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τ-INJECTIVE SUBMODULES OF INDECOMPOSABLE INJECTIVE MODULES

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ABSTRACT. Let τ be a hereditary torsion theory and let p be a prime ideal of a commutative ring R. We study the existence of (minimal) τ-injective submodules of the injective hull of R/p.

1. Introduction and preliminaries

The problem of determining the structure of injective hulls of modules has been approached since they were defined by Eckmann and Schopf [7]. In the past decades, several attempts have been made to deal with this difficult problem and some results have been obtained in particular cases (e.g., [12], [14], [19]). Connected to this, the study of indecomposable injective modules (e.g., [8], [9], [10], [17]) is of a special importance.

We are interested in considering the injective hull $E(R/p)$ of R/p for some prime ideal p of a commutative ring R and using the torsion theoretic framework in order to obtain information on $E(R/p)$. In recent years, there have been established results on the relationship between injective hulls or indecomposable injectives and (τ-closed) prime ideals of some rings in the context of a hereditary torsion theory (e.g., [1], [13]). The purpose of this paper is to study the existence of (minimal) τ-injective submodules of $E(R/p)$.

We first consider arbitrary τ-injective submodules and, for a domain R, we show that if $E_\tau(R/p) \neq E(R/p)$, then there exists a countable family of proper τ-injective submodules of $E(R/p)$ that strictly contain the τ-injective hull $E_\tau(R/p)$ of R/p. Also, for a noetherian domain R, we prove that $0 \neq p \in \text{Spec}(R)$ is τ-closed in R if and only if $\text{Ann}_{E(R/p)}p^m$ is τ-injective for every integer $m \geq 1$. As far as minimal τ-injective submodules are concerned, we show that $E(R/p)$ has at most one minimal

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τ-injective submodule. If this does exist, then it is necessarily $E_\tau(R/p)$ and this happens if and only if R/p is either τ-torsion or τ-cocritical. Finally, we establish a direct sum decomposition theorem and we show that every non-zero direct summand of a τ-completely decomposable module has a minimal τ-injective direct summand. Some of the present results generalize for an arbitrary torsion theory properties established in [2] and [3] for the Dickson torsion theory [6].

Now let us give some basic notation and terminology. Throughout the paper we denote by R an associative ring with non-zero identity and all modules are left unital R-modules. Also τ will always be a hereditary torsion theory on the category of left R-modules. For a submodule B of a module A, $B \trianglelefteq A$ denotes the fact that A is an essential extension of B. We denote by Spec(R) the set of all prime ideals of a commutative ring R. For a prime ideal p of a commutative ring R, we denote by $\dim p$ the (Krull) dimension of the ring R/p.

Let A be a module and let B be a submodule of A. Then B is called τ-dense (respectively τ-closed) in A if A/B is τ-torsion (respectively τ-torsionfree). The τ-closure of B in A is the unique minimal τ-closed submodule of A containing B. A non-zero module A is said to be τ-cocritical if A is τ-torsionfree and each of its non-zero submodules is τ-dense in A.

A module A is said to be τ-injective if it is injective with respect to every monomorphism having a τ-torsion cokernel. For any module A, $E(A)$ and $E_\tau(A)$ denote the injective and the τ-injective hull of A respectively. In this paper, a non-zero module that is the τ-injective hull of each of its non-zero submodules is called minimal τ-injective. Such modules are the torsion theoretic analogues of indecomposable injective modules. It is clear that every minimal τ-injective module is either τ-torsion or τ-cocritical. Also, a module is τ-torsionfree minimal τ-injective if and only if it is τ-cocritical τ-injective [11, Proposition 14.9]. A module A is said to be τ-completely decomposable if it is a direct sum of minimal τ-injective submodules [16, p.77].

For additional information on torsion theories the reader is referred to [11] and [21].

2. τ-injective submodules

We begin with several preliminary technical lemmas.

LEMMA 2.1. *Let A be a τ-injective module and let B be a proper essential τ-injective submodule of A. Suppose that it does not exist any* *proper τ-injective submodule of A that strictly contains B. Then A/B is τ-cocritical.*

Proof. Since B is a proper essential τ-injective submodule of A, A/B is τ-torsionfree. Now let C be a submodule of A that strictly contains B and let C' be the τ-closure of C in A. Since C' is τ-closed in A, it follows that C' is τ-injective. Then by hypothesis we have $C' = A$. Thus A/C is τ-torsion, showing that A/B is τ-cocritical. □

For the rest of this section the ring R is assumed to be commutative.

LEMMA 2.2. [4, Lemma 2.1] *Let A be a τ-cocritical module. Then for every $0 \neq a \in A$, $\mathrm{Ann}_R a = \mathrm{Ann}_R A \in \mathrm{Spec}(R)$ and $R/\mathrm{Ann}_R A$ is τ-cocritical.*

LEMMA 2.3. *Let A be a τ-cocritical faithful module. Then $E_\tau(A) = E(A) \cong E(R)$.*

Proof. Let $0 \neq a \in A$. By Lemma 2.2 we have $\mathrm{Ann}_R a = \mathrm{Ann}_R A = 0$ and $R \cong Ra$ is τ-cocritical. Then every τ-injective module is injective, whence $E(A) = E_\tau(A) = E_\tau(Ra) = E(Ra) \cong E(R)$. □

THEOREM 2.4. *Let R be a domain, let A be a τ-injective module with $\mathrm{Ann}_R A = 0$ and let B be a proper essential τ-injective submodule of A with $\mathrm{Ann}_R B \neq 0$. Then there exists a countable family of proper τ-injective submodules of A that strictly contain B.*

Proof. Suppose that it does not exist any proper τ-injective submodule of A that strictly contains B. By Lemma 2.1, A/B is τ-cocritical. Then by Lemma 2.2, $p = \mathrm{Ann}_R(A/B) \in \mathrm{Spec}(R)$. Suppose that $p = 0$. Then A/B is faithful τ-cocritical and by Lemma 2.3 we have $E(R) \cong E_\tau(A/B)$. It follows that R is τ-cocritical, hence every τ-injective module is injective. But then B is a direct summand of A, a contradiction. Therefore $p \neq 0$.

Now let $0 \neq d \in p$ and $0 \neq r \in \mathrm{Ann}_R B$. Then $dr \in \mathrm{Ann}_R A = 0$, a contradiction. Therefore there exists a proper τ-injective submodule D_1 of A that strictly contains B. If $\mathrm{Ann}_R D_1 = 0$, there exists a proper τ-injective submodule D_2 of D_1 that strictly contains B. If $\mathrm{Ann}_R D_1 \neq 0$, there exists a proper τ-injective submodule D_2 of A that strictly contains D_1. Now the result follows. □

For the reader's convenience we recall the following result.

THEOREM 2.5. [4, Theorem 2.5] *Let $p \in \mathrm{Spec}(R)$ be such that R/p is τ-cocritical. Then $E_\tau(R/p) = \mathrm{Ann}_{E(R/p)} p$ and there exists an R-isomorphism between $E_\tau(R/p)$ and the field of fractions of R/p.*

Corollary 2.6. *Let R be a domain and let $0 \neq p \in \mathrm{Spec}(R)$ be such that R/p is τ-cocritical. Then there exists a countable family of proper τ-injective submodules of $E(R/p)$ that strictly contain $E_\tau(R/p)$.*

Proof. By Theorem 2.5, $E_\tau(R/p) = \mathrm{Ann}_{E(R/p)}p$. Then by [20, Lemma 2.31] we have $\mathrm{Ann}_R(E_\tau(R/p)) = p$. Since $\mathrm{Ann}_R(E(R/p)) = 0$, we have $E_\tau(R/p) \neq E(R/p)$ and the result follows from Theorem 2.4. □

Note that Theorem 2.4 is not suitable for obtaining a result as Corollary 2.6 when $p = 0$. Indeed, then we have $\mathrm{Ann}_R E(R) = 0$, but also $\mathrm{Ann}_R E_\tau(R) = 0$. In order to discuss that case, we need a variation of the ideas from the proof of Theorem 2.4.

Theorem 2.7. *Let R be a domain and let B be a non-zero proper τ-injective submodule of $E(R)$. Then there exists a proper τ-injective submodule of $E(R)$ that strictly contains B.*

Proof. Suppose that it does not exist any proper τ-injective submodule of $E(R)$ that strictly contains B. By Lemma 2.1, $E(R)/B$ is τ-cocritical. Then by Lemma 2.2, $p = \mathrm{Ann}_R(E(R)/B) \in \mathrm{Spec}(R)$. Suppose that $p = 0$. Then $E(R)/B$ is faithful τ-cocritical and by Lemma 2.3 we have $E(R) \cong E_\tau(E(R)/B)$, hence $E(R)$ is minimal τ-injective, a contradiction. Therefore $p \neq 0$. Now let $0 \neq d \in p$. Seeing $E(R)$ as the field of fractions of R, let $\frac{a}{b} \in E(R) \setminus B$. But then we also have $\frac{a}{b} = d \cdot \frac{a}{bd} \in B$, a contradiction. Therefore there exists a proper τ-injective submodule of $E(R)$ that strictly contains B. □

Corollary 2.8. *Let R be a domain such that $E_\tau(R) \neq E(R)$. Then:*

(i) *There exists a totally ordered countable family of proper τ-injective submodules of $E(R)$ that strictly contain $E_\tau(R)$.*

(ii) *If every τ-dense ideal of R is finitely generated, then $E(R)$ is the union of a totally ordered family of proper τ-injective submodules that contain $E_\tau(R)$.*

Proof. (i) By Theorem 2.7.

(ii) Let $\mathcal{F}$ be the family of all proper τ-injective submodules A of $E(R)$ that contain $E_\tau(R)$. Clearly, $\mathcal{F}$ is non-empty. By Theorem 2.7, $\mathcal{F}$ does not have a maximal element. Now suppose that $E(R)$ is not the union of a totally ordered subset of $\mathcal{F}$. Let $(D_j)_{j\in J}$ be a totally ordered subset of $\mathcal{F}$ and $D = \bigcup_{j\in J} D_j$. Let I be a τ-dense ideal of R and let $f : I \to D$ be a homomorphism. Since I is finitely generated, $f(I) \subseteq D_k$ for some $k \in J$. By the τ-injectivity of D_k, there exists a homomorphism $g : R \to D_k$ that extends f. Thus D is τ-injective. We also have $D \neq E(R)$. Hence $D \in \mathcal{F}$ and D is an upper bound of $(D_j)_{j \in J}$. By Zorn's lemma, $\mathcal{F}$ has a maximal element, a contradiction. Now the conclusion follows. □

By now we have worked under the hypothesis that, for $p \in \operatorname{Spec}(R)$, the τ-injective hull and the injective hull of R/p do not coincide. Let us see what happens when they are the same.

THEOREM 2.9. *Let $p \in \operatorname{Spec}(R)$ be such that $E_\tau(R/p) = E(R/p)$. Then $E(R/p)$ is minimal τ-injective.*

Proof. Let A be a non-zero submodule of $E_\tau(R/p)$ and let $0 \neq a \in A$. Then there exists $r \in R$ such that $0 \neq ra \in R/p$. Hence $Ann_R(ra) = p$ and we have $Rra \cong R/p$. It follows that $E_\tau(Rra) \cong E_\tau(R/p) = E(R/p)$, whence we get $E_\tau(Rra) = E(R/p)$. We also have $E_\tau(Rra) \trianglelefteq E_\tau(A) \trianglelefteq E_\tau(R/p) = E(R/p)$. Thus $E_\tau(A) = E(R/p)$ and, consequently, $E(R/p)$ is minimal τ-injective. □

In the sequel we are interested in studying certain particular submodules of $E(R/p)$, where R is noetherian and $0 \neq p \in \operatorname{Spec}(R)$.

Following [22, p.83], for each integer $m \geq 1$ denote

$$A_m = \{x \in E(R/p) \mid p^m x = 0\}.$$

Note that $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_m \subseteq A_{m+1} \subseteq \cdots$. If R is noetherian, then we have $E(R/p) = \bigcup_{m=1}^{\infty} A_m$ [22, p.83]. If R is a domain, then each A_m is a proper submodule of $E(R/p)$, because otherwise $p^m \subseteq \operatorname{Ann}_R A_m = \operatorname{Ann}_R E(R/p) = 0$.

We have seen in Theorem 2.5 that if $p \in \operatorname{Spec}(R)$ is such that R/p is τ-cocritical, then $A_1 = \operatorname{Ann}_{E(R/p)} p = E_\tau(R/p)$, hence A_1 is clearly τ-injective. In the case of a noetherian ring, we study the τ-injectivity of the submodules A_m depending on the prime ideal p. Clearly, every $p \in \operatorname{Spec}(R)$ is either τ-dense or τ-closed in R.

THEOREM 2.10. *Let R be noetherian and let $0 \neq p \in \operatorname{Spec}(R)$.*

(i) *If p is τ-dense in R, then each proper submodule A_m of $E(R/p)$ is not τ-injective.*
(ii) *If p is τ-closed in R, then each A_m is τ-injective.*

Proof. (i) Since R is noetherian and R/p is τ-torsion, we have $E_\tau(R/p) = E(R/p)$. Then by Theorem 2.9, $E(R/p)$ is minimal τ-injective and the conclusion follows.

(ii) Let $\mathcal{P}$ be the set of all τ-closed prime ideals of R. Using [21, Chapter VI, Corollary 6.15], it is easy to show that τ is generated by the class consisting of all modules isomorphic to factor modules U/V, where U and V are ideals of R containing a prime ideal q of R with $q \notin \mathcal{P}$.

Let $m \geq 1$. We show that $E(R/p)/A_m$ is τ-torsionfree. Suppose the contrary. Then there exists a non-zero submodule B of $E(R/p)/A_m$ such that $B \cong U/V$, where U and V are ideals of R containing a prime ideal q of R with $q \notin \mathcal{P}$. Then $qU \subseteq V$, hence $qB = 0$. Note that $q \not\subseteq p$, because otherwise $q \subseteq p$ and $p \in \mathcal{P}$ imply $q \in \mathcal{P}$, a contradiction.

On the other hand, there exists an element $x \in E(R/p) \setminus A_m$ such that $x + A_m \in B$. Then $p^m x \neq 0$ and $qx \subseteq A_m$, hence $q \subseteq \mathrm{Ann}_R(p^m x) \subseteq p$ [20, Lemma 2.31], a contradiction. Therefore A_m is τ-injective. □

COROLLARY 2.11. *Let R be a noetherian domain and let $0 \neq p \in \mathrm{Spec}(R)$. Then:*

(i) *p is τ-closed in R if and only if each A_m is τ-injective.*
(ii) *If p is τ-closed in R, then $E(R/p)$ is the union of a totally ordered countable family of proper τ-injective submodules.*
(iii) *If R/p is τ-cocritical, then $E(R/p)$ is the union of a totally ordered countable family of proper τ-injective submodules that contain $E_\tau(R/p)$.*

COROLLARY 2.12. *Let R be a noetherian domain and let $0 \neq p \in \mathrm{Spec}(R)$ be such that R/p is τ-torsionfree, but not τ-cocritical. Then for each $m \geq 1$, there exist τ-injective modules B_k such that*

$$A_m \subset \cdots \subset B_k \subset \cdots \subset B_1 \subset A_{m+1}.$$

Proof. Let us show first that $A_m \neq A_{m+1}$ for each m. Suppose that there exists m such that $A_m = A_{m+1}$. Then $A_{m+i} = A_m$ for every positive integer i, whence $E(R/p) = A_m$, a contradiction.

Now let B be a τ-injective module such that $A_m \subset B \subsetneq A_{m+1}$. Suppose that it does not exist any proper τ-injective submodule of B that strictly contains A_m. By Lemma 2.1, B/A_m is τ-cocritical. Then by Lemma 2.2 we have $q = \mathrm{Ann}_R(B/A_m) \in \mathrm{Spec}(R)$ and R/q is τ-cocritical. Then $q \not\subseteq p$, because otherwise R/p would be either τ-torsion or τ-cocritical.

On the other hand, let $b \in B \setminus A_m$. Then $p^m b \neq 0$ and $qb \subseteq A_m$, hence $q \subseteq \mathrm{Ann}_R(p^m b) \subseteq p$ [20, Lemma 2.31], a contradiction. Now the result follows. □

3. Minimal τ-injective submodules

Now let us continue our discussion on the existence of τ-injective submodules of indecomposable injective modules by considering the minimal τ-injective ones.

LEMMA 3.1. *Let A be an indecomposable injective module. Then A has at most one minimal τ-injective submodule.*

Proof. Suppose that B and C are minimal τ-injective submodules of A. Then $E(B) = E(C) = A$. But B and C are τ-injective, hence they are τ-closed in A. Then $B \cap C$ is τ-closed in A, hence τ-injective. Thus $B \cap C = B = C$. □

For the rest of this section the ring R is assumed to be commutative. For $E(R/p)$ we shall be able to tell exactly which is that minimal τ-injective submodule. To this end, let us first decide when $E_\tau(R/p)$ is minimal τ-injective.

THEOREM 3.2. *Let $p \in$ Spec(R). Then $E_\tau(R/p)$ is minimal τ-injective if and only if R/p is either τ-torsion or τ-cocritical.*

Proof. (i)$\Longrightarrow$(ii) Clear.

(ii)$\Longrightarrow$(i) Suppose first that R/p is τ-torsion and let B be a non-zero submodule of $E_\tau(R/p)$. Then $E_\tau(B)$ is τ-dense in $E_\tau(R/p)$, hence it is a direct summand. But $E_\tau(R/p)$ is uniform, hence we have $E_\tau(R/p) = E_\tau(B)$. Thus $E_\tau(R/p)$ is minimal τ-injective.

If R/p is τ-cocritical, then clearly $E_\tau(R/p)$ is minimal τ-injective. □

The following form of τ-torsionfree minimal τ-injective modules will be useful.

PROPOSITION 3.3. [4, Proposition 2.3] *If A is a τ-torsionfree minimal τ-injective module, then $A \cong E_\tau(R/p)$, where $p = \text{Ann}_R A \in$ Spec(R).*

COROLLARY 3.4. *Let $p \in$ Spec(R). Then:*

(i) *If $E(R/p)$ has a minimal τ-injective submodule, then this is $E_\tau(R/p)$.*

(ii) *$E(R/p)$ is minimal τ-injective if and only if $E_\tau(R/p) = E(R/p)$.*

Proof. (i) By Theorem 3.2 and Proposition 3.3.

(ii) By (i) and Theorem 2.9. □

Now we are able to discuss the existence of τ-injective submodules inside the τ-injective hull of R/p. Note that by Theorem 3.2 this problem makes sense if R/p is neither τ-torsion nor τ-cocritical.

THEOREM 3.5. *Let $p \in \operatorname{Spec}(R)$ be such that R/p is τ-torsionfree, but not τ-cocritical. Then there exist τ-injective modules D_k such that*

$$\cdots \subset D_k \subset \cdots \subset D_1 \subset E_\tau(R/p)$$

and $D_k \cong E_\tau(R/p)$ for each k.

Proof. Note that $E_\tau(R/p)$ is not minimal τ-injective by Theorem 3.2 and we have $E_\tau(R/p) \neq E(R/p)$ by Theorem 2.9.

Now let D be a non-zero proper τ-injective submodule of $E_\tau(R/p)$ and let $0 \neq a \in D$. Then there exists $r \in R$ such that $0 \neq ra \in R/p$. Hence $\operatorname{Ann}_R(ra) = p$ and we have $Rra \cong R/p$. Put $D_1 = E_\tau(Rra)$. Then $D_1 \subseteq D \subset E_\tau(R/p)$ and $D_1 \cong E_\tau(R/p)$. Now repeat the argument for D_1 instead of $E_\tau(R/p)$. The requested modules D_k are obtained inductively. □

In what follows we consider τ-completely decomposable modules, that is, direct sums of minimal τ-injective modules. An important result concerning direct sum decomposition theorems for the τ-injective hull of a module is [18, Proposition 2], where it is given an equivalent condition for the τ-injective hull of a finitely generated module to be a direct sum of uniform submodules. We shall establish conditions under which the τ-injective hull of a module, not necessarily finitely generated, is a direct sum of minimal τ-injective modules. Recall that every minimal τ-injective module is uniform, but the converse does not hold in general. For instance, if $p \in \operatorname{Spec}(R)$ is such that $E_\tau(R/p) \neq E(R/p)$, then $E(R/p)$ is uniform, but not minimal τ-injective. The next result generalizes the corresponding one for indecomposable injective modules [20, Theorem 4.9].

THEOREM 3.6. *Let A be a module and let $B = B_1 \cap \cdots \cap B_n$ be an irredundant intersection of submodules of A such that each $E_\tau(A/B_i)$ is a minimal τ-injective module. Then $E_\tau(A/B)$ is τ-completely decomposable.*

More precisely, $E_\tau(A/B) \cong \bigoplus_{i=1}^n E_\tau(A/B_i)$ and any two such direct sum decompositions are isomorphic.

Proof. Let $f : A \to \bigoplus_{i=1}^n E_\tau(A/B_i)$ be the homomorphism defined by $f(a) = (a + B_1, \dots, a + B_n)$. Then f induces a monomorphism $g : A/B \to \bigoplus_{i=1}^n E_\tau(A/B_i)$. For each $i \in \{1, \dots, n\}$, let $q_i : E_\tau(A/B_i) \to \bigoplus_{i=1}^n E_\tau(A/B_i)$ denote the canonical injection. Since the intersection $B = B_1 \cap \cdots \cap B_n$ is irredundant, for every i there exists $b_i \in B_1 \cap \cdots \cap B_{i-1} \cap B_{i+1} \cap \cdots \cap B_n$ such that $b_i \notin B_i$. Then $g(b_i + B) = q_i(b_i + B_i)$ is a non-zero element of $g(A/B) \cap q_i(A/B_i)$. But $E_\tau(A/B_i)$ is minimal τ-injective, hence so is $q_i(E_\tau(A/B_i))$. Then $q_i(E_\tau(A/B_i))$ is a τ-injective hull of $g(A/B) \cap q_i(A/B_i)$. Hence

$$\begin{aligned}\bigoplus_{i=1}^{n} E_\tau(A/B_i) &= \bigoplus_{i=1}^{n} q_i(E_\tau(A/B_i)) \\ &= E_\tau(\bigoplus_{i=1}^{n}(g(A/B) \cap q_i(A/B_i))) \\ &= E_\tau(g(A/B)) \cong E_\tau(A/B)\,.\end{aligned}$$

Since the endomorphism ring of a minimal τ-injective module is local [11, Proposition 8.16], the Krull-Remak-Schmidt-Azumaya Theorem finishes the proof. □

Now let us illustrate this decomposition result.

EXAMPLE 3.7. Let n be a positive integer and consider the hereditary torsion theory τ_n generated by all modules of Krull dimension at most n. Then a prime ideal p of R is τ_n-dense (respectively τ_n-closed) in R if and only if $\dim p \leq n$ (respectively $\dim p \geq n+1$) [4, Lemma 3.1]. Also, for R noetherian, a module A is τ_n-torsion (respectively τ_n-torsionfree) minimal τ_n-injective if and only if $A \cong E(R/p)$ (respectively $A \cong E_{\tau_n}(R/p)$), where p is a prime ideal of R with $\dim p \leq n$ (respectively $\dim p = n+1$) [4, Theorem 3.5].

Consider the polynomial ring $R = K[X_1, \ldots, X_{n+2}]$, where K is a field and $n \geq 2$. Let $p = (X_1X_2, X_1X_3)$. If $p_1 = (X_1)$ and $p_2 = (X_2, X_3)$, then $p = p_1 \cap p_2$ is an irredundant intersection of the prime ideals p_1 and p_2 of R. We have $\dim p_1 = n+1$ and $\dim p_2 = n$. Then R/p_2 is τ_n-torsion. Since R is noetherian, it follows that R/p_1 is τ_n-cocritical [4, Corollary 3.3] and $E_{\tau_n}(R/p_2) = E(R/p_2)$. Also, $E_{\tau_n}(R/p_1)$ and $E_{\tau_n}(R/p_2)$ are minimal τ_n-injective. Then by Theorem 3.6,

$$\begin{aligned}&E_{\tau_n}(K[X_1, \ldots, X_{n+2}]/(X_1X_2, X_1X_3)) \\ &\cong E_{\tau_n}(K[X_1, \ldots, X_{n+2}]/(X_1)) \oplus E(K[X_1, \ldots, X_{n+2}]/(X_2, X_3))\,.\end{aligned}$$

Using the ring isomorphisms $K[X_1, \ldots, X_{n+2}]/(X_1) \cong K[X_2, \ldots, X_{n+2}]$ and $K[X_1, \ldots, X_{n+2}]/(X_2, X_3) \cong K[X_1, X_4, \ldots, X_{n+2}]$, it follows by Theorem 2.5 that we have the R-isomorphism

$$\begin{aligned}&E_{\tau_n}(K[X_1, \ldots, X_{n+2}]/(X_1X_2, X_1X_3)) \\ &\cong K(X_2, \ldots, X_{n+2}) \oplus E(K[X_1, X_4, \ldots, X_{n+2}])\,,\end{aligned}$$

where $K(X_2, \dots, X_{n+2})$ is the field of fractions of $K[X_2, \dots, X_{n+2}]$. Moreover, $K(X_2, \dots, X_{n+2})$ is τ_n-torsionfree minimal τ_n-injective and $E(K[X_1, X_4, \dots, X_{n+2}])$ is τ_n-torsion minimal τ_n-injective.

In [15] it was proved that every non-zero direct summand of a direct sum of indecomposable injective modules has an indecomposable injective direct summand. We shall generalize this result to the torsion theoretic context, but first we need several preliminary lemmas.

LEMMA 3.8. [5, Lemma 3.5] *If A is a τ-completely decomposable module and B is a τ-injective submodule of A, then B is a direct summand of A.*

LEMMA 3.9. *Let A be a τ-completely decomposable module, let B be a direct summand of A and let C be a τ-injective submodule of A such that $B \cap C = 0$. Then $B \oplus C$ is a direct summand of A.*

Proof. Write $A = B \oplus D$ for some submodule D of A and let $p : A \to D$ be the canonical projection. Since $B \cap C = 0$ we have $p(C) \cong C$. Hence $p(C)$ is a τ-injective submodule of A. Now by Lemma 3.8, $p(C)$ is a direct summand of A and, consequently, of D. But $B \oplus C = B \oplus p(C)$. Therefore $B \oplus C$ is a direct summand of A. □

LEMMA 3.10. [15, Lemma 2.1] *Let X, Y, Z be submodules of a module such that $X \oplus Y = X \oplus Z$. Then there exists an isomorphism $f : Y \to Z$ such that $f(B) \cap C = (X \oplus B) \cap C$ for every submodule B of Y and for every submodule C of Z.*

THEOREM 3.11. *Let A be a τ-completely decomposable module. Then every non-zero direct summand of A has a minimal τ-injective direct summand.*

Proof. Let $A = \bigoplus_{i \in I} A_i$, where each A_i is a minimal τ-injective submodule of A and let B be a non-zero direct summand of A. Denote by $\mathcal{P}$ the family of all finite subsets J of I such that $(\bigoplus_{j \in J} A_j) \cap B \neq 0$. Note that $\mathcal{P}$ is non-empty. Denote by k the least (finite) cardinal of the elements of $\mathcal{P}$, say $k = |K|$ and take $K = \{i_1, \dots, i_k\}$. Also write $A = B \oplus C$ for some submodule C of A.

Suppose first that $k = 1$. Since $(A_{i_1} \cap B) \cap (A_{i_1} \cap C) = 0$, $A_{i_1} \cap B \neq 0$ and A_{i_1} is uniform, we have $A_{i_1} \cap C = 0$. Then by Lemma 3.9, it follows that $A_{i_1} \oplus C$ is a direct summand of A, say $A = A_{i_1} \oplus C \oplus D$. But we also have $A = B \oplus C$. Then there exists an isomorphism $f : A_{i_1} \oplus D \to B$. Hence $B = f(A_{i_1} \oplus D) = f(A_{i_1}) \oplus f(D)$. Therefore $f(A_{i_1})$ is a minimal τ-injective direct summand of B.

Suppose now that $k > 1$. Denote $M = A_{i_1} \oplus \cdots \oplus A_{i_{k-1}}$ and $L = \bigoplus_{i \in I \setminus \{i_1, \dots, i_{k-1}\}} A_i$. Clearly, $M \cap B = 0$ by the choice of k. By Lemma 3.9, since M is τ-injective, $M \oplus B$ is a direct summand of A, say $A = M \oplus B \oplus N$. On the other hand, we have $A = M \oplus L$. By Lemma 3.10, it follows that there exists an isomorphism $g : L \to B \oplus N$ such that $g(A_{i_k}) \cap B = (M \oplus A_{i_k}) \cap B$. But since $(g(A_{i_k}) \cap B) \cap (g(A_{i_k}) \cap N) = 0$, $g(A_{i_k}) \cap B = (\bigoplus_{i \in K} A_i) \cap B \neq 0$ and $g(A_{i_k})$ is uniform, we have $g(A_{i_k}) \cap N = 0$. Now repeat the argument used for $k = 1$. Then B will have a minimal τ-injective direct summand isomorphic to $g(A_{i_k})$. $\square$

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