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# UNIFORM ASYMPTOTICS IN THE EMPIRICAL MEAN RESIDUAL LIFE PROCESS

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ABSTRACT. In [5], Csörgő and Zitikis exposed the strong uniform-over-[0, $\infty$) consistency, and weak uniform-over-[0, $\infty$) approximation of the empirical mean residual life process by employing weight functions.

We carry on the uniform asymptotic behaviors of the empirical mean residual life process over the whole positive half line by representing the process as an integral form. We compare our results with those of Yang [15], Hall and Wellner [8], and Csörgő and Zitikis [5].

## 1. Introduction

Our motivation for this paper is looking at the empirical mean residual life process as an integral process over the whole positive half line. Yang [15], and Hall and Wellner [8] initiated investigations of the asymptotic behaviors of the empirical mean residual life process. They obtained results on the basis of a uniform on compact topology. In 1996, Csörgő and Zitikis [5] exposed the study of the mean residual life process over the whole positive half line. They establish the strong uniform-over-[0, $\infty$) consistency, and weak uniform-over-[0, $\infty$) approximation of the empirical mean residual life process by employing weight functions.

In this paper, we carry on a different approach of studying the uniform asymptotic behaviors of the empirical mean residual life process over the whole positive half line. More specifically, we view the empirical

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mean residual life process as an integral process over the whole positive half line and prove the Glivenko-Cantelli type uniform consistency, the Donsker-type weak convergence and the Strassen type relative compactness of the process. In the context of Glivenko-Cantelli type process, we obtain the uniform convergence over the whole positive half line in almost sure and in the mean sense. In the vein of Donsker-type process, we get an asymptotic Gaussianity regarded as random elements of the space of cadlag functions over the whole positive half line. In the point of Strassen type process, we get the relative compactness of the process and specify the set of limit points.

Our tool is a unified approach for an empirical process theory that can be represented as an integral form. In particular, our method of deriving uniform consistency and asymptotic Gaussianity of the empirical mean residual life process deviates the use of weight functions that were essentially used in Csörgő and Zitikis [5].

In Section 2, we illustrate the concepts of the empirical mean residual life process and describe the main results. In Section 3, we compare our results with those of Yang [15], Hall and Wellner [8], and Csörgő and Zitikis [5]. In Section 4, we supply the proofs of the results. Finally, in Section 5, we review the results of an integral process indexed by the real line.

## 2. The main results

We begin by introducing a mean residual life function. Let $\xi$ be a nonnegative random variable with the distribution function $F$ that, defined on a probability space $(\Omega, \mathcal{A}, P)$, represents a life time. Then the mean residual life function of $\xi$ is given by

$$M(x) := E(\xi - x \mid \xi > x) \text{ for } x \geq 0.$$

Let $\xi_1, \xi_2, \ldots$ be a sequence of independent copies of $\xi$ and $F_n(x)$ be the empirical distribution of $F$. Let $\xi^*$ denote the random variable having the distribution function $F_n$ when $\xi_1, \ldots, \xi_n$ are fixed. We consider the empirical mean residual life process $\{\mathbb{M}_n(x) : x \geq 0\}$ given by

$$\mathbb{M}_n(x) := E^*(\xi^* - x \mid \xi^* > x) \text{ for } x \geq 0,$$

where $E^*$ denotes the conditional expectation when $\xi_1, \ldots, \xi_n$ are fixed. Let $\xi_{1:n} \leq \cdots \leq \xi_{n:n}$ denote the order statistics of $\xi_1, \ldots, \xi_n$.

Our goal is to establish that the centered empirical mean residual life process

$$\{(\mathbb{M}_n - M)(x) : x \geq 0\}$$

holds uniform consistency, Glivenko [7] and Cantelli [4],

$$\{n^{1/2}(\mathbb{M}_n - M)(x) : x \geq 0\}$$

satisfies uniform asymptotic Gaussianity, Donsker [6] and

$$\{(2 \log \log n)^{-1/2} n^{1/2}(\mathbb{M}_n - M)(x) : x \geq 0, n \geq 3\}$$

is uniformly relative compact, Strassen [11].

We view the process $\{(\mathbb{M}_n - M)(x) : x \geq 0\}$ as an integral process. See Stute and Wang [13] and Stute [12] for a strong law of large numbers and central limit theorem for Kaplan-Meier integral. See also Bae and Kim [1], Bae and Kim [2], and Bae and Kim [3] for uniform asymptotic properties for Kaplan-Meier integral process.

We now introduce a class of functions $\{\varphi_x : x \geq 0\}$ defined by

$$\varphi_x(u) := (u - x)1_{(x,\ \infty)}(u).$$

Then we notice that for each $x \geq 0$ the function $\varphi_x(u)$ is bounded by an envelope $\varphi_0(u) := u1_{(0,\infty)}(u)$ and

$$\int_x^\infty (1 - F(u))du = \int \varphi_x(u)F(du).$$

Let $x_F$ be the, possibly infinite, end of the support of $F$, defined by

$$x_F := \inf\{x : F(x) = 1\}.$$

Then we get the integral representations of $M$ and $\mathbb{M}_n$ as

$$M(x) := \frac{1_{[0,\ x_F)}(x)}{1 - F(x)} G(x) \text{ for } x \geq 0$$

and

$$\mathbb{M}_n(x) := \frac{1_{[0,\ \xi_{n:n})}(x)}{1 - F_n(x)} G_n(x) \text{ for } x \geq 0,$$

where

$$G(x) := \int \varphi_x(u)F(du)$$

and

$$G_n(x) := \int \varphi_x(u)F_n(du).$$

Notice further that, for each $x \geq 0$

$$
\begin{aligned}
(1) \qquad & (\mathbb{M}_n - M)(x) \\
&= \frac{1_{[0,\ \xi_{n:n})}(x)}{1 - F_n(x)} G_n(x) - \frac{1_{[0,\ \xi_{n:n})}(x)}{1 - F_n(x)} G(x) \\
&\quad + \frac{1_{[0,\ \xi_{n:n})}(x)}{1 - F_n(x)} G(x) - \frac{1_{[0,\ x_F)}(x)}{1 - F(x)} G(x) \\
&= \frac{1_{[0,\ \xi_{n:n})}(x)}{1 - F_n(x)} \left[G_n(x) - G(x)\right] + R_n(x) G(x),
\end{aligned}
$$

where

$$R_n(x) := \frac{1_{[0,\ \xi_{n:n})}(x)}{1 - F_n(x)} - \frac{1_{[0,\ x_F)}(x)}{1 - F(x)}.$$

We shall use the representation (1) in studying the uniform asymptotic behaviors of $\mathbb{M}_n - M$ viewed as a random process defined on the whole positive half line.

Firstly, we state a uniform consistency of $\mathbb{M}_n$ to $M$.

THEOREM 1. *Suppose that $E\xi < \infty$. Then the empirical mean residual life process*

$$\{\mathbb{M}_n(x) : x \geq 0\}$$

*is a uniform strong consistent estimator of $M$. That is,*

$$\sup_{x \geq 0} |\mathbb{M}_n(x) - M(x)| \to 0$$

*almost surely. The convergence can be strengthened to convergence in the mean.*

$$E \sup_{x \geq 0} |\mathbb{M}_n(x) - M(x)| \to 0.$$

Let $D[0, \infty)$ be the space of the cadlag functions defined on $[0, \infty)$. We endow the space with the Skorohod topology. We use the following weak convergence. See Van der Vaart and Wellner [14] for a recent reference.

DEFINITION 1. A sequence of $D[0, \infty)$-valued random functions $\{Y_n : n \geq 1\}$ converges in law to a $D[0, \infty)$-valued Borel measurable random function $Y$ whose law concentrates on a separable subset of $D[0, \infty)$, denoted by $Y_n \Rightarrow Y$, if

$$Eg(Y) = \lim_{n \to \infty} Eg(Y_n)$$

for all $g \in C(D[0, \infty), || \cdot ||)$, where $C(D[0, \infty), || \cdot ||)$ is the set of real bounded, continuous functions.

Let $\mathbb{Z} = \{\mathbb{Z}(x) : x \geq 0\}$ be the mean zero Gaussian process with covariance function

$$\mathrm{Cov}(\mathbb{Z}(x), \mathbb{Z}(y)) = \int_{x\vee y}^{\infty} (u-x)(u-y)F(du).$$

Let $B_F := \{B(F(x)) : x \geq 0\}$ denote the stretched-out Brownian bridge. Consider a Gaussian process $\mathbb{W} := \{\mathbb{W}(x) : x \geq 0\}$ defined as follows:

$$(2) \qquad \mathbb{W}(x) := \frac{\mathbb{Z}(x)1_{[0,\ x_F)}(x)}{1-F(x)} + \frac{B_F(x)\int_x^\infty (u-x)F(du)1_{[0,\ x_F)}(x)}{(1-F(x))^2}.$$

Secondly, we state a limiting Gaussian property for the process

$$\left\{n^{1/2}(\mathbb{M}_n - M)(x) : x \geq 0\right\}.$$

THEOREM 2. *Suppose that $E\xi^2 < \infty$. Then we have*

$$n^{1/2}(\mathbb{M}_n - M) \Rightarrow \mathbb{W}$$

*as random elements of $D[0,\infty)$.*

Let $C$ be the space of real valued bounded continuous functions defined on $[0,\ \infty)$ equipped with the sup norm. Let

$$\mathcal{U} = \{g : \int g^2 dF \leq 1\}.$$

Let

$$\mathcal{U}([0,\ \infty)) := \{\eta \in C : \eta(x) \mapsto \sigma^2(x,g) \text{ for } g \in \mathcal{U}\ \},$$

where

$$\sigma^2(x,g) = \frac{1_{[0,\ x_F)}(x)\int_x^\infty (u-x)g(u)F(du)}{1-F(x)}\left[\frac{\int_x^\infty (u-x)F(du)}{1-F(x)} - 1\right].$$

Thirdly, we state the uniform law of the iterated logarithm for the empirical mean residual life process.

THEOREM 3. *Suppose that $E\xi^2 < \infty$. Then almost surely the sequence*

$$\left\{(2\log\log n)^{-1/2} n^{1/2}(\mathbb{M}_n - M)(x) : x \geq 0, n \geq 3\right\}$$

*is relatively compact and the set of its limit points is $\mathcal{U}([0,\ \infty))$.*

## 3. Comparison of the results

We firstly compare our results to those of Yang [15], and Hall and Wellner [8] where they developed uniform consistency and weak uniform approximation over the compact interval.

The following result in Yang [15], and Hall and Wellner [8] is regained as a corollary of Theorem 1.

COROLLARY 1. (Yang [15], Hall and Wellner [8]) *For every fixed* $x_0 < x_F$, *as* $n \to \infty$,

$$\sup_{x \in [0,x_0]} |\mathbb{M}_n - M|(x) \to 0$$

*almost surely.*

*Proof.* For every fixed $x_0 < x_F$, we get

$$\sup_{x \in [0,x_0]} |\mathbb{M}_n - M|(x) \leq \sup_{x \geq 0} |\mathbb{M}_n(x) - M(x)|.$$

Theorem 1 completes the proof. □

Let $\{\mathbb{W}_n : n \geq 1\}$ be a sequence of independent copies of Gaussian process $\mathbb{W}$ given in (2). We observe that the Gaussian process $\mathbb{W}_n$ has the same distribution as that of

$$\frac{1}{1-F(x)} B_n(F(x))M(x) - \frac{1}{1-F(x)} \int_x^\infty B_n(F(u))du$$

for all $x \in [0, x_F)$ and $\mathbb{W}_n(x) := 0$, where $\{B_n : n \geq 1\}$ denotes the sequence of Brownian bridges as in Csörgő and Zitikis [5].

The following result in Yang [15], Hall and Wellner [8] is regained as a corollary of Theorem 2.

COROLLARY 2. (Yang [15], Hall and Wellner [8]) *For every fixed* $x_0 < x_F$, *as* $n \to \infty$,

$$\sup_{x \in [0,x_0]} n^{1/2} |\mathbb{M}_n - M - \mathbb{W}_n|(x) \to 0$$

*in probability.*

*Proof.* For every fixed $x_0 < x_F$, we get

$$\sup_{x \in [0,x_0]} n^{1/2} |\mathbb{M}_n - M - \mathbb{W}_n|(x) \leq \sup_{x \geq 0} n^{1/2} |\mathbb{M}_n - M - \mathbb{W}_n|(x).$$

Theorem 2 and an application of continuous mapping theorem complete the proof. □

We secondly compare our results to those of Csörgő and Zitikis [5] where they developed uniform consistency and weak uniform approximation over the whole positive half line by describing classes of weight functions. See Csörgő and Zitikis [5] for the comparison with those of Yang [15], and Hall and Wellner [8].

In order to relate our result to those of Csörgő and Zitikis [5] we illustrate the notion of the weight functions. Assume that weight function $q : [0,1] \to [0,\infty]$ is measurable and, for every $\delta > 0$,

$$\sup\{q(t) : t \in [0, 1-\delta]\} < \infty. \tag{3}$$

Assume, as well as (3), that the function

$$t \mapsto \frac{q(t)}{1-t} \log_2 \frac{1}{1-t} \tag{4}$$

is non-decreasing in a neighborhood of 1. Here $\log_2 x := \log(e \vee \log(e \vee x))$.

The following result in Csörgő and Zitikis [5] is regained as a corollary of Theorem 1.

COROLLARY 3. (Corollary 1.2 in Csörgő and Zitikis [5]) *If $q$ is the function*

$$t \mapsto C \cdot (1-t)/\log_2\{1/(1-t)\},$$

*then*

$$\sup_{x \geq 0} q(F(x))|\mathbb{M}_n - M|(x) \to 0$$

*almost surely.*

*Proof.* For the choice of such $q$, we see that the function in (4) becomes a constant function $C$. Therefore we get

$$q(F(x)) = C \cdot (1-F(x))/\log_2\{1/(1-F(x))\}.$$

Now, observing the limiting behavior of the right hand side near $x_F$, we conclude that

$$\sup_{x \geq 0} q(F(x)) < \infty.$$

Therefore we have

$$\sup_{x\geq 0} q(F(x))|\mathbb{M}_n - M|(x) \leq \sup_{x \geq 0} q(F(x)) \cdot \sup_{x \geq 0} |\mathbb{M}_n - M|(x).$$

Theorem 1 completes the proof. □

Assume, as well as (3), that the function

$$t \mapsto \frac{q(t)}{1-t} \tag{5}$$

is non-decreasing in a neighborhood of 1, and $E\xi^2 < \infty$.

The following result in Csörgő and Zitikis [5] is regained as a corollary of Theorem 2.

COROLLARY 4. (Corollary 1.6 in Csörgő and Zitikis [5]) *If* $q$ *is the function*

$$t \mapsto C \cdot (1-t),$$

*then*

$$\sup_{x \geq 0} q(F(x)) n^{1/2} |\mathbb{M}_n - M - \mathbb{W}_n|(x) \to 0$$

*in probability.*

*Proof.* As in the proof of Corollary 3 we see that

$$\sup_{x \geq 0} q(F(x)) < \infty.$$

Then, from Theorem 2, we conclude that

$$\begin{aligned} &\sup_{x \geq 0} q(F(x)) n^{1/2} |\mathbb{M}_n - M - \mathbb{W}_n|(x) \\ \leq\ &\sup_{x \geq 0} q(F(x)) \cdot \sup_{x \geq 0} n^{1/2} |\mathbb{M}_n - M - \mathbb{W}_n|(x) \to 0 \end{aligned}$$

in probability. The proof is completed. □

REMARK 1. We do not know whether our Theorem 1 (Theorem 2) implies Theorem 1.1 (Theorem 1.2) in Csörgő and Zitikis [5]. However, for the choice of $q$ and $F$ satisfying

$$\sup_{x \geq 0} q(F(x)) < \infty,$$

we can get the almost sure convergence of

$$\sup_{x \geq 0} q(F(x)) |\mathbb{M}_n - M|(x) \to 0$$

and convergence in probability of

$$\sup_{x \geq 0} q(F(x)) n^{1/2} |\mathbb{M}_n - M - \mathbb{W}_n|(x) \to 0$$

by using our Theorem 1 and Theorem 2. See the proof of Corollary 3 and Corollary 4.

## 4. Proof of the results

Observe that for $x \geq 0$

$$
\begin{aligned}
(6) \qquad & R_n(x) \\
= \;& \frac{1_{[0,\ \xi_{n:n})}(x)}{1-F_n(x)} - \frac{1_{[0,\ \xi_{n:n})}(x) + 1_{[\xi_{n:n},\ x_F)}(x)}{1-F(x)} \\
= \;& 1_{[0,\ \xi_{n:n})}(x)\left[\frac{1}{1-F_n(x)} - \frac{1}{1-F(x)}\right] - \frac{1_{[\xi_{n:n},\ x_F)}(x)}{1-F(x)} \\
= \;& 1_{[0,\ \xi_{n:n})}(x)\frac{(F_n(x)-F(x))}{(1-F_n(x))(1-F(x))} - \frac{1_{[\xi_{n:n},\ x_F)}(x)}{1-F(x)} \\
:= \;& R_{n1}(x) - R_{n2}(x).
\end{aligned}
$$

We begin by the following Lemma 1.

LEMMA 1. *As* $n \to \infty$,

$$n^{1/2} 1_{[\xi_{n:n},\ x_F)}(\cdot) \Rightarrow 0$$

*as random elements of* $D[0,\ \infty)$. *Hence,*

$$\sup_{x \geq 0} 1_{[\xi_{n;n},\ x_F)}(x) \to 0$$

*almost surely and in the mean.*

*Proof.* Observe first that for $x < x_F$

$$
\begin{aligned}
En^{1/2} 1_{[\xi_{n:n},\ x_F)}(x) &= n^{1/2} P(\xi_{n:n} \leq x) \\
&= n^{1/2} [F_\xi(x)]^n \\
&\to 0
\end{aligned}
$$

since $0 \leq F_\xi(x) < 1$. This together with Cramer-Wold device verifies the finite dimensional distributions of $\{n^{1/2} 1_{[\xi_{n:n},\ x_F)}(x) : x \geq 0\}$ converge to those of 0. In order to prove the weak convergence of $n^{1/2} 1_{[\xi_{n:n},\ x_F)}(\cdot)$, we need to prove a tightness of the following form. Given $\epsilon > 0$ we can find a $\delta > 0$ such that

$$\limsup_{n\to\infty} P\left\{\sup_{|y-x|<\delta} n^{1/2} |1_{[\xi_{n:n},\ x_F)}(x) - 1_{[\xi_{n:n},\ x_F)}(y)| > \epsilon\right\} < \epsilon.$$

See Theorem 10.2 in Pollard [10]. Observe that

$$1_{[\xi_{n:n},\ x_F)}(x) = 1_{[0,\ x]}(\xi_{n:n}) \text{ for } x \geq 0.$$

Now, let $\epsilon > 0$. Choose $0 < \delta < x_F$ so that $0 < y - x < \delta$. Then, we get

$$\begin{aligned} & P\left\{\sup_{y-x<\delta} n^{1/2}|1_{[0,\ x]}(\xi_{n:n}) - 1_{[0,\ y]}(\xi_{n:n})| > \epsilon\right\} \\ = & P\left\{\sup_{y-x<\delta} n^{1/2}1_{(x,\ y]}(\xi_{n:n}) > \epsilon\right\} \\ \leq & P\left\{\sup_{x\geq 0} n^{1/2}1_{(x,\ x+\delta]}(\xi_{n:n}) > \epsilon\right\}. \end{aligned}$$

By reasons of symmetry, or stationarity, the last probability reduces to $P\left\{n^{1/2}1_{(0,\ \delta]}(\xi_{n:n}) > \epsilon\right\}$. Next, observe that

$$\begin{aligned} \epsilon P\left\{n^{1/2}1_{(0,\ \delta]}(\xi_{n:n}) > \epsilon\right\} & \leq n^{1/2}E1_{(0,\ \delta]}(\xi_{n:n}) \\ & = n^{1/2}P(0 < \xi_{n:n} \leq \delta) \\ & \leq n^{1/2}P(\xi_{n:n} \leq \delta) \\ & = n^{1/2}[F_\xi(\delta)]^n \\ & \to 0 \end{aligned}$$

since $0 \leq F_\xi(\delta) < 1$. The proof that $n^{1/2}1_{[\xi_{n:n}, x_F)}(\cdot) \Rightarrow 0$ is completed. The fact that convergence of $\sup_{x\geq 0} 1_{[\xi_{n;n},\ x_F)}(x)$ to zero in probability follows from a continuous mapping theorem. Almost sure convergence can be deduced form the monotonicity of $\sup_{x\geq 0} 1_{[\xi_{n;n},\ x_F)}(x)$ (monotone decreasing in $n$) and convergence in probability. Finally the convergence in the mean follows from the uniform integrability. □

LEMMA 2. *Suppose that $E\xi < \infty$. Then,*

$$\sup_{x\geq 0}|F_n - F|(x) \to 0$$

*almost surely and in the mean. Hence,*

$$\sup_{x\geq 0}|R_n|(x) \to 0$$

*almost surely and in the mean.*

*Proof.* The almost sure and mean convergence of $\sup_{x\geq 0}|F_n - F|(x)$ to zero can be gained as a corollary of Proposition 1 by applying $\varphi(x,\cdot) = 1_{[0,\ x]}(\cdot)$. This together with the uniform convergence of $1_{[0,\ \xi_{n:n})}(\cdot)$ to $1_{[0,\ x_F)}(\cdot)$ of Lemma 1 implies that $\sup_{x\geq 0}|R_n|(x) \to 0$ almost surely and in the mean. The proof is completed. □

LEMMA 3. *Suppose that $E\xi^2 < \infty$. Then,*

$$n^{1/2}(F_n - F) \Rightarrow B_F(\cdot)$$

*as random elements of $D[0, \infty)$. Hence,*

$$n^{1/2}R_n(\cdot) \Rightarrow \frac{B_F(\cdot)}{(1-F)^2} 1_{[0,\, x_F)}(\cdot).$$

*as random elements of $D[0, \infty)$.*

*Proof.* By applying Proposition 2 with $\varphi(x, \cdot) = 1_{[0,\, x]}(\cdot)$ we get $n^{1/2}(F_n - F) \Rightarrow B_F(\cdot)$ as random elements of $D[0, \infty)$. Now, the result that

$$n^{1/2}R_{n1}(\cdot) \Rightarrow \frac{B_F(\cdot)}{(1-F)^2} 1_{[0,\, x_F)}(\cdot)$$

is easily followed. The fact that $n^{1/2}R_{n2}(\cdot) \Rightarrow 0$ follows from Lemma 1. The proof, from (6), is completed. □

LEMMA 4. *Suppose that $E\xi^2 < \infty$. Then almost surely the sequence*

$$\{(2 \log \log n)^{-1/2} n^{1/2}(F_n - F)(x) : x \geq 0, n \geq 3\}$$

*is relatively compact and the set of its limit points is*

$$\left\{ y \in C : y(x) \mapsto \int_0^x g dF \text{ for } g \text{ with } \textstyle\int g^2 dF \leq 1 \right\}.$$

*Hence almost surely the sequence*

$$\left\{ (2 \log \log n)^{-1/2} n^{1/2} R_n(x) : x \geq 0, n \geq 3 \right\}$$

*is relatively compact and the set of its limit points is*

$$\left\{ y \in C : y(x) \mapsto \frac{1_{[0,\, x_F)}(x) \int_0^x g dF}{(1 - F(x))^2} \text{ for } g \text{ with } \textstyle\int g^2 dF \leq 1 \right\}.$$

*Proof.* The uniform law of the iterated logarithm for $F_n - F$ can be obtained as a corollary of Proposition 3 by applying $\varphi(x, \cdot) = 1_{[0,\, x]}(\cdot)$. The result for $R_n$ can be deduced from (6). The proof is completed. □

*Proof of Theorem. 1.* The result follows from (1) and Lemma 2. □

*Proof of Theorem. 2.* The result follows from (1) and Lemma 3. □

*Proof of Theorem. 3.* The result follows from (1) and Lemma 4. □

## 5. Uniform asymptotic behaviors of the integral process

In this section we review a unified approach of the uniform asymptotic behaviors of the integral representation of the real line indexed empirical process.

Let $\xi$ be a random variable defined on a probability space$(\Omega, \mathcal{A}, P)$ with the distribution function $F$, and let $\xi_1, \xi_2, \ldots$ be a sequence of independent copies of $\xi$. Let $\mathcal{B}$ denote the Borel $\sigma$-field on $\mathbb{R}$. Let $\varphi : \mathbb{R} \otimes \Omega \to \mathbb{R}$ be a function which is measurable $\mathcal{B} \otimes \mathcal{A}$ and assume that the range of $\varphi$ is *separable* in the sense that there is a measurable set $N \subset \Omega$ with $P(N) = 0$ such that for any open set $\mathcal{O} \subset \mathbb{R}$ and any closed set $F \subset \mathbb{R}$,

$$[\varphi(x, \xi) \in F \text{ for all } x \in \mathcal{O} \cap \mathbb{Q}] \setminus [\varphi(x, \xi) \in F \text{ for all } x \in \mathcal{O}] \subset N.$$

We introduce the integral form of the empirical process

$$\mathbb{G}_n(x) := \int \varphi(x, u) \mathbf{P}_n(du) = n^{-1} \sum_{i=1}^{n} \varphi(x, \xi_i) \text{ for } x \in \mathbb{R},$$

and the integral form of the mean functions

$$\mathbb{G}(x) := \int \varphi(x, u) P_\xi(du) \text{ for } x \in \mathbb{R},$$

where $P_\xi$ denote the probability measure induced by the distribution function $F$ and $\mathbf{P}_n(\cdot) = n^{-1} \sum_{i=1}^{n} \delta_{\xi_i}(\cdot)$ denote the empirical measure.

We firstly state a uniform strong law of the large numbers for the process

$$\{(\mathbb{G}_n - \mathbb{G})(x) : x \in \mathbb{R}\}$$

of Glivenko-Cantelli type under the assumption of $\int |\varphi(x, u)| P_\xi(du) < \infty$ for each $x \in \mathbb{R}$.

The following proposition is real line indexed modification of Corollary 3 in Bae and Kim [1].

PROPOSITION 1. *Suppose* $\int |\varphi(x, u)| P_\xi(du) < \infty$ *for each* $x \in \mathbb{R}$. *Then*

$$\sup_{x \in \mathbb{R}} |\mathbb{G}_n - \mathbb{G}|(x) \to 0$$

*almost surely. The convergence can be strengthened to convergence in the mean.*

$$E \sup_{x \in \mathbb{R}} |\mathbb{G}_n - \mathbb{G}|(x) \to 0.$$

Consider

$$\left\{\mathbb{D}_n(x) := n^{1/2} \int \varphi(x,u)(\mathbf{P}_n - P_\xi)(du) : x \in \mathbb{R}\right\}.$$

Let $\mathbb{Z}_\varphi = \{\mathbb{Z}_\varphi(x) : x \in \mathbb{R}\}$ be the mean zero Gaussian process with covariance function

$$\text{Cov}(\mathbb{Z}_\varphi(x), \mathbb{Z}_\varphi(y)) = \int \varphi(x,u)\varphi(y,u)F(du). \tag{7}$$

Establishing a uniform central limit theorem for the process $D_n$ means showing that $\mathcal{L}(D_n(x) : x \in \mathbb{R}) \to \mathcal{L}(\mathbb{Z}_\varphi(x) : x \in \mathbb{R})$, where the processes are considered as random elements of $D(\mathbb{R})$, the space of the cadlag functions defined on $\mathbb{R}$, equipped with the Skorohod metric on $D(\mathbb{R})$. $\mathbb{Z}_\varphi$ is a Gaussian process whose sample paths are continuous.

Secondly, we state a uniform central limit theorem for the process

$$\{n^{1/2}(\mathbb{G}_n - \mathbb{G})(x) : x \in \mathbb{R}\}$$

of Donsker type under the assumption of $\int \varphi^2(x,u)P_\xi(du) < \infty$ for each $x \in \mathbb{R}$. The following proposition is real line indexed modification of Corollary 2 in Bae and Kim [2]. See also Theorem 3.1 of Ossiander [9].

PROPOSITION 2. *Suppose $\int \varphi^2(x,u)P_\xi(du) < \infty$ for each $x \in \mathbb{R}$. Then*

$$n^{1/2}(\mathbb{G}_n - \mathbb{G}) \Rightarrow \mathbb{Z}_\varphi$$

*as random elements of $D(\mathbb{R})$. The limiting process*

$$\mathbb{Z}_\varphi = \{\mathbb{Z}_\varphi(x) : x \in \mathbb{R}\}$$

*is a mean zero Gaussian process and the covariance function is given in* (7). *The sample paths of $\mathbb{Z}_\varphi$ are continuous.*

Let $C = C(\mathbb{R})$ be the space of real valued bounded continuous functions defined on $\mathbb{R}$ equipped with the sup norm. Let

$$\mathcal{U} := \{g : \int g^2 dF \le 1\}$$

and

$$\mathcal{U}_\varphi(\mathbb{R}) = \left\{y \in C : y(x) \mapsto \int \varphi(x,u)g(u)F(du) \text{ for } g \in \mathcal{U}\right\}.$$

Thirdly, we state the uniform law of the iterated logarithm for the process

$$\left\{(2\log\log n)^{-1/2} n^{1/2}(\mathbb{G}_n - \mathbb{G})(x) : n \ge 3, x \in \mathbb{R}\right\}$$

of Strassen type under the assumption of $\int \varphi^2(x,u)P_\xi(du) < \infty$ for each $x \in \mathbb{R}$. The following proposition is real line indexed modification of Corollary 2 in Bae and Kim [3].

PROPOSITION 3. *Suppose that* $\int \varphi^2(x,u)P_\xi(du) < \infty$ *for each* $x \in \mathbb{R}$. *Then almost surely the sequence*

$$\left\{(2\log\log n)^{-1/2}n^{1/2}\int \varphi(x,u)(\mathbf{P}_n - P_\xi)(du) : x \in \mathbb{R}, n \geq 3\right\}$$

*is relatively compact with respect to the sup norm and the set of its limit points is* $\mathcal{U}_\varphi(\mathbb{R})$.

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