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INVEXITY AS NECESSARY OPTIMALITY CONDITION IN NONSMOOTH PROGRAMS

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Abstract. This paper gives conditions under which necessary optimality conditions in a locally Lipschitz program can be expressed as the invexity of the active constraint functions or the type I invexity of the objective function and the constraint functions on the feasible set of the program. The results are nonsmooth extensions of those of Hanson and Mond obtained earlier in differentiable case.

1. Introduction

Let f and g_i, $i = 1, 2, \ldots, m$, be functions defined on an Euclidean space R^n. Consider the following Mathematical Programming Problem (P)

$$\min \quad f(x) \tag{1}$$

$$\text{subject to} \quad g_i(x) \leq 0, \quad i = 1, 2, \ldots, m. \tag{2}$$

Let S be the set of all feasible points of (P) (i.e., the set of all x satisfying (2)). Take a point $x_0 \in S$ and denote by $I(x_0)$ the index set of the active constraints i.e.,

$$I(x_0) = \{i : g_i(x_0) = 0\}. \tag{3}$$

It is well known ([7]) that the Kuhn-Tucker condition is sufficient for x_0 to be a minimizer of Problem (P) if f and g_i, $i \in I(x_0)$, are differentiable convex functions. In 1981 Hanson [4] showed that this property remains

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valid if the convexity requirement is replaced by a weaker condition, later called invexity. Recall that differentiable functions f and g_i, $i = 1, 2, \dots, m$, are invex on an arbitrary set S at a point $x_0 \in R^n$ if for all $x \in S$

$$f(x) - f(x_0) \geq f'_{x_0}(\eta(x)), \tag{4}$$

$$g_i(x) - g_i(x_0) \geq g'_{ix_0}(\eta(x)), \quad i = 1, 2, \dots, m, \tag{5}$$

where f'_{x_0} and g'_{ix_0} stand for the Fréchet derivatives of f and g_i at x_0, and $\eta : S \to R^n$ is a suitable map. Such a map η will be called a scale (for f and g_i).

In 1985 Martin [8] extended invexity to KT-invexity and proved that every Kuhn-Tucker point of Problem (P) is a global minimizer if and only if this problem is KT-invex. (The result of Martin was generalized to vector optimization in [10]). Recently, Hanson ([5], Theorem 3.1) showed that the invexity of the active constraint functions is also a necessary optimality condition if the set of Kuhn-Tucker multipliers exists and is bounded. In 1987 Hanson and Mond [6] introduced a notion of type I invexity of f and g_i which is different from invexity in that the left side of (5) is replaced by $-g_i(x_0)$ while the right side of (5) remains unchanged. They showed in Theorem 2.2 of [6] that the type I invexity of the objective function and the constraint functions with a nontrivial scale is a necessary optimality condition if the number of the active constraints is less than n (the dimension of variable x).

The notion of invexity was extended to locally Lipschitz functions by Craven [3]. Recently, Sach, Lee and Kim [13, 14] defined generalized invexity and discussed its roles in vector optimization problems. Reiland [11] pointed out that under the invexity assumption the Kuhn-Tucker condition also assures the optimality property in nonsmooth programs involving locally Lipschitz functions. However, the problem of showing that the invexity (or the type I invexity) of functions involved in these programs can be served as a necessary optimality condition is still open. The aim of this paper is to give an answer to this problem. Our results are extensions of the corresponding results of [5, 6] to nonsmooth case. These extensions are useful since many practical problems encountered in economics, engineering design $\cdots$ can be described only by nondifferentiable functions (see [2]); and hence, earlier results for differentiable case can not be applied to such problems.

The organization of this paper is as follows: Section 2 recalls the definitions of Clarke subdifferentials and Clarke directional derivatives of locally Lipschitz functions and gives versions of invexity of these functions.

Section 3 discusses the consistency of systems of sublinear inequalities which are given by support functions of nonempty compact convex sets. The results of this section are needed for proving the invexity properties in subsequent sections. Section 4 proves that under a constraint qualification condition, which is equivalent to the assumption of nonemptiness and boundedness of the set of the Kuhn-Tucker multipliers, the active constraint functions are invex on S at an optimal point x_0. Section 5 contains two theorems: the first one considers the type I invexity of the objective function and the constraint functions on some subset of S, and the second one gives a condition which is equivalent to the type I invexity of these functions on the whole set S with a nontrivial scale. Observe that unlike Theorem 2.2 of [6], where the number of the active constraints must be less than the dimension of the variable x and the objective and constraint functions must be differentiable, the second of these theorems is established under an assumption not depending on the number of the active constraints and is valid for nonsmooth functions.

2. Preliminaries

Let A be a subset of an Euclidean space R^n. The symbols cl$A$, co$A$ and cone$A$ are used to denote the closure of A, the convex hull of A and the cone generated by A. The scalar product of two vectors $a \in R^n$ and $x \in R^n$ is denoted by $\langle a, x\rangle$.

Let $f : R^n \to R$ (the real line) be a locally Lipschitz function. This means that for any $x_0 \in R^n$ there are positive numbers α and β such that

$$\|x - x_0\| < \alpha, \quad \|x' - x_0\| < \alpha \Rightarrow |f(x) - f(x')| \leq \beta\|x - x'\|.$$

Let $f^0(x_0, x)$ be the Clarke directional derivative of f at x_0 in direction $x \in R^n$:

$$f^0(x_0, x) = \limsup_{\lambda\downarrow 0,\ x'\to x_0} (1/\lambda)[f(x' + \lambda x) - f(x')].$$

The set

$$\partial f(x_0) := \{a \in R^n : f^0(x_0, x) \geq \langle a, x\rangle \quad \forall x \in R^n\}$$

is the Clarke subdifferential of f at x_0. It is known from [2] that $\partial f(x_0)$ is a nonempty compact convex set satisfying the following condition:

$$f^0(x_0, .) = \max_{a\in\partial f(x_0)} \langle a, \cdot\rangle. \tag{6}$$

Throughout this paper, unless otherwise specified, we assume that f and g_i, $i = 1, 2, \ldots, m$, are locally Lipschitz functions.

Let S be an arbitrary subset of R^n and x_0 be a given point of R^n. The following definition is introduced by Craven [3].

DEFINITION 2.1. Functions f and g_i, $i = 1, 2, \ldots, m$, are invex on S at x_0 if for all $x \in S$

$$f(x) - f(x_0) \geq f^0(x_0, \eta(x)), \tag{7}$$

$$g_i(x) - g_i(x_0) \geq g_i^0(x_0, \eta(x)), \quad i = 1, 2, \ldots, m, \tag{8}$$

where $\eta : S \to R^n$ is a suitable map. Such a map is called a scale (or more precisely, a scale for the invexity of f and g_i on S at x_0).

Observe that the invexity notion of nonsmooth functions is different from that of differentiable functions only in that the Fréchet derivatives in the right side of (4) and (5) are replaced by the corresponding Clarke directional derivatives. The same is true for the case of type I invexity given in the following definition.

DEFINITION 2.2. Functions f and g_i, $i = 1, 2, \ldots, m$, are type I invex on S at x_0 if for all $x \in S$

$$f(x) - f(x_0) \geq f^0(x_0, \eta(x)), \tag{9}$$

$$-g_i(x_0) \geq g_i^0(x_0, \eta(x)), \quad i = 1, 2, \ldots, m, \tag{10}$$

where $\eta : S \to R^n$ is a suitable map.

A map $\eta : S \to R^n$ satisfying (9) and (10) for all $x \in S$ is called a scale (or more precisely, a scale for the type I invexity of f and g_i on S at x_0). A scale η is nontrivial if $\eta(x) \neq 0$ for all $x \in S$. A nontrivial scale which is a constant map on S is called a nontrivial constant scale. Thus a map $\eta : S \to R^n$ is a nontrivial constant scale if and only if there is a nonzero vector $\overline{\eta} \in R^n$ such that $\eta(x) = \overline{\eta}$ for all $x \in S$.

We emphasize that in each of Definitions 2.1 and 2.2 map η must be the same for all functions f and g_i.

3. Consistency of systems of sublinear inequalities

The general theory of convex inequalities can be found in [7, 12]. Here we are interested only in sublinear inequalities which are given by support functions of nonempty compact convex sets and which are needed for proving some results of Sections 4 and 5.

Given an index set $I = \{0, 1, \dots, k\}$ and functions $\varphi_i : R^n \to R$, $i \in I$, we say that the system of inequalities

$$\varphi_i(\xi) \leq 0, \quad i \in I, \tag{11}$$

is consistent if it has a solution (i.e., (11) is satisfied for some $\xi \in R^n$).

Throughout this section we assume that

$$\varphi_i(\xi) = \max_{b \in B_i} \langle b, \xi \rangle, \tag{12}$$

where B_i is a nonempty compact convex set of R^n. Then system (11) is always consistent since it has a solution $\xi = 0$. To give a necessary and sufficient condition for this system to have a nonzero solution we first introduce the set

$$B = \mathrm{co} \bigcup_{i \in I} B_i. \tag{13}$$

PROPOSITION 3.1. *System* (11) *has a nonzero solution if and only if*

$$R^n \neq \mathrm{cl\ cone}\ B. \tag{14}$$

Proof. Since cl cone$B$ is a closed convex cone we see that

$$\begin{aligned} (14) \text{ holds} &\Leftrightarrow \exists \xi \neq 0, \forall x \in \mathrm{cl\ cone} B, \langle x, \xi \rangle \leq 0 \\ &\Leftrightarrow \exists \xi \neq 0, \forall i \in I, \varphi_i(\xi) \leq 0. \end{aligned}$$

□

We now give a criterion for the consistency of the following system

$$\varphi_i(\xi) \leq 0, \quad i \in I \setminus I_1, \tag{15}$$

$$\varphi_j(\xi) < 0, \quad j \in I_1, \tag{16}$$

where I_1 is a nonempty subset of I. Obviously, any solution of this system must be a nonzero vector and it is also a solution of system (11).

Let us set

$$C = \mathrm{co} \bigcup_{i \in I \setminus I_1} B_i, \tag{17}$$

$$C_1 = \mathrm{co} \bigcup_{j \in I_1} B_j. \tag{18}$$

Observe from Corollary 9.8.2 of [12] that C and C_1 are convex and compact sets.

PROPOSITION 3.2. *System* (15), (16) *is consistent if and only if*

$$-C_1 \cap \mathrm{cl\ cone}\ C = \emptyset. \tag{19}$$

Proof. We see that

$$\begin{aligned} &(19) \text{ holds} \\ &\Leftrightarrow \exists \xi \in R^n, \forall x_1 \in C_1, \forall x \in \mathrm{cl\ cone}\ C, -\langle x_1, \xi \rangle > 0 \geq \langle x, \xi \rangle \qquad (20) \\ &\Leftrightarrow \exists \xi \in R^n \text{ such that (15) and (16) are satisfied.} \end{aligned}$$

(When showing the forward implication "$\Rightarrow$" in (20) we use a separation theorem.) □

COROLLARY 3.1. *System* (16) *is consistent if and only if* $0 \notin C_1$.

Proof. Apply Proposition 3.2 with $B_i = \{0\}, i \in I \setminus I_1$. □

Now let β_i be a real number. Consider the following nonhomogeneous system

$$\varphi_i(\xi) \leq \beta_i, \qquad i \in I. \tag{21}$$

Let

$$B_i' = B_i \times \{-\beta_i\} \subset R^n \times R, \tag{22}$$

$$B' = \mathrm{co} \bigcup_{i \in I} B_i'. \tag{23}$$

PROPOSITION 3.3. *System* (21) *is consistent if and only if*

$$(0, 1) \notin \mathrm{cl\ cone}\ B' \quad (\textit{0 being the origin of } R^n). \tag{24}$$

Proof. As in [7, p.32] we introduce an additional variable $r \in R$ and we set $\xi' = (\xi, r) \in R^n \times R$. Then it is clear that system (21) is consistent if and only if system

$$\varphi_i'(\xi') := \varphi_i(\xi) - r\beta_i \leq 0, \quad i = 0, 1, \ldots, k, \tag{25}$$

$$\varphi_{k+1}'(\xi') := -r < 0 \tag{26}$$

is consistent. Let B_{k+1}' be the set which consists of the unique element $(0, -1) \in R^n \times R$. Then obviously $\varphi_i'(\xi') = \max_{b' \in B_i'} \langle b', \xi' \rangle$, $i = 0, 1, \ldots, k+1$. Applying Proposition 3.2 to system (25), (26) yields condition (24), as desired. □

4. Invexity and necessary optimality condition

We begin by recalling a known result of [2]. As we said in Section 2, unless otherwise specificed all functions f and g_i are assumed to be locally Lipschitz.

THEOREM 4.1. *If x_0 is a minimizer of Problem (P), then the Fritz John condition is satisfied: there are nonnegative numbers $\lambda_i, i = 0, 1, \ldots, m$, not all zero, such that*

$$0 \in \lambda_0 \partial f(x_0) + \sum_{i=1}^{m} \lambda_i \partial g_i(x_0), \tag{27}$$

$$0 = \lambda_i g_i(x_0), \qquad i = 1, 2, \ldots, m. \tag{28}$$

Observe that in case $I(x_0) \neq \emptyset$ the Fritz John condition can be reformulated as follows: there are nonnegative numbers λ_0 and λ_i, $i \in I(x_0)$, not all zero, such that

$$0 \in \lambda_0 \partial f(x_0) + \sum_{i \in I(x_0)} \lambda_i \partial g_i(x_0). \tag{29}$$

Under additional conditions the multiplier λ_0 must be different from zero and hence we may set $\lambda_0 = 1$. We now give such a condition, called the condition (CQ): if $I(x_0) \neq \emptyset$ then

$$0 \notin A(x_0) := \operatorname{co} \bigcup_{i \in I(x_0)} \partial g_i(x_0). \tag{30}$$

Observe from Corollary 9.8.2 of [12] that the set $A(x_0)$ being the convex hull of a finite number of compact sets is (convex and) compact.

The following result is a direct consequence of Theorem 4.1 and condition (CQ).

THEOREM 4.2. *Assume that condition (CQ) holds. If x_0 is a minimizer of Problem (P) then the Kuhn-Tucker condition is satisfied: there are nonnegative numbers λ_i, $i = 1, 2, \ldots, m$, satisfying (28) and the following condition*

$$0 \in \partial f(x_0) + \sum_{i=1}^{m} \lambda_i \partial g_i(x_0). \tag{31}$$

Observe that in case $I(x_0) \neq \emptyset$ the Kuhn-Tucker condition means that there are nonnegative numbers λ_i, $i \in I(x_0)$, such that

$$0 \in \partial f(x_0) + \sum_{i \in I(x_0)} \lambda_i \partial g_i(x_0). \tag{32}$$

We shall give conditions equivalent to condition (CQ). For this purpose we have to introduce some definitions. If $I(x_0) \neq \emptyset$ we shall use the symbol $\lambda = (\lambda_i,\ i \in I(x_0))$ to denote the vector with coordinates λ_i, $i \in I(x_0)$, and the symbol $E(x_0)$ to denote the set of all vectors $\lambda = (\lambda_i,\ i \in I(x_0))$ with nonnegative coordinates λ_i such that (32) is satisfied. Observe that $E(x_0)$ is a closed convex set, but it may be empty or unbounded. We shall see that the nonemptiness and boundedness of $E(x_0)$ at the minimizer x_0 of (P) is equivalent to condition (CQ).

Let us introduce the following two conditions:

Condition (CQ)$'$: If $I(x_0) \neq \emptyset$ then $E(x_0)$ is a nonempty bounded set.

Condition (CQ)$''$: If $I(x_0) \neq \emptyset$ then there is a point $x \in R^n$ such that

$$g_i^0(x_0, x) < 0, \qquad i \in I(x_0). \tag{33}$$

LEMMA 4.1. *Let $x_0 \in S$ be such that $I(x_0) \neq \emptyset$. Then*

condition (CQ)$'$ $\Rightarrow$ *condition* (CQ) $\Leftrightarrow$ *condition* (CQ)$''$.

If, in addition, x_0 is a minimizer of Problem (P), *then all three conditions are equivalent.*

Proof. The equivalence of conditions (CQ) and (CQ)$''$ is derived from Corollary 3.1 and equality (6). Implications (CQ)$'$ $\Rightarrow$ (CQ) and (CQ) $\Rightarrow$ (CQ)$'$ can be obtained from Theorem 5.1 of [9]. For reader's convenience we give a direct proof of these implications.

(CQ)$'$ $\Rightarrow$ (CQ). Since $E(x_0) \neq \emptyset$ we can pick a vector $\lambda = (\lambda_i,\ i \in I(x_0)) \in E(x_0)$. Assume to the contrary that $0 \in A(x_0)$ then there are nonnegative numbers λ_i', $i \in I(x_0)$, not all zero, such that

$$0 \in \sum_{i \in I(x_0)} \lambda_i' \partial g_i(x_0). \tag{34}$$

Let γ be an arbitrary positive number. Multiplying both sides of (34) by γ and summing up the obtained inclusion and inclusion (32) we get

$$0 \in \partial f(x_0) + \sum_{i \in I(x_0)} (\lambda_i + \gamma \lambda_i') \partial g_i(x_0).$$

Thus vector $\overline{\lambda} = (\lambda_i + \gamma\lambda_i',\ i \in I(x_0))$ belongs to $E(x_0)$. Since $\gamma > 0$ is arbitrarily chosen, this shows that $E(x_0)$ is unbounded. It is impossible.

(CQ) $\Rightarrow$ (CQ)$'$ (under the assumption that x_0 is a minimizer of Problem (P)). By Theorem 4.2 the set $E(x_0)$ is nonempty. To prove the boundedness of this set we assume to the contrary that there is a sequence of vectors $\lambda^k = (\lambda_i^k,\ i \in I(x_0))$ such that $\lambda_i^k \geq 0,\ i \in I(x_0)$, $\|\lambda^k\| \to \infty$ and

$$0 \in \partial f(x_0) + \sum_{i \in I(x_0)} \lambda_i^k \partial g_i(x_0).$$

Setting $\gamma^k = \sum_{i \in I(x_0)} \lambda_i^k$ and dividing the last inclusion by γ^k we get that

$$0 \in (1/\gamma^k)\partial f(x_0) + A(x_0).$$

Letting $k \to \infty$ and observing from the known property of Clarke subdifferentials that $\partial f(x_0)$ and $A(x_0)$ are compact sets, we obtain $0 \in A(x_0)$, a contradiction to condition (CQ). □

COROLLARY 4.1. *Let the following Slater condition be satisfied: if* $I(x_0) \neq \emptyset$ *then there is a point* $x \in S$ *such that* $g_i(x) < 0, i \in I(x_0)$. *Let the active constraint functions of Problem* (P) *be invex on* S *at* x_0. *Then condition* (CQ) *holds.*

Proof. From the Slater condition and the invexity property it follows that there is a point $\eta(x) \in R^n$ such that (33) is satisfied with $\eta(x)$ instead of x. Thus condition (CQ)$''$ holds and hence, Lemma 4.1 yields our desired conclusion. □

Combining Theorem 4.2 and Corollary 4.1 we obtain the following result which is established in Theorem 4 of [1] for differentiable case.

THEOREM 4.3. *Assume that the active constraint functions are invex on* S *at* x_0 *and the Slater condition holds at* x_0. *If* x_0 *is a minimizer of Problem* (P) *then the Kuhn-Tucker condition is satisfied.*

THEOREM 4.4. *Assume that condition* (CQ) *holds. If* x_0 *is a minimizer of Problem* (P) *then the active constraint functions are invex on* S *at* x_0.

Proof. Observe from the boundedness of $E(x_0)$ (see Lemma 4.1) that for fixed $x \in S$ there is a positive number β_0 such that for all $\lambda = (\lambda_i,\ i \in I(x_0)) \in E(x_0)$ we have

$$-\beta_0 - \sum_{i \in I(x_0)} \lambda_i \beta_i < 0, \tag{35}$$

where

$$\beta_i := g_i(x) - g_i(x_0), \qquad i \in I(x_0). \tag{36}$$

Without loss of generality, we may assume that

$$I(x_0) = \{1, 2, \dots, k\}, \tag{37}$$

where $1 \le k \le m$. Let us set $I = \{0\} \cup I(x_0) = \{0, 1, 2, \dots, k\}$,

$$g_0(\xi) = f(\xi), \tag{38}$$

$$\varphi_i(\xi) = g_i^0(x_0, \xi), \tag{39}$$

$$B_i = \partial g_i(x_0). \tag{40}$$

To prove the invexity of the active constraint functions, it is enough to show that system (21) is consistent. By Proposition 3.3 this is equivalent to the validity of condition (24), where B' is defined by (22) and (23). To prove it, first observe from (35) and condition (CQ) that B' does not contain the origin of space $R^n \times R$. Indeed, otherwise we have

$$0 \in \sum_{i=0}^{k} \lambda_i' \partial g_i(x_0), \tag{41}$$

$$0 = -\sum_{i=0}^{k} \lambda_i' \beta_i \tag{42}$$

for suitable numbers $\lambda_i' \ge 0$, $i = 0, 1, \dots, k$, with

$$\sum_{i=0}^{k} \lambda_i' = 1. \tag{43}$$

Observe that $\lambda_0' \neq 0$ since otherwise (41) contradicts condition (CQ). Dividing (41) and (42) by λ_0' and setting $\lambda_i = \lambda_i'/\lambda_0'$, we obtain

$$0 \in \partial f(x_0) + \sum_{i=1}^{k} \lambda_i \partial g_i(x_0), \tag{44}$$

$$0 = -\beta_0 - \sum_{i=1}^{k} \lambda_i \beta_i. \tag{45}$$

Inclusion (44) shows that $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \in E(x_0)$ and hence, (45) contradicts (35). We have thus proved that B' does not contain the origin of $R^n \times R$. From Corollary 9.6.1 of [12] it follows that coneB' is a closed set. Assume now that (24) fails to hold i.e. $(0,1) \in \mathrm{cl\ cone}\, B' =$ coneB'. Then there are $\gamma \geq 0$ and $\lambda_i' \geq 0$, $i = 0, 1, 2, \ldots, k$, such that (43) and the following conditions are satisfied:

$$0 \in \gamma \sum_{i=0}^{k} \lambda_i' \partial g_i(x_0), \tag{46}$$

$$1 = -\gamma \sum_{i=0}^{k} \lambda_i' \beta_i. \tag{47}$$

Equality (47) shows that $\gamma \neq 0$ and condition (46) yields $\lambda_0' \neq 0$ (see (30)). Setting $\lambda_i = \lambda_i'/\lambda_0'$ and dividing (46) by $\gamma\lambda_0'$, we obtain again (44) which shows that $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \in E(x_0)$. On the other hand, (47) implies that

$$-\beta_0 - \sum_{i=1}^{k} \lambda_i \beta_i = 1/\gamma\lambda_0' > 0,$$

a contradiction to (35). □

As a consequence of Lemma 4.1 and Theorem 4.4 we obtain the following result which is proved in Theorem 3.1 of [5] for differentiable programs.

COROLLARY 4.2. *Assume that condition* (CQ)$'$ *is satisfied. Then the conclusion of Theorem 4.4 is true.*

COROLLARY 4.3. *Assume that condition* (CQ)$'$ *holds. If x_0 is a minimizer of Problem* (P) *then, for any $j \in I(x_0)$ such that $g_j(x) < 0$ for some $x \in S$, we have*

$$-\partial g_j(x_0) \bigcap \mathrm{cl\ cone}\, \widetilde{C}_j(x_0) = \emptyset$$

where

$$\widetilde{C}_j(x_0) = \mathrm{co} \bigcup_{i \in I(x_0) \setminus \{j\}} \partial g_j(x_0).$$

Proof. By Corollary 4.2 the active constraint functions are invex on S at x_0. Thus we find $\eta(x) \in R^n$ such that $g_i(x) - g_i(x_0) \geq$

$g_i^0(x_0, \eta(x))$, $i \in I(x_0)$. This shows that $\xi := \eta(x)$ is a nonzero solution of system

$$g_i^0(x_0, \xi) \leq 0, \quad i \in I(x_0) \setminus \{j\}, \quad g_j^0(x_0, \xi) < 0.$$

Our desired conclusion is now an immediate consequence of Proposition 3.2. □

Now let us set

$$S_0 = \{x \in S : f(x) > f(x_0)\}. \tag{48}$$

THEOREM 4.5. *Assume that x_0 is a minimizer of Problem* (P). *Assume, in addition, that there is a point $x \in S \setminus S_0$ such that $g_i(x) < 0, i \in I(x_0)$. If the objective function and the active constraint functions are invex on S at x_0 then*

$$0 \in \partial f(x_0). \tag{49}$$

Proof. Since the Slater condition is satisfied, by Theorem 4.3 there are nonnegative numbers λ_i and vectors $a_0 \in \partial f(x_0)$, $a_i \in \partial g_i(x_0)$ such that

$$-a_0 = \sum_{i \in I(x_0)} \lambda_i a_i. \tag{50}$$

We claim that $\lambda_i = 0$ for all $i \in I(x_0)$. In this case, (50) yields $a_0 = 0$ and hence, (49) holds. Assume to the contrary that $\lambda_i \neq 0$ for some $i \in I(x_0)$. Let x be the point appearing in the formulation of the theorem. Then by the invexity assumption we have

$$\sum_{i \in I(x_0)} \lambda_i \langle a_i, \eta(x) \rangle \leq \sum_{i \in I(x_0)} \lambda_i [g_i(x) - g_i(x_0)] < 0, \tag{51}$$

$$\langle a_0, \eta(x) \rangle \leq f(x) - f(x_0) = 0. \tag{52}$$

On the other hand, from (50) and (51) we have $-\langle a_0, \eta(x) \rangle < 0$, a contradiction to (52). The proof is thus complete. □

EXAMPLE 4.1. Let $n = m = 1$ and let $g_1(x) = x$,

$$f(x) = \begin{cases} x & \text{if } \ 0 \leq x, \\ 0 & \text{if } \ -1 \leq x < 0, \\ -x - 1 & \text{if } \ x < -1. \end{cases}$$

Then $x_0 = 0$ is a minimizer of (P), $S = (-\infty, 0]$, $S_0 = (-\infty, -1)$. Obviously, $I(x_0) = \{1\}$, $x := -1/2 \in S \setminus S_0$ and $g_1(x) < 0$. On the other hand, it can be seen that f and g_1 are invex on S at x_0, with η being the identity map. By Theorem 4.5 inclusion (49) holds.

EXAMPLE 4.2. Let $n = m = 1$ and let $g_1(x) = -x$,

$$f(x) = \begin{cases} 0 & \text{if } 2 \le x, \\ -x+2 & \text{if } 1 \le x < 2, \\ x & \text{if } 0 \le x < 1, \\ 2^{-1}x & \text{if } x < 0. \end{cases}$$

Obviously, for $x_0 = 0$ $I(x_0) = \{1\}$, $\partial f(x_0) = [2^{-1}, 1] \not\ni 0$ and x_0 is a minimizer. On the other hand, for $x = 2 \in S \setminus S_0$ we have $g_1(x) = -2 < 0$. By Theorem 4.5 f and g_1 cannot be invex on S at x_0.

5. Type I invexity and necessary optimality condition

Let x_0 be a minimizer of Problem (P). Then the objective function and the constraint functions are type I invex on S at x_0, with the trivial scale $\eta(x) = 0$ for all $x \in S$. This section will give conditions under which nontrivial scales or nontrivial constant scales exist on some subset of S or on the whole set S. Observe that for the differentiable case Hanson and Mond ([6], Theorem 2.2) show that a nontrivial scale on S can be found if the number of the active constraints is less than the dimension of variable x. We shall see in Corollary 5.1 that this fact is a consequence of a more general result (Theorem 5.2) which is established for nonsmooth programs under an assumption not depending on the number of the active constraints.

Let S_0 be defined by (48). We have the following result.

THEOREM 5.1. *Let condition* (CQ) *hold. If x_0 is a minimizer of Problem* (P) *then the objective function and the constraint functions are type I invex on S_0 at x_0, with a nontrivial scale.*

Proof. We first assume that $I(x_0) \neq \emptyset$. Let $x \in S_0$. Since $f(x) > f(x_0)$ and

(53) $$-g_i(x_0) > 0, \quad i \notin I(x_0),$$

there is a positive number β_0 such that

(54) $$f(x) - f(x_0) \ge \beta_0,$$

(55) $$-g_i(x_0) \ge \beta_0, \quad i \notin I(x_0).$$

Because of the boundedness of set $E(x_0)$ (see Lemma 4.1) there are negative numbers β_i, $i \in I(x_0)$, such that for all vectors $\lambda = (\lambda_i,\ i \in I(x_0))$ of $E(x_0)$ we have

$$-\beta_0 - \sum_{i \in I(x_0)} \lambda_i \beta_i < 0. \tag{56}$$

Let $I(x_0)$, I, g_0, φ_i and B_i be as in the proof of Theorem 4.4. Using inequality (56) and condition (CQ), and arguing as in the proof of Theorem 4.4 we can show that system (21) has a nonzero solution ξ. Now let us take $\alpha \in (0,1)$ such that $\alpha g_i^0(x_0, \xi) < \beta_0$ for all $i \notin I(x_0)$. Setting $\eta(x) = \alpha\xi$ we have

$$f(x) - f(x_0) \geq \beta_0 \geq \alpha\beta_0 \geq \alpha f^0(x_0, \xi) = f^0(x_0, \eta(x)), \tag{57}$$

$$-\, g_i(x_0) = 0 > \alpha\beta_i \geq \alpha g_i^0(x_0, \xi) = g_i^0(x_0, \eta(x)), \quad i \in I(x_0), \tag{58}$$

$$-\, g_i(x_0) \geq \beta_0 \geq \alpha g_i^0(x_0, \xi) = g_i^0(x_0, \eta(x)), \quad i \notin I(x_0). \tag{59}$$

Thus f and g_i, $i = 1, 2, \ldots, m$, are invex on S_0 at x_0, with the nontrivial scale η. This conclusion is proved under the assumption that $I(x_0) \neq \emptyset$.

Consider now the case $I(x_0) = \emptyset$. Since functions g_i are continuous and since $g_i(x_0) < 0$ for all $i = 1, 2, \ldots, m$ we must find an open neighborhood U of x_0 such that $g_i(u) < 0$ for all $i = 1, 2, \ldots, m$ and $u \in U$. Then $U \subset S$. Since by assumption x_0 is a minimizer of f on S, it must be a minimizer of f on U. By [14], $0 \in \partial f(x_0)$ or, equivalently, $f^0(x_0, \xi) \geq 0$ for all $\xi \in R^n$. Therefore, we must find $\xi \neq 0$ such that $f^0(x_0, \xi) \leq \beta_0$ where β_0 is a positive number satisfying (54) and (55). Take $\alpha \in (0,1)$ such that $\alpha g_i^0(x_0, \xi) < \beta_0$ for all $i = 1, 2, \ldots, m$. Setting $\eta(x) = \alpha\xi$, we obtain again (57) and (59). Thus f and g_i are invex on S_0 at x_0, with nontrivial scale η. □

REMARK 5.1. Theorem 5.1 fails to hold if we replace S_0 by S. This is shown by an example of Hanson and Mond ([6], p.55), where $f(x) = x$, $g(x) = 1 - x$ and $x_0 = 1$. Another example is now given.

EXAMPLE 5.1. Let $n = m = 1$, $f(x) = -x^2$ and $g_1(x) = x^2 - 1$. Then $x_0 = -1$ is a minimizer of (P), $S = [-1, 1]$, $S_0 = (-1, 1)$. Since $f'_{x_0} = 2$, $g'_{1x_0} = -2$, functions f and g_1 are type I invex on S at x_0 if and only if $f(x) - f(x_0) \geq 2\eta(x)$ and $-g_1(x_0) \geq -2\eta(x)$ for all $x \in S$. If we take $x = 1 \in S \setminus S_0$ then these last inequalities are satisfied only if $\eta(x) = 0$. This shows that f and g_1 cannot be type I invex on the whole set S at x_0 with a nontrivial scale.

Observe from Example 5.1 (and the mentioned example of Hanson-Mond in [6]) that, although condition (CQ) is satisfied the type I invexity of f and g_i, $i = 1, 2, \dots, m$, on the whole set S (with a nontrivial scale) fails to hold. So the type I invexity on the whole set S requires a condition different from condition (CQ). Such a condition is given in Theorem 5.2 below. Before formulating it let us set

$$A_0(x_0) = \operatorname{co} \bigcup_{i \in I_0(x_0)} \partial g_i(x_0), \tag{60}$$

where $g_0 = f, I_0(x_0) = \{0\} \cup I(x_0)$.

THEOREM 5.2. *Let x_0 be a minimizer of Problem* (P). *Then the following statements are equivalent:*

(a) *The objective function and the constraint functions are type I invex on S at x_0, with a nontrivial constant scale.*
(b) *The objective function and the constraint functions are type I invex on S at x_0, with a nontrivial scale.*
(c) $R^n \neq \operatorname{cl} \operatorname{cone} A_0(x_0)$.

Proof. (a) $\Rightarrow$ (b) Obvious.
(b) $\Rightarrow$ (c) Setting $x = x_0$ we obtain from (9) and (10),

$$0 = f(x_0) - f(x_0) \geq f^0(x_0, \eta(x_0)),$$
$$0 = -g_i(x_0) \geq g_i^0(x_0, \eta(x_0)), \quad i \in I(x_0),$$

where $\eta(x_0) \neq 0$ by assumption (b). Applying Proposition 3.1 yields condition (c).

(c) $\Rightarrow$ (a) By Proposition 3.1 there is $\xi \neq 0$ such that

$$f^0(x_0, \xi) \leq 0, \; g_i^0(x_0, \xi) \leq 0, \quad i \in I(x_0). \tag{61}$$

Since $g_i(x_0) < 0$ for $i \notin I(x_0)$ we can take $\alpha > 0$ such that

$$\alpha g_i^0(x_0, \xi) \leq -g_i(x_0), \quad i \notin I(x_0). \tag{62}$$

Observing now that $g_i(x_0) = 0$ for $i \in I(x_0)$ and $f(x) - f(x_0) \geq 0$ for all $x \in S$ we see from (61) and (62) that the nontrivial constant map $\eta \equiv \alpha\xi$ can be taken as a scale for functions f and g_i, $i = 1, 2, \ldots, m$.□

REMARK 5.2. Theorem 5.2 remains true for differentiable programs. In this case, the set $A_0(x_0)$ is nothing else than the convex hull of vectors f'_{x_0} and g'_{ix_0}, $i \in I(x_0)$. Observe that in condition (c) we do not require that the number of the active constraints is less than the dimension of variable x. As we shall see in the proof of the following corollary, the last condition implies (c).

COROLLARY 5.1. *Assume that x_0 is a minimizer of problem* (P) *where all functions involved in this problem are Fréchet differentiable at x_0. If the number of the active constraints is less than n (the dimension of variable x) then statements* (a), (b), *and* (c) *hold.*

Proof. Since x_0 is a minimizer of (P) the Fritz John condition (see Theorem 4.1 with f'_{x_0} and g'_{ix_0} in place of $\partial f(x_0)$ and $\partial g_i(x_0)$, respectively) shows that vectors f'_{x_0} and g'_{ix_0}, $i \in I(x_0)$, are linearly dependent. On the other hand, the number of these vectors is less than or equal to n. Hence the (closed) linear subspace generated by them cannot coincide with R^n. Thus statement (c) of Theorem 5.2 holds. Since (c) implies by Theorem 4.2 the statements (a) and (b), our proof is thus complete. □

Let us observe that in Example 5.1 condition (c) is violated. Hence by Theorem 5.2 f and g_1 cannot be type I invex on S at x_0 with a nontrivial scale. The same is true for the mentioned example of Hanson-Mond [6] and another example of Hanson given in ([5, pp.600–601]).

In Example 4.1, $\partial f(x_0) = [0, 1]$ and $\partial g_1(x_0) = \{1\}$. Thus $A_0(x_0) = [0, 1]$ and condition (c) holds. Hence by Theorem 5.2 f and g_1 are type I invex on S with a nontrivial constant scale.

COROLLARY 5.2. *Let x_0 be a minimizer of Problem* (P) *and for some $j \in I(x_0)$*

$$-\partial g_j(x_0) \bigcap \operatorname{cl} \operatorname{cone} C_j(x_0) = \emptyset, \tag{63}$$

where

$$C_j(x_0) = \operatorname{co} \bigcup_{i \in I_0(x_0) \setminus \{j\}} \partial g_j(x_0).$$

Then the objective function and the constraint functions are type I invex on S at x_0, with a nontrivial scale. Conversely, if the objective function *and the active constraint functions are invex on S at x_0 and if for $j \in I(x_0)$ there is a point $x \in S \setminus S_0$ such that $g_j(x) < 0$ then* (63) *holds.*

Proof. Let g_0, φ_i, B_i, $I(x_0)$ and I be as in the proof of Theorem 4.4 and let $I_1 = \{j\}$. If (63) holds then by Proposition 3.2 it follows that system (15), (16) (and hence, system (11)) has a nonzero solution ξ. By Proposition 3.1 statement (c) (and hence, (b)) of Theorem 5.2 holds.

Turning to the proof of the second part of the corollary let us observe that, for the point x appearing in its formulation, we have

$$\begin{aligned}
&g_0(x) - g_0(x_0) = f(x) - f(x_0) = 0,\\
&g_i(x) - g_i(x_0) \le 0, \quad i \in I(x_0) \setminus \{j\},\\
&g_j(x) - g_j(x_0) < 0.
\end{aligned}$$

Taking account of these conditions and making use of the invexity assumption we derive that there is a point $\eta(x) \in R^n$ such that $\xi := \eta(x)$ is a solution of system (15), (16). Applying Proposition 3.2 we obtain (63), as desired. □

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