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THE SOLVABILITY CONDITIONS FOR A CLASS OF CONSTRAINED INVERSE EIGENVALUE PROBLEM OF ANTISYMMETRIC MATRICES

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ABSTRACT. In this paper, a class of constrained inverse eigenvalue problem for antisymmetric matrices and their optimal approximation problem are considered. Some sufficient and necessary conditions of the solvability for the inverse eigenvalue problem are given. A general representation of the solution is presented for a solvable case. Furthermore, an expression of the solution for the optimal approximation problem is given.

1. Introduction

We consider a class of constrained inverse eigenvalue problem: Find a real $n \times n$ matrix $A \in S$ such that

$$AZ = Z\Lambda_0 \tag{1.1}$$

and

$$\lambda(A)\backslash\lambda(\Lambda_0) \subset D_\alpha, \tag{1.2}$$

where S is a given set of real $n \times n$ matrices. $z_1, z_2, \ldots, z_m (m < n)$ are given n-dimensional complex vectors and $Z = [z_1 \; z_2 \cdots z_m]$. $\lambda_1, \lambda_2, \ldots, \lambda_m$ are given complex constants and $\Lambda_0 = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_m)$. $\lambda(A)$ and $\lambda(\Lambda_0)$ denote the set of eigenvalues of A and Λ_0 respectively. $\lambda(A)\backslash\lambda(\Lambda_0)$ denotes the difference of $\lambda(A)$ and $\lambda(\Lambda_0)$. $D_\alpha = \{z||z| \le \alpha, \alpha > 0\}$ is a given closed disk.

With no constraint (1.2) ($\alpha = +\infty$) a class of matrix inverse eigenvalue problem is obtained. That is (1.1). For important results on the

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discussions of the inverse eigenvalue problem (1.1) associated with several kinds of different sets S, such as antisymmetric matrices, centrosymmetric matrices, symmetric and anti-persymmetric matrices and Hermitian-generalized Hamiltonian matrices, we refer the reader to [3–6].

For the case when S is the set of all real $n \times n$ symmetric matrices this kind of problem was discussed in the literature ten years ago [2], but it seems to have been neglected since. In this paper, we will discuss this kind of problem for the case when S is the set of all real $n \times n$ antisymmetric matrices. It is necessary to point out that we will consider given multiple eigenvalues in this paper. Only simple eigenvalues, however, have been discussed in [3]. In addition, the solution set of problem (1.1) in [3] is a linear manifold. The solution set of (1.1) with constraint (1.2) in this paper is only a closed convex set (See Corollary 2.1). Moreover, some results in [3] is actually a special case which is contained in this paper.

Let R^n and C^n denote the sets of n-dimensional real vectors and n-dimensional complex vectors respectively. $C^{n\times m}, R^{n\times m}, SR^{n\times n}, ASR^{n\times n}$ and $OR^{n\times n}$ denote the sets of complex $n \times m$ matrices, real $n \times m$ matrices, real $n \times n$ symmetric matrices, real $n \times n$ antisymmetric matrices and real $n \times n$ orthogonal matrices respectively. The notation $R_r^{n\times m}$ denote all real $n \times m$ matrices with rank r. A^T, A^+, $R(A)$, $N(A)$ and $\text{rank}(A)$ denote the transpose, the Moore-Penrose generalized inverse, the column space, the null space and the rank of a matrix A respectively. I_n is the identity matrix of order n. The notation $\bigoplus \sum_{j=1}^{k} A_{jj} = A_{11} \bigoplus A_{22} \bigoplus \cdots \bigoplus A_{kk}$ denotes the direct sum of the matrices $A_{11}, A_{22}, \ldots, A_{kk}$, $A_{jj} \in R^{n_j \times n_j}$. Define matrix inner product $(A, B) = \text{tr}(B^T A)$, $A, B \in R^{n\times m}$. Then $R^{n\times m}$ is a Hilbert inner product space. The norm of matrix produced by the inner product is Frobenius norm, i.e., $\|A\| = \sqrt{(A, A)} = (\text{tr}(A^T A))^{\frac{1}{2}}$. Define vector inner product $\langle x, y\rangle = y^* x$, $x, y \in C^n$, where $y^* = \bar{y}^T$, $\bar{y}$ is the component-wise conjugate. The norm of vector produced by inner product is Euclidean norm, i.e., $\|x\|_2 = \sqrt{\langle x, x\rangle} = \sqrt{x^* x}$, $x \in C^n$. Two vectors $x, y \in C^n$ are called orthogonal if $\langle x, y\rangle = 0$. The notation $V^\perp$ stands for the orthogonal complement of the linear subspace V.

It is well known that each eigenvalue of a real antisymmetric matrix A is either zero or pure imaginary. If a real antisymmetric matrix A has imaginary eigenvalues, they must occur in complex conjugate pairs. Notice that the imaginary part of zero is zero, then the constrained inverse eigenvalue problem studied in this paper can be described as the following problem.

PROBLEM 1. Given $Z = [z_1 \cdots z_m] \in C^{n\times m}$ with $m < n$. Given $\Lambda_0 = \mathrm{diag}(\lambda_1, \ldots, \lambda_m) \in C^{m\times m}$, where λ_j is pure imaginary or zero, $j = 1, 2, \ldots, m$. Given a real number $\alpha > 0$.

(1) Find a matrix $A \in ASR^{n\times n}$ such that

$$AZ = Z\Lambda_0. \tag{1.3}$$

Denote the set of all the solutions of (1.3) by $\varphi(Z, \Lambda_0)$.

(2) Find the subset $\varphi(Z, \Lambda_0, \alpha) \subset \varphi(Z, \Lambda_0)$ such that the imaginary parts of all of the remaining eigenvalues of any matrix in $\varphi(Z, \Lambda_0, \alpha)$ are located in the interval $[-\alpha, \alpha]$.

We also discuss the so-called optimal approximation problem associated with the solution set $\varphi(Z, \Lambda_0, \alpha)$ of problem 1. The problem is as follows.

PROBLEM 2. Given $B \in R^{n\times n}$. Find a matrix $\widehat{A} \in \varphi(Z, \Lambda_0, \alpha)$ such that

$$\|B - \widehat{A}\| = \inf_{\forall A \in \varphi(Z, \Lambda_0, \alpha)} \|B - A\|, \tag{1.4}$$

where $\|\cdot\|$ is the Frobenius norm.

The paper is organized as follows. In section 2 we will give and testify the necessary and sufficient conditions for the solvability of problem 1 and the expression of the general solution of problem 1. In section 3 we will prove that there exists a unique solution of problem 2 and give the expression of the solution for problem 2.

2. The solvability conditions and general solutions of problem 1 in real field

From now on the lower-case English letter i will always be denoted a formal symbol satisfying the relation $i^2 = -1$.

LEMMA 2.1. *Let $A \in ASR^{n\times n}$. Let $x + iy$ be an eigenvector associated with an eigenvalue λi, where $x, y \in R^n$ and $\lambda \in R$ and $\lambda \neq 0$. Then*

(i) *$x \neq 0$, $y \neq 0$ and* $A[x\ y] = [x\ y] \begin{bmatrix} 0 & \lambda \\ -\lambda & 0 \end{bmatrix}$.

(ii) *$\langle x, y\rangle = 0$ and* $\|x\|_2 = \|y\|_2$.

Proof. If $A \in ASR^{n\times n}$ and $A(x+iy) = \lambda i(x+iy)$, then $A(x-iy) = -\lambda i(x-iy)$. Hence, $Ax = -\lambda y$ and $Ay = \lambda x$, i.e.,

$$A[x\ y] = [x\ y]\begin{bmatrix} 0 & \lambda \\ -\lambda & 0 \end{bmatrix}.$$

This, together with $\lambda \neq 0$, yields $x \neq 0$ and $y \neq 0$.

Since $A(x\pm iy) = \pm\lambda i(x\pm iy)$ and A is a normal matrix and $\lambda i \neq -\lambda i$, we have $(x-iy)^*(x+iy) = (x^Tx - y^Ty) + i(y^Tx + x^Ty) = 0$. Notice that $y^Tx = x^Ty$, we obtain $\langle x, y\rangle = 0$ and $\|x\|_2 = \|y\|_2$. This establishes the lemma. □

Next we consider the problem 1 in real field.

Let

$$(2.1)\qquad X = [X_1 \cdots X_p\ X_{p+1}] \in R^{n\times m}, \qquad \Lambda = \bigoplus\sum_{j=1}^{p+1} \Lambda^{(j)} \in R^{m\times m},$$

where every column of X is a non-vanishing vector, $m < n$; $X_j = [X_j^{(1)} \cdots X_j^{(m_j)}] \in R^{n\times 2m_j}$, $X_j^{(k)} = [x_j^{(k)}\ y_j^{(k)}] \in R^{n\times 2}$, $j = 1,\ldots,p$, $k = 1,\ldots,m_j$, $2\sum_{j=1}^{p} m_j = t$; $X_{p+1} \in R^{n\times(m-t)}$; $\Lambda^{(j)} = \underbrace{\Lambda_j \bigoplus \cdots \bigoplus \Lambda_j}_{m_j}$,

$\Lambda_j = \begin{bmatrix} 0 & \lambda_j \\ -\lambda_j & 0 \end{bmatrix} \in R^{2\times 2}$, $\lambda_j \neq 0$, $j = 1,\ldots,p$; $\lambda_l \neq \lambda_q$, $l, q = 1,\ldots,p$, $l \neq q$; $\Lambda^{(p+1)} = 0 \in R^{(m-t)\times(m-t)}$.

Based on Lemma 2.1, we obtain the equivalent formulation of problem 1 in real number field.

PROBLEM I$_0$. Given $X \in R^{n\times m}$ and $\Lambda \in R^{m\times m}$ are the same as (2.1). Given a real number $\alpha > 0$.

(1) Find a matrix $A \in ASR^{n\times n}$ such that

$$(2.2)\qquad AX = X\Lambda.$$

Denote the set of all the solutions of (2.2) by $\varphi(X,\Lambda)$.

(2) Find the subset $\varphi(X,\Lambda,\alpha) \subset \varphi(X,\Lambda)$ such that the imaginary parts of all of the remaining eigenvalues of any matrix in $\varphi(X,\Lambda,\alpha)$ are located in the interval $[-\alpha,\alpha]$.

We are now in a position to state the first main theorem of this section.

THEOREM 2.1. *Let $X \in R_r^{n\times m}$ and $\Lambda \in R^{m\times m}$ satisfy (2.1). Then the solution set $\varphi(X,\Lambda)$ in problem I_0 is nonempty if and only if*

$$X_j^T X_k = 0, \quad j \neq k, \quad j,k = 1,2,\dots,p+1; \tag{2.3}$$

$$\|x_j^{(k)}\|_2 = \|y_j^{(k)}\|_2, \quad \langle x_j^{(k)}, y_j^{(k)}\rangle = 0, \quad j=1,\dots,p, \quad k=1,\dots,m_j; \tag{2.4}$$

$$\begin{gathered}\langle x_j^{(k)}, x_j^{(l)}\rangle = \langle y_j^{(k)}, y_j^{(l)}\rangle, \qquad \langle x_j^{(k)}, y_j^{(l)}\rangle + \langle x_j^{(l)}, y_j^{(k)}\rangle = 0,\\ j=1,\dots,p; \quad k\neq l, \quad k,l=1,2,\dots,m_j.\end{gathered} \tag{2.5}$$

Moreover, the elements in the solution set $\varphi(X,\Lambda)$ can be expressed as

$$A = A_0 + U_2 G U_2^T, \qquad \forall G \in ASR^{(n-r)\times(n-r)}, \tag{2.6}$$

where $A_0 = X\Lambda X^+$, $R(U_2) = N(X^T)$ and $U_2^T U_2 = I_{n-r}$.

Proof. We give the proof only for the case $r < m$. The proof for $r = m$ is similar.

(Necessity) Suppose that problem I_0 has a solution $A \in \varphi(X,\Lambda)$. Then $X^TAX = X^TX\Lambda$. From $A^T = -A$ and $\Lambda^T = -\Lambda$, we have

$$X^TX\Lambda = \Lambda X^TX. \tag{2.7}$$

It is not difficult to show that (2.7) is equivalent to (2.3), (2.4), and (2.5).

(Sufficiency) Let the singular value decomposition of X as follows:

$$X = U\begin{bmatrix}\Sigma & 0\\ 0 & 0\end{bmatrix}V^T = U_1\Sigma V_1^T, \tag{2.8}$$

where $U = [U_1 \quad U_2] \in OR^{n\times n}$, $V = [V_1 \quad V_2] \in OR^{m\times m}$, $U_1 \in R^{n\times r}$, $R(U_2) = N(X^T)$, $V_1 \in R^{m\times r}$, $\Sigma = \mathrm{diag}(\sigma_1,\dots,\sigma_r)$, $\sigma_1 \geq \cdots \geq \sigma_r > 0$.

Suppose that (2.3), (2.4), and (2.5) hold. Then (2.7) holds. From (2.7) and (2.8), we have

$$V_1\Sigma^2V_1^T\Lambda = \Lambda V_1\Sigma^2V_1^T. \tag{2.9}$$

It follows from (2.9), $V_2^TV_1 = 0$, and $V_1^TV_1 = I_r$ that

$$V_2^T\Lambda V_1 = 0, \tag{2.10}$$

and

$$\Sigma^{-1}V_1^T\Lambda V_1\Sigma = \Sigma V_1^T\Lambda V_1\Sigma^{-1}. \tag{2.11}$$

Let $A_0 = X\Lambda X^+$. It follows from (2.8), (2.11), and $\Lambda^T = -\Lambda$ and $X^+ = V_1\Sigma^{-1}U_1^T$ that

$$A_0^T = (X\Lambda X^+)^T = -X\Lambda X^+ = -A_0. \tag{2.12}$$

From (2.8), (2.10), and $X^+X = V_1V_1^T = I_m - V_2V_2^T$, we get

$$A_0X = X\Lambda X^+X = X\Lambda - X\Lambda V_2V_2^T = X\Lambda. \tag{2.13}$$

Now (2.12) and (2.13) show that $A_0 \in \varphi(X, \Lambda)$. This means that the solution set $\varphi(X, \Lambda)$ of problem I_0 is nonempty.

In the following we show that if (2.3), (2.4), and (2.5) hold, then the elements in the solution set $\varphi(X, \Lambda)$ can be expressed in the form of (2.6).

For any $A \in \varphi(X, \Lambda)$, let

$$U^TAU = \widetilde{A} = \begin{bmatrix} \widetilde{A}_{11} & \widetilde{A}_{12} \\ \widetilde{A}_{21} & \widetilde{A}_{22} \end{bmatrix}, \tag{2.14}$$

where $\widetilde{A}_{11} \in R^{r\times r}$ and U is given by (2.8). Since $AX = X\Lambda$, we have $\widetilde{A}U^TXV = U^TX\Lambda V$, i.e.,

$$\begin{bmatrix} \widetilde{A}_{11}\Sigma & 0 \\ \widetilde{A}_{21}\Sigma & 0 \end{bmatrix} = \begin{bmatrix} \Sigma V_1^T\Lambda V_1 & \Sigma V_1^T\Lambda V_2 \\ 0 & 0 \end{bmatrix} \tag{2.15}$$

This, together with (2.10), yields that

$$\widetilde{A}_{11} = \Sigma V_1^T\Lambda V_1\Sigma^{-1} \quad \text{and} \quad \widetilde{A}_{21} = 0.$$

Then, by $A^T = -A$, we get

$$\widetilde{A}_{11} = \Sigma V_1^T\Lambda V_1\Sigma^{-1}, \tag{2.16}$$

$$\widetilde{A}_{12} = -\widetilde{A}_{21}^T = 0, \tag{2.17}$$

and

$$\widetilde{A}_{22} = -\widetilde{A}_{22}^T. \tag{2.18}$$

Let $G = \widetilde{A}_{22}$. It follows from (2.14) and

$$U_1\Sigma V_1^T\Lambda V_1\Sigma^{-1}U_1^T = X\Lambda X^+ = A_0$$

that

$$\begin{aligned} A &= U\widetilde{A}U^T \\ &= U\begin{bmatrix} \Sigma V_1^T\Lambda V_1\Sigma^{-1} & 0 \\ 0 & \widetilde{A}_{22} \end{bmatrix}U^T \\ &= U_1\Sigma V_1^T\Lambda V_1\Sigma^{-1}U_1^T + U_2\widetilde{A}_{22}U_2^T \\ &= A_0 + U_2GU_2^T, \end{aligned}$$

where matrix $G \in ASR^{(n-r)\times(n-r)}$ is arbitrary, $R(U_2) = N(X^T)$ and $U_2^TU_2 = I_{n-r}$. This completes the proof. □

In order to study the second question of problem I_0, we need the following two lemmas.

LEMMA 2.2([1]). *$A \in ASR^{n\times n}$ if and only if there is a real orthogonal matrix $Q \in OR^{n\times n}$ such that*

$$Q^T A Q = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & A_k & 0 \\ 0 & \cdots & 0 & 0 \end{bmatrix},$$

where each $A_j \in R^{2\times 2}$ has the form

$$A_j = \begin{bmatrix} 0 & \lambda_j \\ -\lambda_j & 0 \end{bmatrix}, \quad j = 1, 2, \ldots, k.$$

LEMMA 2.3. *Let $A \in ASR^{n\times n}$. Let $\lambda \in R$ and $\lambda \neq 0$. Let $x_j + iy_j \in C^n$ with $x_j, y_j \in R^n$, $j = 1, 2, \ldots, k$, $k \geq 2$. Suppose that λi is an eigenvalue of A and $x_j + iy_j$ is an eigenvector associated with λi, $j = 1, 2, \ldots, k$. Then the following statements are true.*

- (i) *If $x_1 + iy_1$, $x_2 + iy_2$, ..., $x_k + iy_k$ are linearly dependent, then so are x_1, y_1, x_2, y_2, ..., x_{k-1}, y_{k-1}, x_k and x_1, y_1, x_2, y_2, ..., x_{k-1}, y_{k-1}, y_k.*
- (ii) *Eigenvectors $x_1 + iy_1$, $x_2 + iy_2$, ..., $x_k + iy_k$ are linearly independent if and only if vectors x_1, y_1, x_2, y_2, ..., x_{k-1}, y_{k-1}, x_k, y_k are linearly independent.*
- (iii) *Vectors x_1, y_1, x_2, y_2, ..., x_k, y_k are linearly independent if and only if vectors $x_1 + iy_1$, $x_1 - iy_1$, $x_2 + iy_2$, $x_2 - iy_2$, ..., $x_k + iy_k$, $x_k - iy_k$ are linearly independent.*

Proof. Using induction and Lemma 2.1, it is easy to prove (i) and the necessity of (ii), and the sufficiency of (ii) and (iii) immediately follow from (i) and the necessity of (ii) respectively. We only prove the necessity of (iii).

If vectors x_1, y_1, x_2, y_2, ..., x_k, y_k are linearly independent, then so are vectors x_1, $-y_1$, x_2, $-y_2$, ..., x_k, $-y_k$. By the sufficiency of (ii), eigenvectors $x_1 + iy_1$, $x_2 + iy_2$, ..., $x_k + iy_k$ and $x_1 - iy_1$, $x_2 - iy_2$, ..., $x_k - iy_k$, which associated with the eigenvalue λi and $-\lambda i$ respectively, are linearly independent respectively. Since A is a normal matrix and $\lambda i \neq -\lambda i$, the assertion follows.

Let us introduce a notation. Given a real number $\alpha > 0$. Let $ASR^{n\times n}_{[-\alpha,\alpha]}$ denote all real $n \times n$ antisymmetric matrices with the imaginary parts of all eigenvalues located in the interval $[-\alpha, \alpha]$.

We now state the second main result of this section.

THEOREM 2.2. *Let $X \in R^{n\times m}_r$ and $\Lambda \in R^{m\times m}$ satisfy (2.1). Given a real number $\alpha > 0$. Then the solution set $\varphi(X, \Lambda, \alpha)$ of problem I_0 is nonempty if and only if*

$$X_j^T X_k = 0, \quad j \neq k, \quad j, k = 1, 2, \ldots, p+1; \tag{2.19}$$

$$\|x_j^{(k)}\|_2 = \|y_j^{(k)}\|_2, \quad \langle x_j^{(k)}, y_j^{(k)}\rangle = 0, \quad j = 1, \ldots, p, \quad k = 1, \ldots, m_j; \tag{2.20}$$

$$\begin{gathered}\langle x_j^{(k)}, x_j^{(l)}\rangle = \langle y_j^{(k)}, y_j^{(l)}\rangle, \quad \langle x_j^{(k)}, y_j^{(l)}\rangle + \langle x_j^{(l)}, y_j^{(k)}\rangle = 0, \\ j = 1, \ldots, p, \quad k \neq l, \quad k, l = 1, 2, \ldots, m_j.\end{gathered} \tag{2.21}$$

Moreover, the elements in the set $\varphi(X, \Lambda, \alpha)$ can be expressed as

$$A = A_0 + U_2 G U_2^T, \tag{2.22}$$

where $A_0 = X\Lambda X^+$, $R(U_2) = N(X^T)$ and $U_2^T U_2 = I_{n-r}$, matrix $G \in ASR^{(n-r)\times(n-r)}_{[-\alpha,\alpha]}$ is arbitrary.

Proof. From Theorem 2.1, it suffices to show that if the solution set $\varphi(X, \Lambda)$ of problem I_0 is nonempty then any matrix A of $\varphi(X, \Lambda)$, which can be expressed as

$$A = X\Lambda X^+ + U_2 G U_2^T \tag{2.23}$$

with $G \in ASR^{(n-r)\times(n-r)}$, only has r given eigenvalues and the remaining $n - r$ eigenvalues are the eigenvalues of the matrix G

Let $A \in \varphi(X, \Lambda)$, and let the expression of A be (2.23). Suppose that $\text{rank}(X_{p+1}) = r_1$, where $X_{p+1} \in R^{n\times(m-t)}$ is given by (2.1). Then the matrix A has r_1 given 0 eigenvalues and r_1 given corresponding real eigenvectors which are linearly independent.

By $\text{rank}(X) = \text{rank}([X_1 \cdots X_p \;\; X_{p+1}]) = r$ and (2.3) of Theorem 2.1 and Lemma 2.3, we conclude that $\text{rank}([X_1 \cdots X_p]) = r - r_1$ must be even number, and the matrix A has $r - r_1$ given imaginary eigenvalues and $r - r_1$ given complex eigenvectors which are linearly independent. This implies that the matrix A has r given eigenvalues and r given corresponding eigenvectors which are linearly independent.

Let $\frac{r-r_1}{2} = s$. We may assume, for simplicity, that the r given eigenvalues of the matrix A are $\lambda_1 i, -\lambda_1 i, \ldots, \lambda_s i, -\lambda_s i, \underbrace{0, \ldots, 0}_{r_1}$ and the

r given corresponding eigenvectors, which are linearly independent, are $x_1 + iy_1$, $x_1 - iy_1$, ..., $x_s + iy_s$, $x_s - iy_s$,x_{s+1}, ..., x_{s+r_1}, $x_j \in R^n$, $j = 1, \ldots, s + r_1$, $y_k \in R^n$, $k = 1, \ldots, s$. Denote $\widetilde{X} = [x_1\ y_1 \cdots x_s\ y_s\ x_{s+1} \cdots x_{s+r_1}]$. Then $R(\widetilde{X}) = R(X)$ and vectors $x_1, y_1, \ldots, x_s, y_s, x_{s+1}, \ldots, x_{s+r_1}$ are linearly independent. From $R(\widetilde{X}) = R(X)$ and $R(X) = R(U_1) = R(U_2)^\perp$, we get $R(\widetilde{X}) = R(U_2)^\perp$.

On the other hand, for the matrix $G \in ASR^{(n-r)\times(n-r)}$ which is given by (2.23), according to Lemma 2.2, there exists a real orthogonal matrix $Q_2 = [q_1 \cdots q_{n-r}] \in OR^{(n-r)\times(n-r)}$ such that $GQ_2 = Q_2\widetilde{\Lambda}$, where $\widetilde{\Lambda} = \bigoplus \sum_{j=1}^{k+1} \widetilde{\Lambda}_j$, $\widetilde{\Lambda}_j = \begin{bmatrix} 0 & \mu_j \\ -\mu_j & 0 \end{bmatrix} \in R^{2\times 2}$, $\mu_j \neq 0$, $j = 1, \ldots, k$, $0 \leq k \leq [\frac{n-r}{2}]$ ($[\frac{n-r}{2}]$ denotes the maximal integer number that is not greater than $\frac{n-r}{2}$); $\widetilde{\Lambda}_{k+1} = 0 \in R^{(n-r-2k)\times(n-r-2k)}$. This, together with $X^+U_2 = 0$ and $U_2^TU_2 = I_{n-r}$, gives $AU_2Q_2 = X\Lambda X^+U_2Q_2 + U_2GU_2^TU_2Q_2 = U_2Q_2\widetilde{\Lambda}$. This shows that $\mu_1 i$, $-\mu_1 i$, ...,$\mu_k i$, $-\mu_k i$, $\underbrace{0, \ldots, 0}_{n-r-2k}$ are eigenvalues of the matrix A, and $U_2q_1 + U_2q_2i$, $U_2q_1 - U_2q_2i$, ..., $U_2q_{2k-1} + U_2q_{2k}i$, $U_2q_{2k-1} - U_2q_{2k}i$, U_2q_{2k+1}, ..., U_2q_{n-r} are corresponding eigenvectors which are linearly independent. Denote $U_2Q_2 = [U_2q_1\ U_2q_2 \cdots U_2q_{n-r}]$. Clearly, $R(U_2Q_2) = R(U_2)$. This implies that the vectors U_2q_1, U_2q_2, ..., U_2q_{n-r} are linearly independent.

Now $R(\widetilde{X}) = R(U_2)^\perp$ and $R(U_2Q_2) = R(U_2)$ yield that $R(\widetilde{X}) = R(U_2Q_2)^\perp$. Thus, vectors $x_1, y_1, \ldots, x_s, y_s, x_{s+1}, \ldots, x_{s+r_1}, U_2q_1, U_2q_2, \ldots, U_2q_{n-r}$ are linearly independent. By lemma 2.3 again, we obtain that $x_1 + iy_1$, $x_1 - iy_1$, ..., $x_s + iy_s$, $x_s - iy_s$, x_{s+1}, ..., x_{s+r_1}, $U_2q_1 + U_2q_2i$, $U_2q_1 - U_2q_2i$, ..., $U_2q_{2k-1} + U_2q_{2k}i$, $U_2q_{2k-1} - U_2q_{2k}i$, U_2q_{2k+1}, ..., U_2q_{n-r} are n linearly independent eigenvectors of the matrix A associated with eigenvalues $\lambda_1 i$, $-\lambda_1 i$, ..., $\lambda_s i$, $-\lambda_s i$, $\underbrace{0, \ldots, 0}_{r_1}$, $\mu_1 i$, $-\mu_1 i$, ..., $\mu_k i$, $-\mu_k i$, $\underbrace{0, \ldots, 0}_{n-r-2k}$, respectively, and the assertion therefore follows. □

The following corollary is easy to prove.

COROLLARY 2.1. *If* (2.19), (2.20), *and* (2.21) *hold, then the solution set* $\varphi(X, \Lambda, \alpha)$ *of problem* I_0 *is a closed convex set.*

3. The solution of Problem 2

First we introduce two notations and an operator.

Given $B \in ASR^{n\times n}$ and given a real number $\alpha > 0$, let the real Schur decomposition of B be

$$B = Q\Lambda Q^T, \tag{3.1}$$

where $Q \in OR^{n\times n}$; $\Lambda = \Lambda_1 \bigoplus \cdots \bigoplus \Lambda_s \bigoplus \Lambda_{s+1} \bigoplus \cdots \bigoplus \Lambda_k \bigoplus \Theta$, $\Lambda_j = \begin{bmatrix} 0 & \lambda_j \\ -\lambda_j & 0 \end{bmatrix} \in R^{2\times 2}$, $j = 1, \ldots, k$, $\lambda_1 \geq \cdots \geq \lambda_s > \alpha \geq \lambda_{s+1} \geq \cdots \geq \lambda_k > 0$, s is a nonnegative integer and $0 \leq s \leq k$; $\Theta = 0 \in R^{(n-2k)\times(n-2k)}$.

Denote $\Lambda_\alpha = \begin{bmatrix} 0 & \alpha \\ -\alpha & 0 \end{bmatrix}$.

Let $[\Lambda]_\alpha = \Lambda_\alpha \bigoplus \cdots \bigoplus \Lambda_\alpha \bigoplus \Lambda_{s+1} \bigoplus \cdots \bigoplus \Lambda_k \bigoplus \Theta$, where Λ_j ($j = s+1, \ldots, k$) and Θ are the same as (3.1).

Define

$$[B]_\alpha = Q[\Lambda]_\alpha Q^T. \tag{3.2}$$

Obviously, matrix $[B]_\alpha$ is uniquely determined by the matrix B and the real number $\alpha > 0$; $[\Lambda]_\alpha \in ASR^{n\times n}_{[-\alpha,\alpha]}$ and $[B]_\alpha \in ASR^{n\times n}_{[-\alpha,\alpha]}$.

LEMMA 3.1. *Given $B \in ASR^{n\times n}$ and given a real number $\alpha > 0$,*

$$\|B - [B]_\alpha\| = \inf_{\forall Z \in ASR^{n\times n}_{[-\alpha,\alpha]}} \|B - Z\|.$$

Proof. Without loss of generality, suppose that the real Schur decomposition of the matrix B is the same as (3.1).

From the orthogonal invariance of the Frobenius norm, it follows

$$\|B - Z\| = \|Q^T(Q\Lambda Q^T - Z)Q\| = \|\Lambda - Q^T Z Q\|, \quad \forall Z \in ASR^{n\times n}_{[-\alpha,\alpha]}.$$

Notice that for any $Z \in ASR^{n\times n}_{[-\alpha,\alpha]}$, we have $Q^T Z Q \in ASR^{n\times n}_{[-\alpha,\alpha]}$. Hence

$$\|\Lambda - [\Lambda]_\alpha\| = \inf_{\forall Z \in ASR^{n\times n}_{[-\alpha,\alpha]}} \|\Lambda - Q^T Z Q\| = \inf_{\forall Z \in ASR^{n\times n}_{[-\alpha,\alpha]}} \|B - Z\|.$$

Clearly, $\|\Lambda - [\Lambda]_\alpha\| = \|\Lambda - Q^T(Q[\Lambda]_\alpha Q^T)Q\|$ and $Q[\Lambda]_\alpha Q^T = [B]_\alpha \in ASR^{n\times n}_{[-\alpha,\alpha]}$. Therefore,

$$\|B - [B]_\alpha\| = \inf_{\forall Z \in ASR^{n\times n}_{[-\alpha,\alpha]}} \|B - Z\|.$$

The lemma is established. □

THEOREM 3.1. *Given $B \in R^{n\times n}$, and given a real number $\alpha > 0$, let $X \in R_r^{n\times m}$ and $\Lambda \in R^{m\times m}$ satisfy (2.1). If (2.19), (2.20), and (2.21) hold, then there is a unique solution $\widehat{A}$ for Problem 2 and $\widehat{A}$ can be represented as*

$$\widehat{A} = A_0 + U_2\widehat{G}U_2^T, \tag{3.3}$$

where $A_0 = X\Lambda X^+$; $R(U_2) = N(X^T)$, $U_2^TU_2 = I_{n-r}$; $\widehat{G} = [U_2^TB_2U_2]_\alpha$ and $B_2 = \frac{1}{2}(B - B^T)$.

Proof. Because $\varphi(X,\Lambda,\alpha)$ is a closed convex set, Problem 2 has a unique solution $\widehat{A}$ in $\varphi(X,\Lambda,\alpha)$. This means that there exists a unique matrix $\widehat{G} \in ASR_{[-\alpha,\alpha]}^{(n-r)\times(n-r)}$ such that

$$\widehat{A} = A_0 + U_2\widehat{G}U_2^T \tag{3.4}$$

and

$$\|B - \widehat{A}\| = \inf_{\forall A\in\varphi(X,\Lambda,\alpha)} \|B - A\|, \tag{3.5}$$

where A_0 and U_2 are given by (2.22).

Obviously, for given $B \in R^{n\times n}$, there exist unique $B_1 \in SR^{n\times n}$ and $B_2 \in ASR^{n\times n}$ such that

$$B = B_1 + B_2, \qquad (B_1, B_2) = 0, \tag{3.6}$$

where $B_1 = \frac{1}{2}(B + B^T)$ and $B_2 = \frac{1}{2}(B - B^T)$.

For any $A \in \varphi(X,\Lambda,\alpha)$, by (2.22) and (3.6), we have

$$\begin{aligned}\|B - A\|^2 &= \|B - (A_0 + U_2GU_2^T)\|^2\\ &= \|B_1\|^2 + \|B_2 - A_0 - U_2GU_2^T\|^2,\end{aligned}$$

i.e.,

$$\|B - A\|^2 = \|B_1\|^2 + \|B_2 - A_0 - U_2GU_2^T\|^2. \tag{3.7}$$

Taking $U_1 \in R^{n\times r}$ such that $U_2^TU_1 = 0$ and $U_1^TU_1 = I_r$. Denote $U = [U_1 \ \ U_2]$. Then $U \in OR^{n\times n}$. Applying the orthogonality of the matrix U and $A_0U_2 = 0$ and $U_2^TA_0 = 0$, we obtain

$$\begin{aligned}\|B_2 - A_0 - U_2GU_2^T\|^2 &= \|U^T(B_2 - A_0 - U_2GU_2^T)U\|^2\\ &= \|U_1^T(B_2 - A_0)U_1\|^2 + \|U_1^TB_2U_2\|^2\\ &\quad + \|U_2^TB_2U_1\|^2 + \|U_2^TB_2U_2 - G\|^2.\end{aligned}$$

This, together with (3.7), (3.5), and (3.4), implies that $\|B - \widehat{A}\| = \inf_{\forall A \in \varphi(X,\Lambda,\alpha)} \|B - A\|$ is equivalent to

$$\|U_2^T B_2 U_2 - \widehat{G}\| = \inf_{\forall G \in ASR^{(n-r)\times(n-r)}_{[-\alpha,\alpha]}} \|U_2^T B_2 U_2 - G\|. \tag{3.8}$$

It follows from Lemma 3.1 that $\widehat{G} = [U_2^T B_2 U_2]_\alpha$. This result and (3.4) imply (3.3). This completes the proof. □

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