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PARTIALLY ASHPHERICAL MANIFOLDS WITH NONZERO EULER CHARACTERISTIC AS PL FIBRATORS

Young Ho Im and Yongkuk Kim

Abstract. Approximate fibrations form a useful class of maps. By definition fibrators provide instant detection of maps in this class, and PL fibrators do the same in the PL category. We show that every closed s-hopfian t-aspherical manifold N with sparsely Abelian, hopfian fundamental group and $\chi(N) \neq 0$ is a codimension-$(t+1)$ PL fibrator.

1. Introduction

We seek to identify homotopy types by means of which a proper map defined on an arbitrary manifold of a given dimension can be quickly recognized as an approximate fibration, simply because all point preimages have the specified homotopy type. More precisely, the goal is to present closed n-manifolds N which force proper maps $p : M \to B$ to be approximate fibrations, when M is a connected $(n+k)$-manifold and each $p^{-1}(b)$ has the homotopy type (or, more generally, the shape) of N. Such a manifold N is called a codimension-k fibrator.

It is well known that every closed s-hopfian manifold N with hopfian $\pi_1(N)$ and $\chi(N) \neq 0$ is a codimension-2 fibrator (see [9, Proposition 2.4]). In general, these codimension-2 fibrators need not to be codimension-k ($k > 2$) fibrators. For example, the real projective plane $\mathbb{RP}^2$ is not a codimension-3 fibrator although $\pi_1(\mathbb{RP}^2) \cong \mathbb{Z}_2$ is hopfian and $\chi(\mathbb{RP}^2) = 1 \neq 0$ [5].

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Normally cohopficity and sparsely Abelianness on $\pi_1(N)$ are indispensable for those codimension-2 fibrators to be codimension-k ($k > 2$) fibrators. In this paper, without considering normally cohopficity on $\pi_1(N)$, we show that a manifold N with $\chi(N) \neq 0$ can be a codimension-k ($k > 2$) fibrator. Since it is well-known that all closed aspherical manifolds with nonzero Euler characteristic always have normally cohopfian fundamental groups [11], to fit our purpose, we consider a partially aspherical manifolds with nonzero Euler characteristic. Of course, not every manifolds N with $\chi(N) \neq 0$, such as some 4-manifolds N with $\pi_1(N) = \mathbb{Z}_2 * \mathbb{Z}_2$, have normally cohopfian $\pi_1(N)$.

By looking at the covering spaces of M instead of considering M itself, we have to take care of degree of maps between different manifolds with same dimensions (see Lemma 2.1 below and the proof of Proposition 2.6). As a result, we get the main result, Theorem 2.9, which promises that every closed s-hopfian t-aspherical manifold N with sparsely Abelian, hopfian $\pi_1(N)$ and $\chi(N) \neq 0$ is a codimension-$(t+1)$ fibrator.

2. Manifolds with nonzero Euler characteristic as PL fibrators

Throughout this paper, the symbols χ, $\approx$ and $\cong$ denote Euler characteristic, homeomorphism and isomorphism in that order, and homology groups will be computed with integer coefficients unless specified.

We begin by presenting the notation and fundamental terminology to be employed throughout: M is a connected $(n+k)$-manifold and $p : M \to B$ is a proper map of M to a space B such that each $p^{-1}(b)$ has the homotopy type (or, more generally, the shape) of a closed, connected n-manifold. Such a map p will be called a *codimension-k* map. When N is a fixed PL n-manifold, M is a PL manifold, B is a polyhedron, and $p : M \to B$ is a PL map, then p is said to be *N-like* if each $p^{-1}(b)$ collapses to an n-complex homotopy equivalent to N (denoted by $p^{-1}(b) \sim N$). (This PL tameness feature, which seems just as effective as requiring $p^{-1}(b)$ actually to be an n-manifold, imposes significant homotopy-theoretic relationships, revealed in [4, Lemma 2.4], between N and preimages of links in B.) We call N a *codimension-k PL fibrator* if, for every PL $(n+k)$-manifold M and N-like PL map $p : M \to B$, p is an approximate fibration. Similarly, we call N a *codimension-k orientable PL fibrator* if this holds for all orientable, PL $(n+k)$-manifolds M, which we abbreviate by writing that N is a codimension-k PL o-fibrator. Finally, if N is a codimension-k PL fibrator (respectively, codimension-k

PL o-fibrator) for all $k > 0$, we simply call N a *PL fibrator* (respectively, *PL o-fibrator*).

An ANR Y is said to be *t-aspherical* if $\pi_i(Y) \cong 0$ whenever $1 < i \leq t$.

A group G is said to be: *hopfian* if each epimorphism $G \to G$ is an isomorphism; *cohopfian* if each monomorphism $G \to G$ is an isomorphism; and *normally cohopfian* if each monomorphism $G \to G$ with image a normal subgroup of G is an isomorphism. A group G is *sparsely Abelian* if it contains no non-trivial Abelian normal subgroup A such that G/A is isomorphic to a normal subgroup of G.

A codimension-k map $p : M^{n+k} \to B$ is said to have Property $\mathcal{R}_i^{\cong}$ ($\mathcal{R}_i^{\geq}, \mathcal{R}_i^{\leq}$) if for each $x \in B$, a retraction $\mathcal{R} : p^{-1}(U) \to p^{-1}(x)$ defined on some open neighborhood U of x in B induces an isomorphism(epimorphism, monomorphism, resp.) $(\mathcal{R}|p^{-1}(y))_* : H_i(p^{-1}(y)) \to H_i(p^{-1}(x))$ for all $y \in U$. The (absolute) degree of a map is computed with integer coefficients and is understood to be a nonnegative number. Explicitly, a map $f : N \to N'$ between closed, orientable n-manifolds is said to have *degree* d if there are choices of generators $\gamma \in H_n(N;\mathbb{Z}), \gamma' \in H_n(N';\mathbb{Z})$ such that $f_*(\gamma) = d\gamma'$, where $d \geq 0$ is an integer.

A closed, orientable manifold N is said to be *hopfian* if every degree 1 map $N \to N$ which induces an isomorphism at the fundamental group level is a homotopy equivalence. As a result, when $\pi_1(N)$ is a hopfian group, N is a hopfian manifold if and only if all degree 1 maps $N \to N$ are homotopy equivalences.

The *continuity set* of p consists of all $x \in B$ equipped with such a neighborhood U such that the associated $\mathcal{R} : p^{-1}(U) \to p^{-1}(x)$ restricts to an isomorphism $(\mathcal{R}|p^{-1}(y))_* : H_n(p^{-1}(y)) \to H_n(p^{-1}(x))$ for all $y \in U$. Establishing that B equals the continuity set of p is a cornerstone for showing an N-like map p is an approximate fibration.

LEMMA 2.1. [6, Proposition 2.1] *Let $p : M^{n+k} \to \mathbb{R}^k$ be a codimension-k map from an orientable $(n+k)$-manifold M^{n+k} onto Euclidian k-space such that p is an approximate fibration over $\mathbb{R}^k \setminus \mathbf{0}$. Then p has Property $\mathcal{R}_i^{\cong}$ for all $i \leq k-3$, and $\mathcal{R}_{k-2}^{\geq}$. Furthermore, if p has Property $\mathcal{R}_{k-2}^{\leq}$ and $\mathcal{R}_{k-1}^{\geq}$, then for all $y \in \mathbb{R}^k \setminus \mathbf{0}$, the degree of map $\mathcal{R}|p^{-1}(y) : p^{-1}(y) \to p^{-1}(\mathbf{0})$ is one.*

Proof. Let $y \in \mathbb{R}^k \setminus \mathbf{0}$. Since p is an approximate fibration over the homotopy $(k-1)$-sphere $\mathbb{R}^k \setminus \mathbf{0}$, the Serre exact sequence [12, p.519]

$$\cdots \to H_i(p^{-1}(y)) \to H_i(M \setminus p^{-1}(\mathbf{0})) \to H_i(\mathbb{R}^k \setminus \mathbf{0}) \cong 0 \ (i < k-1)$$

gives an isomorphism $H_i(p^{-1}(y)) \to H_i(M \setminus p^{-1}(\mathbf{0}))$ for $i \leq k-3$ and an epimorphism for $i = k-2$. The homology exact sequence of the pair $(M, M \setminus p^{-1}(\mathbf{0}))$ gives an isomorphism $H_i(M \setminus p^{-1}(\mathbf{0})) \to H_i(M)$ for $i \leq k-2$. Therefore, the inclusion $p^{-1}(y) \to M$ induces a composite homomorphism

$$\mathrm{incl}_* : H_i(p^{-1}(y)) \to H_i(M \setminus p^{-1}(\mathbf{0})) \to H_i(M),$$

which is an isomorphism for $i \leq k-3$ and an epimorphism for $i = k-2$. In this setting the inclusion $p^{-1}(\mathbf{0}) \to M$ is a homotopy equivalence: an appropriate deformation retraction of $\mathbb{R}^k$ into a neighborhood of $\mathbf{0}$ fixing a smaller neighborhood of $\mathbf{0}$ can be lifted to a deformation of M into a (preassigned) neighborhood W of $p^{-1}(\mathbf{0})$ fixing a smaller neighborhood of the latter [1, Proposition 1.5], and W can be selected to deform to $p^{-1}(\mathbf{0})$ in M. Consequently, $H_i(M) \cong H_i(p^{-1}(\mathbf{0}))$ for all i. It follows that $H_i(p^{-1}(0)) \to H_i(M)$ is an isomorphism, and p has property $\mathcal{R}_i^{\cong}$ for all $i \leq k-3$ and $\mathcal{R}_{k-2}^{\geq}$.

Furthermore, suppose p has Property $\mathcal{R}_{k-2}^{\leq}$ and $\mathcal{R}_{k-1}^{\geq}$. We demonstrate that $\mathcal{R}|p^{-1}(y)$ is a degree one map by verifying it gives a cohomology isomorphism between $H^n(p^{-1}(\mathbf{0})) \cong H_k(M, M \setminus p^{-1}(\mathbf{0}))$ and $H^n(p^{-1}(y)) \cong H_k(M, M \setminus p^{-1}(y))$, and then applying the universal coefficient theorem to obtain the same for homology. The key step involves showing that p induces an epimorphism $q : H_k(M, M \setminus p^{-1}(y)) \to H_k(\mathbb{R}^k, \mathbb{R}^k \setminus y)$ of pairs.

In the Serre exact sequence $H_{k-1}(p^{-1}(y)) \to H_{k-1}(M \setminus p^{-1}(\mathbf{0})) \to H_{k-1}(\mathbb{R}^k \setminus \mathbf{0}) \to H_{k-2}(p^{-1}(y)) \to H_{k-2}(M \setminus p^{-1}(\mathbf{0}))$, the final homomorphism is a monomorphism, for p has Property $\mathcal{R}_{k-2}^{\leq}$ and $H_{k-2}(M \setminus p^{-1}(\mathbf{0})) \to H_{k-2}(M)$ is an isomorphism. Hence we have an epimorphism

$$p' : H_{k-1}(M \setminus p^{-1}(\mathbf{0})) \to H_{k-1}(\mathbb{R}^k \setminus \mathbf{0}).$$

Consequently, from the following diagram of exact sequences

$$\begin{array}{ccccc}
 & & H_k(M, M \setminus p^{-1}(\mathbf{0})) & & \\
 & & \big\downarrow \partial & & \\
H_{k-1}(p^{-1}(y)) & \xrightarrow{\mathrm{incl}} & H_{k-1}(M \setminus p^{-1}(\mathbf{0})) & \xrightarrow[\text{onto}]{p'} & H_{k-1}(\mathbb{R}^k \setminus \mathbf{0}) \cong \mathbb{Z} \\
 & \text{onto} & \big\downarrow & & \\
 & & H_{k-1}(M) & &
\end{array}$$

we conclude that $H_{k-1}(M \setminus p^{-1}(\mathbf{0})) \cong incl_*(H_{k-1}(p^{-1}(y))) + im(\partial)$. To obtain the epimorphism of pairs, examine the homology ladder

$$\begin{array}{ccccc} \mathbb{Z} \cong H_k(M, M \setminus p^{-1}(\mathbf{0})) & \xrightarrow{\partial} & H_{k-1}(M \setminus p^{-1}(\mathbf{0})) & \longrightarrow & H_{k-1}(M) \\ {\scriptstyle q_*}\downarrow & & {\scriptstyle p'_*}\downarrow & & \\ \mathbb{Z} \cong H_k(\mathbb{R}^k, \mathbb{R}^k \setminus \mathbf{0}) & \xrightarrow{\cong} & H_{k-1}(\mathbb{R}^k \setminus \mathbf{0}). & & \end{array}$$

As $incl_*(H_{k-1}(p^{-1}(y))) \subset ker\ p'_*$, diagram chasing yields that p'_* carries $im(\partial)$ isomorphically onto $H_{k-1}(\mathbb{R}^k \setminus \mathbf{0})$. By a similar argument, p restricts to an isomorphism $H_{k-1}(M, M \setminus p^{-1}(y)) \to H_{k-1}(\mathbb{R}^k \setminus y)$. Specify a closed k-ball $D \subset \mathbb{R}^k$ containing $\mathbf{0}$ and y in its interior. One sees what is needed, the isomorphism between $H^n(p^{-1}(\mathbf{0}))$ and $H^n(p^{-1}(y))$, in the diagram below.

$$\begin{array}{ccccc} H^n(p^{-1}(\mathbf{0})) & \longleftarrow & H^n(p^{-1}(D)) & \longrightarrow & H^n(p^{-1}(y)) \\ \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} \\ H_k(M, M \setminus p^{-1}(\mathbf{0})) & \longleftarrow & H_k(M, M \setminus p^{-1}(D)) & \longrightarrow & H_k(M, M \setminus p^{-1}(y)) \\ \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \text{onto}} \\ H_k(\mathbb{R}^k, \mathbb{R}^k \setminus \mathbf{0}) & \xleftarrow{\cong} & H_k(\mathbb{R}^k, \mathbb{R}^k \setminus D) & \xrightarrow{\cong} & H_k(\mathbb{R}^k, \mathbb{R}^k \setminus y). \end{array}$$

□

REMARK 2.2. In Lemma 2.1 all fibers are not required to have the same homotopy type.

LEMMA 2.3. [4, Lemma 6.1] *Suppose N and N' are closed orientable n-manifolds such that $\beta_i(N) = \beta_i(N') > 0$ for some $0 < i < n$, and suppose $f : N \to N'$ is a map that induces isomorphisms $f_*| :$ free part$\{H_i(N)\} \to$ free part$\{H_i(N')\}$ and $f^*| :$ free part$\{H^{n-i}(N')\} \to$ free part$\{H^{n-i}(N)\}$. Then the degree of map f is one.*

Proof. Let $\eta \in H_n(N)$ and $\eta' \in H_n(N')$ be generators, and $f_*(\eta) = d\eta'$ for some integer d. Take an indivisible element $\xi' \in H_i(N')$. (Recall: $a \in H_{k-1}(M)$ is indivisible if $d \cdot a' = a$ for some $a' \in H_{k-1}(M)$ and $d \in \mathbb{Z}$ implies $d = \pm 1$.) Choose $\xi \in H_i(N)$ for which $f_*(\xi) = \xi'$. Identify $\nu \in H^{n-i}(N)$ such that $\eta \frown \nu = \xi$. Finally, find $\nu' \in H^{n-i}(N')$ such that $(f^*)(\nu') = \nu$. Naturality of Poincaré duality gives

$$\xi' = f_*(\xi) = f_*(\eta \frown f^*(\nu')) = f_*(\eta) \frown \nu' = d \cdot (\eta' \frown \nu'),$$

and indivisibility of ξ implies $d = \pm 1$. Hence, the degree of f is one. □

REMARK 2.4. The argument actually shows that the degree of map f is one merely if it induces isomorphisms of free part of H^{n-i} and if some indivisible element $\xi' \in H_i(N')$ belongs to the image of f_*.

LEMMA 2.5. *Suppose N is a codimension-$(k-1)$ PL o-fibrator with sparsely Abelian fundamental group and $\chi(N) \neq 0$ such that N is t-aspherical, where $k \leq t+1$. Suppose that $p : M^{n+k} \to B^k$ is an N-like PL map defined on an orientable manifold M^{n+k}. Then B is a k-manifold.*

Proof. Focus on a star S about a typical vertex $v \in B$, with corresponding link L. It suffices to show that the link L is a homotopy $(k-1)$-sphere. Being the image of $L' = p^{-1}(L)$ under an approximate fibration, L must be a closed $(k-1)$-manifold [3, Theorem 5.4]. See [4, Lemma 2.1] about its being a sphere for $k \leq 2$. So we will assume $k > 2$. We will show that $\pi_1(L) \cong 1$; then L is a homotopy $(k-1)$-sphere by the t-asphericity of N. In fact, consider the exact sequence

$$\cdots \to \pi_i(N) \to \pi_i(L') \to \pi_i(L) \to \pi_{i-1}(N) \to \cdots .$$

For $i \leq k-2 < t$, we have $\pi_i(N) \cong 0$. In the homotopy exact sequence of the approximate fibration $p \mid L'$,

$$\cdots \to \pi_2(L) \to \pi_1(N) \to \pi_1(L') \to \pi_1(L) \to 1,$$

we have a trivial homomorphism $\pi_2(L) \to \pi_1(N)$ and a monomorphism $\pi_1(N) \to \pi_1(L')$ because $\pi_1(N)$ is sparsely Abelian. Hence,

$$H_i(L) \cong \pi_i(L) \cong \pi_i(L') \cong \pi_i(S') \cong \pi_i(N) \cong 0$$

holds for the same range. Here the third isomorphism is obtained by the fact that the inclusion $L' \to S' = p^{-1}(S)$ induces isomorphisms $\pi_i(L') \to \pi_i(S')$ for $i \leq k-2$. The $(k-2)$-connectivity of L assures that L is a homotopy $(k-1)$-sphere.

Assume that L were not simply connected. Then in the homotopy exact sequence of the approximate fibration $p|L'$, $\pi_1(p^{-1}(z)) \to \pi_1(L')$ could not be surjective for $z \in L$. Form the covering $q : S'_I \to S'$ of S' corresponding to the image of $\pi_1(p^{-1}(z)) \to \pi_1(L') \to \pi_1(S')$. (Recall that the second homomorphism is an isomorphism.) Let N_I denote a cover of N corresponding to the image of

$$\pi_1(p^{-1}(z)) \to \pi_1(S') \to \pi_1((p^{-1}(v)) \cong \pi_1(N).$$

The inclusion $i = k \circ j : p^{-1}(z) \to S'$ lifts to $i_I : p^{-1}(z) \to S'_I$, which induces a π_1-epimorphism. Here $j : p^{-1}(z) \to L'_I$ and $k : L'_I \to S'_I$ are inclusions. Consider

$$\begin{array}{ccc} & & S'_I(\sim N_I) \\ & i_I & \Big\downarrow q \\ p^{-1}(z)(\sim N) & \xrightarrow{\quad i \quad} & S' \end{array}$$

Case 1: $[\pi_1(S'); i_\#(\pi_1(N))] = \infty$.

Since $\pi_1(N)$ is sparsely Abelian, in the homotopy exact sequence of the approximate fibration $p|L'$, $j : p^{-1}(z) \to L'_I$ induces a π_1-monomorphism and $k : L'_I \to S'_I$ induces a π_1-isomorphism, and so i_I induces a π_1-isomorphism. Set $L'_I = q^{-1}(L')$; L'_I is partitioned into copies of the various $p^{-1}(z)$, and the associated quotient map $\mu : L'_I \to L_I$ can be viewed as an N-like PL map, which is an approximate fibration, since locally over the base it looks just like $p|L'$. Inspection of the homotopy exact sequence for μ reveals that $\pi_1(L_I)$ is trivial – $\pi_1(N) \to \pi_1(L'_I)$ is surjective because $\pi_1(L'_I) \to \pi_1(S'_I)$ is an isomorphism. Since N is t-aspherical, we have $\pi_i(L_I) = 0$ for $1 \le i \le k-2$. Being L_I an infinite covering space of L, L_I is contractible by the Hurewicz theorem. The approximate fibration $L'_I \to L_I$ shows that L'_I has the same homotopy type as N. We will show that $H_i(N_I)$ is finitely generated for all i. For $i = 0, 1$, $H_i(N_I)$ is obviously finitely generated. Assume $H_j(N_I)$ is finitely generated for $j < i$. Consider the homology exact sequence for the pair (S'_I, L'_I);

$$H_i(L'_I) \to H_i(S'_I) \to H_i(S'_I, L'_I).$$

$H_i(L'_I) \cong H_i(N)$ and $H_i(S'_I, L'_I) \cong H_c^{n+k-i}(N_I) \cong H_{i-k}(N_I)$ are finitely generated. Hence, $H_i(S'_I) \cong H_i(N_I)$ is finitely generated. By Milnor [10], $\chi(N) = 0$. This is impossible.

Case 2: $[\pi_1(S'); i_\#(\pi_1(N))] < \infty$.

From the above diagram in the case 1, i_I induces a π_1-isomorphism. In the homotopy exact sequence of the approximate fibration $L'_I \to L_I$, L_I is simply connected. Like the argument that L is a homotopy $(k-1)$-sphere when L is simply connected, by the t-asphericity of N and the Poincaré Duality , L_I is a homotopy $(k-1)$-sphere and i_I induces π_i-isomorphism for $1 \le i \le t$. By the Whitehead theorem, $(i_I)_* : H_i(N) \to H_i(S'_I)$ is an isomorphism for $1 \le i \le t$. Lemma 2.1 implies that i_I is a degree one map. Since degree of i_I is 1 and degree of q is positive, $\beta_i(N) \ge \beta_i(N_I) \ge \beta_i(N)$ for all i. Then $\chi(N) = \chi(N_I)$ and q is a homeomorphism. Consequently, $i : N \to S'$ has degree 1 and induces a π_1-isomorphism. Hence, $\pi_1(L) = 1$ and L is a homotopy $(k-1)$-sphere. □

PROPOSITION 2.6. *Suppose N is a closed hopfian n-manifold with sparsely Abelian, hopfian fundamental group and $\chi(N) \neq 0$ such that N is t-aspherical. Then N is a codimension-$(t+1)$ PL o-fibrator.*

Proof. It is known that N is a codimension-2 fibrator provided N is a closed hopfian manifold with hopfian fundamental group and $\chi(N) \neq 0$ [2]. By induction we assume that N is a codimension-$(k-1)$ PL o-fibrator. Suppose $p : M^{n+k} \to B$ is an N-like PL map defined on an orientable manifold M^{n+k}, where $k \leq t+1$. By Lemma 2.5, B is a $k-$manifold. We can restrict $p : M^{n+k} \to B$ so M represents the preimage of the star S of an arbitrary vertex $v \in B$. Let L denote the associated link and $L' = p^{-1}(L)$. From the homotopy exact sequence of the approximate fibration $p|L'$, the inclusion $p^{-1}(z) \to L'$ induces a π_1-isomorphism. Then $p^{-1}(z) \to M \to p^{-1}(v)$ induces a π_1-isomorphism. By the t-asphericity of N and the Whitehead theorem, $p^{-1}(z) \to M \to p^{-1}(v)$ induces isomorphisms $H_i(p^{-1}(z)) \to H_i(p^{-1}(v))$ for all $i \leq i \leq t$. By Lemma 2.1 and [6, Corolary 2.2], p is an approximate fibration. □

COROLLARY 2.7. *Suppose $N = N_1\#N_2(\not\simeq \mathbb{RP}^n\#\mathbb{RP}^n)$ is a closed hopfian n-manifold with $\chi(N) \neq 0$ such that N is t-aspherical and $\pi_1(N_e) \cong \mathbb{Z}_2$ $(e = 1, 2)$. Then N is a codimension-(t+1) PL o-fibrator.*

Proof. It is known that $\pi_1(N)$ is hopfian and $\pi_1(N)$ has no nontrivial abelian normal subgroup, and so it is sparsely Abelian. □

The $(t+1)$-sphere, a PL o-fibrator in codimension-$(t+1)$ but not in codimension-$(t+2)$, illustrates the sharpness of Proposition 2.6. The product of any PL manifold in Corollary 2.7 and the $(t+1)$-sphere gives additional examples.

Our concluding results address PL fibrator properties, not simply PL o-fibrator properties. It involves the following approach for investigating non-orientable manifolds introduced in [8]. Let N be a closed n-manifold which has a 2-to-1 covering. Consider the covering space N_H of N corresponding to H, where $H = \bigcap_{i\in I} H_i$ with $[\pi_1(N) : H_i] = 2$ for $i \in I = \{i : [\pi_1(N) : H_i] = 2\}$. The index set I is finite, and N_H is a closed orientable n-manifold, since every (finite) covering of an n-dimensional orientable manifold is again orientable and all non-orientable manifolds have 2-to-1 orientable coverings. A closed manifold N is *s-hopfian* if

N is hopfian when N is orientable, and N_H is hopfian when N is non-orientable, where N_H is the covering space of N corresponding to $H = \bigcap_{i\in I} H_i$ with $I = \{i : [\pi_1(N) : H_i] = 2\}$. From now on, we reserve the symbols H and N_H to represent the above. Although an orientable N must be hopfian when N_H is, the converse is unknown. In a related setting, index 2 subgroups of hopfian groups need not be hopfian.

LEMMA 2.8. *Suppose N is a codimension-$(k-1)$ PL fibrator with sparsely Abelian fundamental group and $\chi(N) \neq 0$ such that N is t-aspherical, where $k \leq t+1$. Suppose that $p : M^{n+k} \to B^k$ is an N-like PL map defined on a manifold M^{n+k}. Then B is a k-manifold.*

Proof. Focus on a star S about a typical vertex $v \in B$, with corresponding link L. It suffices to show that the link L is a homotopy $(k-1)$-sphere. As in the proof of Lemma 2.5, it suffices to show that $\pi_1(L) \cong 1$. Let $z \in L$. In the homotopy exact sequence of the approximate fibration $p|L'$,

$$\pi_2(L) \to \pi_1(p^{-1}(z)) \to \pi_1(L') \to \pi_1(L) \to 1,$$

we have $\pi_1(L') \cong \pi_1(S') \cong \pi_1(p^{-1}(v))$ by the general position for the first isomorphism.

If L were not simply connected, the inclusion induced homomorphism $\pi_1(p^{-1}(z)) \to \pi_1(L')$ could not be surjective. Form the covering $q : S'_H \to S'$ of S' corresponding to the subgroup H of $\pi_1(S') \cong \pi_1(N)$. Consider

$$\begin{array}{ccccc}
N_K \simeq q^{-1}(p^{-1}(z))_C & \xrightarrow{\ \widetilde{i}\ } & S'_H & \xrightarrow{\ \widetilde{R}\ } & q^{-1}(p^{-1}(v)) \simeq N_H \\
\Big\downarrow q| & & \Big\downarrow q & & \Big\downarrow q| \\
p^{-1}(z) & \xrightarrow{\ i\ } & S' & \xrightarrow{\ R\ } & p^{-1}(v),
\end{array}$$

where N_H is the cover of N corresponding to H and N_K is the n-manifold corresponding to a component of $q^{-1}(p^{-1}(z))$. Then N_K is a covering space of N_H and orientable. Since $\pi_1(N)$ is sparsely Abelian, i induces a π_1-monomorphism, and so does $\widetilde{i}$. To show that i induces a π_1-isomorphism, it suffices to show that $\widetilde{i}$ induces a π_1−epimorphism. Suppose otherwise. Take the covering $\Theta : S'_{HI} \to S'_H$ corresponding to

$\widetilde{i}_{\#}(\pi_1(q^{-1}(p^{-1}(z))_C))$. Now consider

$$\begin{array}{ccccc}
 & i^* & S'_{HI} & \xrightarrow{R^*} & \Theta^{-1}(q^{-1}(p^{-1}(v))) \simeq N_{HI} \\
 & & \Big| \Theta & & \Big| \Theta| \\
N_K \simeq q^{-1}(p^{-1}(z))_C & \xrightarrow{\widetilde{i}} & S'_H & \xrightarrow{\widetilde{R}} & q^{-1}(p^{-1}(v)) \simeq N_H \\
\Big| q| & & \Big| q & & \Big| q| \\
p^{-1}(z) & \xrightarrow{i} & S' & \xrightarrow{R} & p^{-1}(v).
\end{array}$$

Here, i^* and R^* are liftings of $\widetilde{i}$ and $\widetilde{R} \circ \Theta$, respectively.

If Θ is the infinite covering map, then we have the conclusion by following the proof of the case 1 in Lemma 2.5. So we consider that Θ is the finite covering map. Since i^* induces a π_1-isomorphism, L_{HI} is the homotopy $(k-1)$-sphere as in the proof of Lemma 2.5. According to Lemma 2.1, $R^* \circ i^*$ is a degree one map. Because of $\chi(N) \neq 0$, N_H, N_K and N_{HI} are the same homotopy type. Hence, $\widetilde{R} \circ \widetilde{i}$ and $R \circ i$ are degree one maps and induce π_1-isomorphisms. □

As a result, we have the following main theorem.

THEOREM 2.9. *Suppose N is a closed $s-$hopfian n-manifold with sparsely Abelian, hopfian fundamental group and $\chi(N) \neq 0$ such that N is t-aspherical. Then N is a codimension-$(t+1)$ PL fibrator.*

Proof. It is known that N is a codimension-2 fibrator provided N is a closed *s-hopfian* manifold with hopfian fundamental group and $\chi(N) \neq 0$ [9, Proposition 2.4]. By the induction we assume that N is a codimension-$(k-1)$ PL fibrator$(k \geq 3)$. Suppose $p : M^{n+k} \to B$ is an N-like PL map defined on a manifold M^{n+k}, where $k \leq t+1$. By Lemma 2.8, B is a k-manifold. Upon forming the cover $\theta : M_H \to M$ corresponding to the image of $H \subset \pi_1(N) \cong \pi_1(p^{-1}(v))$ in $\pi_1(M)$, we see $p \circ \theta : M_H \to B$ is an N_H-like PL map and M_H is orientable, since it covers all possible 2-1 coverings of M. Following the proof of Proposition 2.6, $p \circ \theta$ and p are approximate fibrations. □

COROLLARY 2.10. *Every closed $(n-2)$-aspherical n-manifold N with sparsely Abelian, hopfian fundamental group and $\chi(N) \neq 0$ is a codimension-$(n-1)$ PL fibrator.*

Proof. Such a manifold is a closed $s-$hopfian manifold. □

COROLLARY 2.11. *Every closed aspherical manifold N with hopfian fundamental group and $\chi(N) \neq 0$ is a PL fibrator.*

Proof. The fundamental group of such a manifold has no nontrivial Abelian normal subgroup [11] so that $\pi_1(N)$ is sparsely Abelian. □

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Young Ho Im
Department of Mathematics
Pusan National University
Pusan 609-735, Korea
E-mail: yhim@pusan.ac.kr

Yongkuk Kim
Department of Mathematics
Kyungpook National University
Taegu 702-701, Korea
E-mail: yongkuk@knu.ac.kr