}(\Omega)$. This definition has the disadvantage that the product of two distributions in $\mathcal{D}(\Omega)$ is not necessarily a distribution in $\mathcal{D}(\Omega)$.

From now on, we use Definition 3 for the product of distributions and prove the following commutative extension of equation (2).

THEOREM 2. *The commutative product* $(x^r \ln^p |x|).(x^{-r-1} \ln^q |x|)$ *exists and*

$$(x^r \ln^p |x|).(x^{-r-1} \ln^q |x|) = x^{-1} \ln^{p+q} |x| \tag{7}$$

for $r = 0, \pm 1, \pm 2, \dots$ *and* $p, q = 0, 1, 2, \dots$.

Proof. We first of all prove equation (7) when $r = 0$. Putting

$$(\ln^p |x|)_n = \ln^p |x| * \delta_n(x) = \int_{-1/n}^{1/n} \ln^p |x - t| \delta_n(t)\, dt,$$

$$(x^{-1} \ln^q |x|)_n = (x^{-1} \ln^q |x|) * \delta_n(x) = (q+1)^{-1} \int_{-1/n}^{1/n} \ln^{q+1} |x - t| \delta_n'(t)\, dt,$$

we have

$$\int_{-1}^{1} (\ln^p |x|)_n (x^{-1} \ln^q |x|)_n \, dx = 0 \tag{8}$$

since the integrand is odd.

Next, if ψ is an arbitrary continuous function, we have

$$\begin{aligned}
&\int_{-1}^{1} x (\ln^p |x|)_n (x^{-1} \ln^q |x|)_n \psi(x)\, dx \\
&= (q+1)^{-1} \int_{-1/n}^{1/n} \int_{-1/n}^{1/n} \int_{-1/n}^{1/n} x \ln^p |x - s| \delta_n(s) \\
&\quad \times \ln^{q+1} |x - t| \delta_n'(t) \psi(x)\, ds\, dt\, dx \\
&\quad + \int_{1/n}^{1} x (\ln^p |x|)_n (x^{-1} \ln^q |x|)_n \psi(x)\, dx \\
&\quad + \int_{-1}^{-1/n} x (\ln^p |x|)_n (x^{-1} \ln^q |x|)_n \psi(x)\, dx \\
&=: I_1 + I_2 + I_3.
\end{aligned} \tag{9}$$

Making the substitutions $ns = u$, $nt = v$ and $nx = w$, we have

$$I_1 = (q+1)^{-1}n^{-1}\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} w\ln^p|(w-u)/n|\rho(u) \times \ln^{q+1}|(w-v)/n|\rho'(v)\psi(w/n)\,du\,dv\,dw$$

and it follows immediately that

$$\lim_{n\to\infty} I_1 = 0. \tag{10}$$

To deal with I_2 and I_3, we note that when $x > 1/n$

$$\begin{aligned}(\ln^p|x|)_n &= \int_{-1/n}^{1/n} \ln^p|x-s|\delta_n(s)\,ds \\ &= \int_{-1}^{1} \ln^p|x-u/n|\rho(u)\,du \\ &= \int_{-1}^{1}\Big[\ln|x| - \sum_{i=1}^{\infty}\frac{u^i}{in^ix^i}\Big]^p\rho(u)\,du \\ &= \ln^p|x|\int_{-1}^{1}\rho(u)\,du + O(n^{-1}) \\ &= \ln^p|x| + O(n^{-1})\end{aligned} \tag{11}$$

and

$$\begin{aligned}&(q+1)(x^{-1}\ln^q|x|)_n \\ &= \int_{-1/n}^{1/n}\ln^{q+1}|x-t|\delta_n'(t)\,dt \\ &= n\int_{-1}^{1}\ln^{q+1}|x-v/n|\rho'(v)\,dv \\ &= n\int_{-1}^{1}\Big[\ln|x| - \sum_{i=1}^{\infty}\frac{v^i}{in^ix^i}\Big]^{q+1}\rho'(v)\,dv \\ &= n\ln^{q+1}|x|\int_{-1}^{1}\rho'(v)\,dv \\ &\quad - n(q+1)\ln^q|x|\sum_{i=1}^{\infty}\int_{-1}^{1}\frac{v^i}{in^ix^i}\rho'(v)\,dv + O(n^{-1}) \\ &= \frac{(q+1)\ln^q|x|}{x} + O(n^{-1}).\end{aligned} \tag{12}$$

Choosing η with $1/n < \eta < 1$ and using equations (11) and (12), we have

$$\begin{aligned}
&\Big|\int_{1/n}^{\eta} x(\ln^p|x|)_n(x^{-1}\ln^q|x|)_n\psi(x)\,dx\Big| \\
&\le \int_{1/n}^{\eta} |x(\ln^p|x|)_n(x^{-1}\ln^q|x|)_n\psi(x)|\,dx \\
&\le K\int_{1/n}^{\eta} |\ln^{p+q}x|\,dx + O(n^{-1})\eta \\
&\le K\sum_{k=0}^{p+q}\frac{(p+q)!}{(p+q-k)!}[\eta|\ln^{p+q-k}\eta| + n^{-1}\ln^{p+q-k}n] + O(n^{-1})\eta \\
&= O(\eta|\ln^{p+q}\eta|) + O(n^{-1}\ln^{p+q}n) + O(n^{-1})\eta,
\end{aligned}$$

where $K = \max\{\psi(x) : x \in [-1,1]\}$. It follows that

$$\lim_{n\to\infty}\Big|\int_{1/n}^{\eta} x(\ln^p|x|)_n(x^{-1}\ln^q|x|)_n\psi(x)\,dx\Big| = O(\eta|\ln^{p+q}\eta|). \tag{13}$$

Similarly,

$$\lim_{n\to\infty}\Big|\int_{-\eta}^{-1/n} x(\ln^p|x|)_n(x^{-1}\ln^q|x|)_n\psi(x)\,dx\Big| = O(\eta|\ln^{p+q}\eta|). \tag{14}$$

Now let φ be an arbitrary function in $\mathcal{D}$ with support contained in the interval $[-1,1]$. By the mean value theorem

$$\varphi(x) = \varphi(0) + x\varphi'(\xi x),$$

where $0 < \xi < 1$. Thus

$$\begin{aligned}
&\langle(\ln^p|x|)_n(x^{-1}\ln^q|x|)_n, \varphi(x)\rangle \\
&= \int_{-1}^{1}(\ln^p|x|)_n(x^{-1}\ln^q|x|)_n\varphi(x)\,dx \\
&= \varphi(0)\int_{-1}^{1}(\ln^p|x|)_n(x^{-1}\ln^q|x|)_n\,dx \\
&\quad + \int_{-1/n}^{1/n} x(\ln^p|x|)_n(x^{-1}\ln^q|x|)_n\varphi'(\xi x)\,dx \\
&\quad + \int_{1/n}^{\eta} x(\ln^p|x|)_n(x^{-1}\ln^q|x|)_n\varphi'(\xi x)\,dx \\
&\quad + \int_{\eta}^{1} x(\ln^p|x|)_n(x^{-1}\ln^q|x|)_n\varphi'(\xi x)\,dx
\end{aligned}$$

$$+\int_{-\eta}^{-1/n} x(\ln^p|x|)_n(x^{-1}\ln^q|x|)_n\varphi'(\xi x)\,dx$$
$$+\int_{-1}^{-\eta} x(\ln^p|x|)_n(x^{-1}\ln^q|x|)_n\varphi'(\xi x)\,dx.$$

Using the equations (8), (9), (10), (13) and (14) and noting that the sequence $\{(x^{-1}\ln^q|x|)_n\}$ converges uniformly to the function $x^{-1}\ln^q|x|$ on the intervals $[\eta,1]$ and $[-1,-\eta]$, it follows that

$$\lim_{n\to\infty}\langle(\ln^p|x|)_n(x^{-1}\ln^q|x|)_n,\varphi(x)\rangle = O(\eta|\ln^{p+q}\eta|) + \int_{\eta}^{1}\ln^{p+q}|x|\varphi'(\xi x)\,dx + \int_{-1}^{-\eta}\ln^{p+q}|x|\varphi'(\xi x)\,dx.$$

However, since η can be made arbitrarily small, it follows that

$$\begin{aligned}\lim_{n\to\infty}&\langle(\ln^p|x|)_n(x^{-1}\ln^q|x|)_n,\varphi(x)\rangle\\ &=\int_{-1}^{1}\ln^{p+q}|x|\varphi'(\xi x)\,dx\\ &=\int_{-1}^{1}x^{-1}\ln^{p+q}|x|[\varphi(x)-\varphi(0)]\,dx\\ &=\langle x^{-1}\ln^{p+q}|x|,\varphi(x)\rangle\end{aligned}$$

on using equation (1). This proves equation (7) on the interval $[-1,1]$ for $r=0$ and $p,q=0,1,2,\ldots$ but equation (7) clearly holds on any closed interval not containing the origin.

It now follows as in the proof of Theorem 2 that equation (7) holds for $r,p,q=0,1,2,\ldots$. However, since we are now dealing with a commutative product, equation (7) must also hold for $r=-1,-2,\ldots$ and $p,q=0,1,2,\ldots$. This completes the proof of the theorem. □

The product $(\operatorname{sgn}x\,x^r\ln^p|x|)(\operatorname{sgn}x\,x^s\ln^q|x|)$ also exists by Definition 2 and

$$(\operatorname{sgn}x\,x^r\ln^p|x|)(\operatorname{sgn}x\,x^s\ln^q|x|) = x^{r+s}\ln^{p+q}|x| \tag{15}$$

for $r+s,p,q=0,1,2,\ldots$.

We now prove the following extension of equation (15).

THEOREM 3. *The commutative product*

$$(\operatorname{sgn}x\,x^r\ln^p|x|).(\operatorname{sgn}x\,x^{-r-1}\ln^q|x|)$$

exists and

$$(\operatorname{sgn} x\, x^r \ln^p|x|).(\operatorname{sgn} x\, x^{-r-1}\ln^q|x|) = x^{-1}\ln^{p+q}|x| \tag{16}$$

for $r = 0, \pm 1, \pm 2, \ldots$ *and* $p, q = 0, 1, 2, \ldots$.

Proof. We put

$$\begin{aligned}
&(\ln^p x_+)_n \\
&= \ln^p x_+ * \delta_n(x) \\
&= \begin{cases} \int_{-1/n}^{1/n} \ln^p(x-s)\delta_n(s)\,ds, & x \ge 1/n, \\ \int_{-1/n}^{x} \ln^{q+1}(x-s)\delta_n(s)\,ds, & -1/n < x < 1/n, \\ 0, & x \le -1/n, \end{cases} \\
&(\ln^p x_-)_n \\
&= \ln^p x_- * \delta_n(x) \\
&= \begin{cases} \int_{1/n}^{-1/n} \ln^p(s-x)\delta_n(s)\,ds, & x \le -1/n, \\ \int_{x}^{1/n} \ln^p(s-x)\delta_n(s)\,ds, & -1/n < x < 1/n, \\ 0, & x \ge 1/n, \end{cases} \\
&(x_+^{-1}\ln^q x_+)_n \\
&= (x_+^{-1}\ln^q x_+) * \delta_n(x) \\
&= \begin{cases} (q+1)^{-1}\int_{-1/n}^{1/n} \ln^{q+1}(x-t)\delta_n'(t)\,dt, & x \ge 1/n, \\ (q+1)^{-1}\int_{-1/n}^{x} \ln^{q+1}(x-t)\delta_n'(t)\,dt, & -1/n < x < 1/n, \\ 0, & x \le -1/n, \end{cases} \\
&(x_-^{-1}\ln^q x_-)_n \\
&= (x_-^{-1}\ln^q x_-) * \delta_n(x) \\
&= \begin{cases} (q+1)^{-1}\int_{-1/n}^{1/n} \ln^{q+1}(t-x)\delta_n'(t)\,dt, & x \le -1/n, \\ (q+1)^{-1}\int_{x}^{1/n} \ln^{q+1}(t-x)\delta_n'(t)\,dt, & -1/n < x < 1/n, \\ 0, & x \ge 1/n, \end{cases}
\end{aligned}$$

so that the support of

$$(\ln^p x_-)_n(x_+^{-1}\ln^q x_+)_n - (\ln^p x_+)_n(x_-^{-1}\ln^q x_-)_n$$

is contained in the interval $[-1/n, 1/n]$. We then have

$$\int_{-1/n}^{1/n} [(\ln^p x_-)_n(x_+^{-1}\ln^q x_+)_n - (\ln^p x_+)_n(x_-^{-1}\ln^q x_-)_n]\,dx = 0 \tag{17}$$

since the integrand is odd.

Next, if ψ is an arbitrary continuous function, we have

$$\int_{-1/n}^{1/n} x(\ln^p x_-)_n(x_+^{-1}\ln^q x_+)_n\psi(x)\,dx$$
$$= (q+1)^{-1}\int_{-1/n}^{1/n}\int_{-1/n}^{x}\int_{x}^{1/n} x\ln^p(s-x)\delta_n(s)$$
$$\times \ln^{q+1}(x-t)\delta_n'(t)\psi(x)\,ds\,dt\,dx$$
$$= (q+1)^{-1}n^{-1}\int_{-1}^{1}\int_{-1}^{w}\int_{u}^{1} w\ln^p|(u-w)/n|\rho(u)$$
$$\times \ln^{q+1}|(w-v)/n|\rho'(v)\psi(w/n)\,du\,dv\,dw$$

on making the substitutions $ns = u$, $nt = v$ and $nx = w$. It follows immediately that

$$\lim_{n\to\infty}\int_{-1/n}^{1/n} x(\ln^p x_-)_n(x_+^{-1}\ln^q x_+)_n\psi(x)\,dx = 0. \tag{18}$$

Similarly,

$$\lim_{n\to\infty}\int_{-1/n}^{1/n} x(\ln^p x_+)_n(x_-^{-1}\ln^q x_-)_n\psi(x)\,dx = 0. \tag{19}$$

Now let φ is an arbitrary function in $\mathcal{D}$, we have

$$\langle(\ln^p x_-)_n(x_+^{-1}\ln^q x_+)_n - (\ln^p x_+)_n(x_-^{-1}\ln^q x_-)_n, \varphi(x)$$
$$= \varphi(0)\int_{-1/n}^{1/n}[(\ln^p x_-)_n(x_+^{-1}\ln^q x_+)_n - (\ln^p x_+)_n(x_-^{-1}\ln^q x_-)_n]\varphi(x)\,dx$$
$$+ \int_{-1/n}^{1/n}[(\ln^p x_-)_n(x_+^{-1}\ln^q x_+)_n - (\ln^p x_+)_n(x_-^{-1}\ln^q x_-)_n]\varphi'(\xi x)\,dx$$

and it follows from equations (17), (18) and (19) that

$$\lim_{n\to\infty}\langle(\ln^p x_-)_n(x_+^{-1}\ln^q x_+)_n - (\ln^p x_+)_n(x_-^{-1}\ln^q x_-)_n, \varphi(x)\rangle = 0. \tag{20}$$

Putting

$$(\operatorname{sgn} x\,\ln^p|x|)_n = (\operatorname{sgn} x\,\ln^p|x|) * \delta_n(x) = (\ln_+^p x)_n - (\ln_-^p x_-)_n,$$
$$(\operatorname{sgn} x\,x^{-1}\ln^q|x|)_n = (\operatorname{sgn} x\,x^{-1}\ln^q|x|) * \delta_n(x)$$
$$= (x_+^{-1}\ln_+^q x)_n - (x_-^{-1}\ln_-^q x_-)_n,$$

we have

$$\begin{aligned}&(\operatorname{sgn} x\ \ln^p|x|)_n(\operatorname{sgn} x\, x^{-1}\ln^q|x|)_n-(\ln^p|x|)_n(x^{-1}\ln^q|x|)_n\\ &=[(\ln^p x_+)_n-(\ln^p x_-)_n][(x_+^{-1}\ln^q x_+)_n+(x_-^{-1}\ln^q x_-)_n]\\ &\quad-[(\ln^p x_+)_n+(\ln^p x_-)_n][(x_+^{-1}\ln^q x_+)_n-(x_-^{-1}\ln^q x_-)_n].\end{aligned}$$

Multiplying out and using equation (20) it follows that

$$\begin{aligned}\lim_{n\to\infty}\langle(\operatorname{sgn} x\ \ln^p|x|)_n&(\operatorname{sgn} x\, x^{-1}\ln^q|x|)_n\\ &-(\ln^p|x|)_n(x^{-1}\ln^q|x|)_n,\varphi(x)\rangle=0.\end{aligned}$$

Using equation (7), we have proved equation (16) for the case $r=0$ and $p,q=0,1,2,\ldots$. The general case follows by induction as in the proof of equation (7). □

References

[1] J. -F. Colombeau, *New generalized functions and multiplication of distributions*, North-Holland Publishing Co., 1984, xii+375.

[2] B. Fisher, *The product of distributions*, Quart. J. Math. Oxford Ser. **22** (1971), no. 2, 291–298.

[3] ———, *The product of the distributions $x_+^{-r-\frac{1}{2}}$ and $x_-^{-r-\frac{1}{2}}$*, Proc. Cambridge Philos. Soc. **71** (1971), 123–130.

[4] ———, *The product of the distributions x^{-r} and $\delta^{(r-1)}(x)$*, Proc. Cambridge Philos. Soc. **72** (1972), 201–204.

[5] ———, *Some results on the product of distributions,* Proc. Cambridge Philos. Soc. **73** (1973), 317–325.

[6] I. M. Gel'fand and G. E. Shilov, *Generalized Functions*, Vol. I, Academic Press, 1964.

[7] L. Schwartz, *Théorie des Distributions*, Tome I, Actualités Sci. Ind., no. 1091, Hermann & Cie., Paris 1950, 148.

Brian Fisher
Department of Mathematics
University of Leicester
Leicester, LE1 7RH, England
E-mail: fbr@le.ac.uk

Kenan Taş
Department of Mathematics and Computer Science
Faculty of Arts and Sciences
Cankaya University
Ankara, Turkey
E-mail: kenan@cankaya.edu.tr