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ON A VORTICITY MINIMIZATION PROBLEM FOR THE STATIONARY 2D STOKES EQUATIONS

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ABSTRACT. This paper is concerned with a boundary control problem for the vorticity minimization, in which the flow is governed by the stationary two dimensional Stokes equations. We wish to find a mathematical formulation and a relevant process for an appropriate control along the part of the boundary to minimize the vorticity due to the flow. After showing the existence and uniqueness of an optimal solution, we derive the optimality conditions. The differentiability of the state solution in regard to the control parameter shall be conjunct with the necessary conditions for the optimal solution. For the minimizer, an algorithm based on the conjugate gradient method shall be proposed.

1. Introduction

In this paper, we are concerned with a vorticity minimization problem for a flow which is governed by the two dimensional stationary Stokes equations. Let us describe the boundary control problem for the Stokes equations that models the minimization of the vorticity in a fluid flow. Let Ω be a bounded domain in $\mathbb{R}^2$ with $\mathcal{C}^2$ boundary. For practical purposes, we assume that the boundary $\partial\Omega \equiv \Gamma$ is composed of two disjoint parts with positive measures; the homogeneous part Γ_0 and the control part Γ_c such that $\Gamma = \Gamma_0 \cup \Gamma_c$. We are concerned with the following Stokesian flow in Ω with the control effected over the boundary Γ_c:

$$-\nu\Delta\vec{u} + \nabla p = \vec{f} \quad \text{in } \Omega, \tag{1.1}$$

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and

$$\nabla \cdot \vec{u} = 0 \quad \text{in}\, \Omega \tag{1.2}$$

along with the Dirichlet boundary condition

$$\vec{u} = \begin{cases} \vec{g} & \text{on } \Gamma_c, \\ \vec{0} & \text{on } \Gamma_0 . \end{cases} \tag{1.3}$$

Here, we denote the gradient operator by ∇ and the Laplacian operator by $\Delta \equiv \nabla^2$. The vector field $\vec{u}$ represents the velocity of the flow, p the pressure and $\nu > 0$ the inverse of the dimensionless Reynolds number. In the Newtonian fluid based on the constitutive laws, this Stokes system appears when the flow is stationary and the Reynolds number is relatively small so that the nonlinear convective term $(\vec{u}\cdot\nabla)\vec{u}$ in the whole Navier-Stokes system is too small to be neglected.

We describe the control parameter by the boundary velocity $\vec{g}$ along the part Γ_c of the boundary. For the compatibility and regularity for the solutions of the equations (1.1)-(1.3), the control parameter $\vec{g}$ should satisfy

$$\text{support of } \vec{g} \subset \Gamma_c\,, \quad \text{and} \quad \int_{\Gamma_c} \vec{g}\cdot\vec{n}\, ds = 0\,, \tag{1.4}$$

where $\vec{n}$ is the unit normal vector along Γ_c.

The modeling boundary control problem for the vorticity minimization is formulated as follows:

Find the optimal boundary control $\vec{g}$ along Γ_c minimizing the cost functional

$$(\mathrm{P}): \quad J(\vec{u},\vec{g}) = \frac{1}{2}\int_\Omega |\nabla \times \vec{u}|^2\, dx + \frac{\alpha}{2}\int_{\Gamma_c} |\vec{g}|^2\, ds\,, \tag{1.5}$$

where $\vec{u}$ is subject to the two dimensional Stokes system (1.1)-(1.4).

Here, $\nabla \times \vec{u}$ denotes the curl operator in $\mathbb{R}^2$. In the expression for $J(\vec{u},\vec{g})$, the first term $\int_\Omega |\nabla \times \vec{u}|^2\, dx$ measures the vorticity stemmed from the fluid flow, and the second term $\int_{\Gamma_c} |\vec{g}|\, ds$ defines the control factor along the boundary Γ_c. It is often demanded for a concession of mathematical rigor for the control. The positive penalty parameter α in (1.5) may be used to switch the relative importance between terms in (1.5). It is also necessary to keep the uniform boundedness for the control terms.

The vorticity introduced by the fluid flow is an important factor dealt with the fluid dynamics and mechanics. It is recognized as the force generating the turbulence. It has been regarded as a major source of the disturbance in the fluid flow, and is closely connected with a variety of technical applications in science and engineering such as aerodynamics and the crystal growth process. In [1], Abergel et. al. discussed some turbulence control problems thorough the distributed control. However, the boundary control for the turbulence minimization raises some significant difficulties in connection with the rigorous mathematical formulation for the control as we shall indicate at the forwarding section. The purpose of this paper is to investigate the mathematical structure for the boundary control dealing with the minimization of the vorticity, which generates the rotational force due to the fluid flow. For this purpose, we begin with a relatively simple dynamical situations. We assume the fluid flow is governed by the two dimensional stationary Stokes equations under the divergence free condition. The divergence free condition stands for the incompressibility of the fluid volume under the constitutive laws. Using the divergence free condition, one may transfer the Stokes problem (1.1)-(1.5) into the stream function formulation, or the stream function-vorticity formulation. In this paper, we examine the velocity-pressure formulation linked with an orthogonal projector.

The plan of the study is as follows. In the remainder of this section, we will introduce some notations and preliminary results that will be useful in what follows. In section 2, we will discuss the existence of an optimal solution and examine the first order necessary conditions for the minimizer. In section 3, we organize the optimality system for the control problem, and for a minimizer, an algorithm based upon the conjugate gradient method shall be provided.

1.1. Notations and preliminaries

Throughout this paper, we will denote the generic constants by C whose values depend on the context. Let Ω be a bounded domain in $\mathbb{R}^2$ with $\mathcal{C}^2$ boundary. Let us denote by $H^s(\mathcal{O})$, $s \in \mathbb{R}$, the standard Sobolev space of order s, with respect to the set $\mathcal{O}$, which is either the flow domain Ω, or its boundary Γ, or part of its boundary. We denote the inner product on $H^s(\mathcal{O})$ by $(\cdot\,,\cdot)_s$ and its norm by $\|\cdot\|_s = \sqrt{(\cdot\,,\cdot)_s}$. Especially, when $s = 0$, $H^0(\mathcal{O}) = L^2(\mathcal{O})$, and $\|\cdot\|_0$ represents L^2-norm.

Also, it will be convenient to have a compact notation for partial derivatives. We shall write ∂_j for $\dfrac{\partial}{\partial x_j}$, and for higher-order derivatives we use multi-index notation. When $\mu = (\mu_1, \mu_2)$ is a multi-index, we set

$$|\mu| = \sum_{j=1}^{2} \mu_j \quad \text{with} \quad \partial^\mu = \left(\frac{\partial}{\partial x_1}\right)^{\mu_1} \left(\frac{\partial}{\partial x_2}\right)^{\mu_2} .$$

Whenever m is a nonnegative integer, the inner product over $H^m(\mathcal{O})$ is defined by

$$(\phi\,,\psi)_m = \sum_{|\mu|\le m} (\partial^\mu \phi\,, \partial^\mu \psi)\,,$$

and its norm by

$$\|\phi\|_m^2 = \sum_{|\mu|\le m} \int_{\mathcal{O}} |\partial^\mu \phi(x)|^2 \, dx\,,$$

where $\mu = (\mu_1\,, \mu_2)$ is a multi-index.

For spaces of vector-valued functions, we will use boldface notation. For example, $\mathbf{H}^m(\Omega) = [H^m(\Omega)]^2$ denotes the space of $\mathbb{R}^2$-valued functions such that each component of an element in $\mathbf{H}^s(\Omega)$ belongs to $H^s(\Omega)$. Of particular interest for our purpose is the space

$$\mathbf{H}^1(\Omega) = \{\vec{v} = (v_1\,, v_2) \in \mathbf{L}^2(\Omega) \,|\, \partial_j v_i \in L^2(\Omega)\,,\ 1 \le i,\, j \le 2\}\,,$$

which is equipped with the norm

$$\|\vec{v}\|_1^2 = \|v_1\|_1^2 + \|v_2\|_1^2 .$$

Whenever $\Gamma_0 \subset \Gamma$ has a positive measure, the space with the homogeneous boundary condition imposed along Γ_0 is defined by $\mathbf{H}^1_{\Gamma_0}(\Omega) = \{\vec{v} \in \mathbf{H}^1(\Omega) \,|\, \vec{v} = \vec{0} \ \text{ on } \Gamma_0\}$, and we let $\mathbf{H}^1_{\Gamma}(\Omega) = \mathbf{H}^1_0(\Omega)$. For details about these spaces, see, e.g., [2], [6], [8], and [12].

Of special use, we define the space of infinitely differentiable functions with the divergence free condition by

$$\mathcal{V}(\Omega) = \{\vec{v} \in \mathbf{C}^\infty(\bar{\Omega}) \mid \nabla \cdot \vec{v} = 0 \text{ in } \Omega\,,\ \ \vec{v} = \vec{0} \text{ on } \Gamma_0\}\,,$$

and its completion on $\mathbf{L}^2(\Omega)$ and $\mathbf{H}^1(\Omega)$ by

$$\mathbf{H} = \{\vec{v} \in \mathbf{L}^2(\Omega) \mid \nabla \cdot \vec{v} = 0 \text{ in } \Omega\,,\ \ \vec{v} = \vec{0} \text{ on } \Gamma_0\}$$

and

$$\mathbf{V} = \{\vec{v} \in \mathbf{H}^1_{\Gamma_0}(\Omega) \mid \nabla \cdot \vec{v} = 0 \text{ in } \Omega\}\,,$$

respectively. Also, we will define the closed subspace of $\mathbf{H}$ and $\mathbf{V}$, where the homogeneous condition is imposed on the whole boundary, by

$$\mathbf{H}_0 = \{\vec{v} \in \mathbf{L}^2(\Omega) \mid \nabla \cdot \vec{v} = 0 \text{ in } \Omega, \text{ and } \vec{v} = \vec{0} \text{ on } \Gamma\}$$

and

$$\mathbf{V}_0 = \{\vec{v} \in \mathbf{H}_0^1(\Omega) \mid \nabla \cdot \vec{v} = 0 \text{ in } \Omega\}.$$

On the space $\mathbf{H}$ the norm and the inner product are denoted by $\|\vec{v}\|_0$ and $(\vec{u}, \vec{v})$ respectively. We also denote the seminorm on $\mathbf{V}$ by $\|\vec{v}\| = \|\nabla\vec{v}\|_0$. According to Poincare's inequality ([6]), this is equivalent to the norm of $\mathbf{H}^1(\Omega)$. Here, the norm on $\mathbf{H}^1(\Omega)$ is given by

$$\|\vec{v}\|_1 = \left(\|\vec{v}\|_0^2 + \|\vec{v}\|^2\right)^{1/2}.$$

Let us denote the dual space of $\mathbf{V}$ by $\mathbf{V}^*$ and the duality between $\mathbf{V}^*$ and $\mathbf{V}$ by $< \cdot, \cdot >_{\mathbf{V}^*}$. Since $\mathbf{V}$ is densely imbedded in $\mathbf{H}$ and $\mathbf{H}$ may be identified with its dual $\mathbf{H}^*$ by Riesz's theorem (cf. [10]), the spaces constitutes the canonical Gelfand's framework for the weak variational formulation in the sense of

$$\mathbf{V} \subset \mathbf{H} \subset \mathbf{V}^*,$$

where the inclusions implicitly define dense embeddings. Especially the space $\mathbf{V}$ is compactly imbedded in $\mathbf{H}$.

For the concerned boundary Γ_c, $\mathbf{H}_0^s(\Gamma_c)$ denotes the space of functions in $\mathbf{H}^s(\Gamma_c)$ with compact support in Γ_c. We shall denote the dual space of $\mathbf{H}_0^s(\Gamma_c)$ by $\mathbf{H}^{-s}(\Gamma_c)$ and the duality between $\mathbf{H}^{-s}(\Gamma_c)$ and $\mathbf{H}_0^s(\Gamma_c)$ by $< \cdot, \cdot >_{-s,\Gamma_c}$. Restricting the domain of integration, we represent the norm $\| \cdot \|_{s,\Gamma_c}$ for $\mathbf{H}^s(\Gamma_c)$. We also define the traces to the control part Γ_c by $\gamma_c^0 : \mathbf{H}_{\Gamma_0}^1(\Omega) \to \mathbf{H}^{1/2}(\Gamma_c)\,; (\vec{v} \mapsto \vec{v}\big|_{\Gamma_c})$, and $\gamma_c^1 : \mathbf{H}_{\Gamma_c}^1(\Omega) \to \mathbf{H}^{-1/2}(\Gamma_0)\,; \left(\vec{v} \mapsto \frac{\partial \vec{v}}{\partial \vec{n}}\Big|_{\Gamma_c}\right)$, respectively.

In view of (1.4), we take the space for the control parameter as

$$\mathbf{W} = \left\{ \vec{g} \in \mathbf{H}_0^{1/2}(\Gamma_c) \,\middle|\, \int_{\Gamma_c} \vec{g} \cdot \vec{n}\, ds = 0 \right\}.$$

It is clear that $\mathbf{W}$ is a closed subspace of $\mathbf{H}^{1/2}(\Gamma_c)$.

The following lemma, which is called the Lax-Milgram lemma, plays an essential role not only in the formulations but also in the exposition of the well-posedness associated with the elliptic systems, whenever the data are appropriately good.

LEMMA 1.1 *Let X be a real separable Hilbert space equipped with the norm $\|\cdot\|_X$. Let $a(\cdot,\cdot) : X \times X \to \mathbb{R}$ be a bilinear form satisfying*

$$|a(x,y)| \le c_1 \|x\|_X \|y\|_X \quad \forall x,\, y \in X\,, \quad (X\text{-continuity})$$

and

$$a(x,x) \ge c_2 \|x\|_X^2 \quad \forall x \in X\,, \quad (X\text{-coercivity}),$$

where c_1 and c_2 are positive constants independent of x, $y \in X$. Then, for each $f \in X^*$, there exists a unique solution $x \in X$ satisfying

$$a(x,y) =< f, y >, \quad \forall y \in X$$

and

$$\|x\|_X \le \frac{1}{c_2} \|f\|_{X^*} \,.$$

For the proof, one may consult with [6], [8], and [11].

Finally, we need to refer to the two dimensional curl operators. Let $\mathcal{D}'(\Omega)$ denote the space of distributions in Ω. In the sense of distributions, two kinds of curl operators are introduced:

$$\vec{\nabla} \times \phi = (\partial_2 \phi, -\partial_1 \phi) \quad \text{for } \phi \in \mathcal{D}'(\Omega)$$

and

$$\nabla \times \vec{v} = \partial_1 v_2 - \partial_2 v_1 \quad \text{for } \vec{v} = (v_1, v_2) \in (\mathcal{D}'(\Omega))^2 \,.$$

One can easily check the following identities hold:

$$\nabla \times (\vec{\nabla} \times \phi) = -\Delta \phi \tag{1.6}$$

and

$$\vec{\nabla} \times (\nabla \times \vec{v}) = -\Delta \vec{v} + \nabla(\nabla \cdot \vec{v}) \,.$$

Hence, it is immediately followed that

$$\vec{\nabla} \times (\nabla \times \vec{v}) = -\Delta \vec{v}\,, \quad \forall \vec{v} \in \mathbf{V} \,. \tag{1.7}$$

REMARK 1.1. In the three dimensional case, there is no corresponding curl operator acting on the scalar function as in (1.6). When $\vec{v} = (v_1, v_2, v_3)$ is a differentiable vector field in $\mathbb{R}^3$, the curl operator is defined by

$$\vec{\nabla} \times \vec{v} = (\partial_2 v_3 - \partial_3 v_2, \partial_3 v_1 - \partial_1 v_3, \partial_1 v_2 - \partial_2 v_1) \,.$$

Then, as in (1.7), it also holds that

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{v}) = -\,\Delta \vec{v} \quad \forall \vec{v} \in \mathbf{V} \,.$$

2. Existence results and the optimality system

In this section, we will show the existence of an optimal solution for the control problem (P) and establish the first order necessary conditions for an optimal solution by using the differentiability of the state solution.

2.1. Existence of an optimal solution

A variational form for the Stokes equations (1.1)-(1.3) can be introduced by the bilinear form $a(\cdot\,,\cdot) : \mathbf{V} \times \mathbf{V} \to \mathbb{R}$, which is defined by

$$a(\vec{u}\,,\vec{v}) = \nu \int_\Omega \nabla\vec{u} \cdot \nabla\vec{v}\,dx\,.$$

Obviously, $a(\cdot,\cdot)$ is coercive over $\mathbf{V}$, and according to the Poincare's inequality it also follows for some positive constants C_1 and C_2

$$|a(\vec{u},\vec{v})| \le C_1\|\vec{u}\|_1\|\vec{v}\|_1 \le C_2\|\vec{u}\|\|\vec{v}\| \quad \forall\,\vec{u}\,,\,\vec{v} \in \mathbf{V}\,.$$

Hence the bilinear form $a(\cdot,\cdot)$ is continuous over $\mathbf{V}$.

By the Green's first identity, we have for every $\vec{v} \in \mathbf{V}_0 = \mathbf{V} \cap \mathbf{H}_0^1(\Omega)$,

$$\begin{aligned} a(\vec{u}\,,\vec{v}) &= \nu \int_\Omega \nabla\vec{u}\cdot\nabla\vec{v}\,dx \\ &= \nu \int_\Omega \Big(\nabla\cdot(\vec{v}\cdot\nabla\vec{u}) - \Delta\vec{u}\cdot\vec{v}\Big)\,dx \\ &= \nu \int_{\partial\Omega} \vec{v}\cdot\frac{\partial\vec{u}}{\partial\vec{n}}\,ds - \nu\int_\Omega \vec{v}\cdot\Delta\vec{u}\,dx \\ &= \nu\int_\Omega \vec{v}\cdot(-\,\Delta\vec{u})\,dx\,. \end{aligned}$$

Thus, the Stokes equations (1.1)-(1.3) can be written by the following variational form:

Seek $\vec{u} \in \mathbf{V}$ satisfying

$$\text{(2.1)} \qquad \begin{cases} a(\vec{u},\vec{v}) = & <\vec{f},\vec{v}> \quad \forall\,\vec{v} \in \mathbf{V}_0\,, \\ \vec{u}(s) = & \vec{g}(s) \qquad\quad \forall\,s \in \Gamma_c\,, \\ \vec{u}(s) = & \vec{0} \qquad\qquad \forall\,s \in \Gamma_0\,. \end{cases}$$

In the following, we demonstrate the well-posedness for the system (2.1) as well as the regularity result.

PROPOSITION 2.1. *Let Ω be a bounded domain in $\mathbb{R}^2$ with C^2-boundary. Let $\vec{f} \in \mathbf{H}^{-1}(\Omega)$ and $\vec{g} \in \mathbf{W}$. Then, the system* (2.1) *has a unique solution $\vec{u} \in \mathbf{V}$, and the system is well posed in a sense*

$$\|\vec{u}\| \leq C(\|\vec{f}\|_{-1} + \|\vec{g}\|_{1/2,\Gamma_c}), \tag{2.2}$$

where the constant C is independent of $\vec{f}$ and $\vec{g}$.

Furthermore, if $\vec{f} \in \mathbf{L}^2(\Omega)$ and $\vec{g} \in \mathbf{W} \cap \mathbf{H}^{3/2}(\Omega)$, $\vec{u}$ belongs to $\mathbf{V} \cap \mathbf{H}^2(\Omega)$, and we have

$$\|\vec{u}\|_2 \leq C(\|\vec{f}\|_0 + \|\vec{g}\|_{3/2,\Gamma_c}). \tag{2.3}$$

Proof. Since $\vec{g} \in \mathbf{W}$, there exists $\vec{u}_0 \in \mathbf{V}$ such that $\gamma_c^0(\vec{u}_0) = \vec{g}$ by Ladyzhenskaya [9] (see also [8], [12]), and $\vec{u} - \vec{u}_0$ belongs to $\mathbf{V}_0$. Hence, substituting in (2.1) by $\vec{v}=\vec{u} - \vec{u}_0$, we have

$$a(\vec{u}, \vec{u} - \vec{u}_0) = < \vec{f}, \vec{u} - \vec{u}_0 > . \tag{2.4}$$

Since the bilinear form $a(\cdot,\cdot)$ is continuous and coercive over the space $\mathbf{V}$, (2.4) has a unique solution by Lemma 1.1.

According to the Poincare's inequality applied to

$$a(\vec{u} - \vec{u}_0, \vec{u} - \vec{u}_0) = < \vec{f}, \vec{u} - \vec{u}_0 > + a(\vec{u}_0, \vec{u} - \vec{u}_0),$$

one can get for some positive constant C_1

$$\|\vec{u} - \vec{u}_0\|^2 \leq C_1(\|\vec{f}\|_{-1}\|\vec{u} - \vec{u}_0\| + \|\vec{u}_0\|\|\vec{u} - \vec{u}_0\|),$$

so that

$$\|\vec{u} - \vec{u}_0\| \leq C_1(\|\vec{f}\|_{-1} + \|\vec{u}_0\|).$$

Hence, from the triangle inequality we have

$$\|\vec{u}\| \leq \|\vec{u} - \vec{u}_0\| + \|\vec{u}_0\| \leq C(\|\vec{f}\|_{-1} + \|\vec{u}_0\|)$$

for constant $C > 0$. If we take infimum over all $\vec{u}_0 \in \mathbf{V}$ such that $\gamma_c^0(\vec{u}_0) = \vec{g}$, then the following is derived from the above inequality

$$\|\vec{u}\| \leq C(\|\vec{f}\|_{-1} + \|\vec{g}\|_{1/2,\Gamma_c}).$$

The regularity result stated in (2.3) is standard in elliptic systems. □

In the next proposition, we show the existence of an optimal solution.

PROPOSITION 2.2. *Given $\vec{f} \in \mathbf{H}^{-1}(\Omega)$, there exists a unique minimal solution $(\vec{u}, \vec{g})$ for the problem (P) such that $\vec{u}$ is the solution of* (1.1)-(1.3) *with $\gamma_c^0(\vec{u}) = \vec{g}$.*

Proof. Let

$$\mathcal{U} := \{(\widetilde{u}, \widetilde{g}) \in \mathbf{V} \times \mathbf{W} \,|\widetilde{u} \text{ is a solution of the system } (1.1)\text{-}(1.3) \text{ such that } \gamma_c^0(\widetilde{u}) = \widetilde{g}\}.$$

Let $\widehat{g}$ be an element of $\mathbf{W}$. Then, by Proposition 2.1, there exists a unique $\widehat{u} \in \mathbf{V}$ such that $\widehat{u}$ satisfies the system (1.1)-(1.3) with $\gamma_c^0(\widehat{u}) = \widehat{g}$. Hence $\mathcal{U}$ is not empty. Let $\{(\vec{u}_n, \vec{g}_n)\} \subset \mathcal{U}$ be a minimizing sequence, which is bounded below. Since $\vec{g}_n$ is uniformly bounded, $\vec{u}_n$ is also uniformly bounded by (2.2). So, one can extract a subsequence (denoted again by the same notation) which converges weakly to $(\vec{u}, \vec{g})$. Using the compact embedding of $\mathbf{V}$ in $\mathbf{H}$ and the continuity of the trace mapping, one can deduce

$$\begin{array}{llll} \vec{g}_n & \rightharpoonup & \vec{g} & \text{in } \mathbf{W}, \\ \vec{u}_n & \rightharpoonup & \vec{u} & \text{in } \mathbf{V}, \\ \gamma_c^0(\vec{u}_n) & \rightharpoonup & \gamma_c^0(\vec{u}) & \text{in } \mathbf{W}, \\ \vec{u}_n & \longrightarrow & \vec{u} & \text{in } \mathbf{H}, \end{array} \tag{2.5}$$

where $\rightharpoonup$ denotes the weak convergence.

Concerned with the vorticity term in the cost functional, especially we have

$$\int_\Omega (\nabla \times \vec{u}_n) \cdot \phi \, dx - \int_\Omega \vec{u}_n \cdot (\vec{\nabla} \times \phi) \, dx = \int_{\Gamma_c} (\vec{u}_n \cdot \vec{\tau}) \phi \, ds \quad \forall \phi \in H^1(\Omega).$$

This can be derived by taking integration by parts. Hence, for every $\vec{v} \in \mathbf{V} \cap \mathbf{H}^2(\Omega)$, relations (2.5) allow us to pass to the limit with the aid of (1.7) that

$$\begin{aligned} \int_\Omega \nabla \times \vec{u}_n \cdot \nabla \times \vec{v} \, dx &= \int_\Omega \vec{u}_n \cdot \vec{\nabla} \times (\nabla \times \vec{v}) \, dx + \int_{\Gamma_c} (\vec{u}_n \cdot \vec{\tau})(\nabla \times \vec{v}) \, ds \\ &= \int_\Omega \vec{u}_n \cdot (-\Delta \vec{v}) \, dx + \int_{\Gamma_c} (\vec{u}_n \cdot \vec{\tau})(\nabla \times \vec{v}) \, ds \\ &\longrightarrow \int_\Omega \vec{u} \cdot (-\Delta \vec{v}) \, dx + \int_{\Gamma_c} (\vec{u} \cdot \vec{\tau})(\nabla \times \vec{v}) \, ds \\ &= \int_\Omega \nabla \times \vec{u} \cdot \nabla \times \vec{v} \, dx, \end{aligned}$$

which yields

$$\nabla \times \vec{u}_n \rightharpoonup \nabla \times \vec{u} \quad \text{in } \mathbf{H}.$$

We also note that the Young's inequality produces

$$\int_\Omega |\nabla \times \vec{u}|^2 \, dx = \int_\Omega (\partial_1 u_2 - \partial_2 u_1)^2 \, dx \le C\|\vec{u}\|^2 .$$

This implies that the cost functional J is strongly continuous, and hence J is lower-semicontinuous. Thus, if we pass the sequence to the limit in $\mathcal{U}$, it follows

$$J(\vec{u},\vec{g}) \le \liminf_{n\to\infty} J(\vec{u}_n,\vec{g}_n) .$$

Therefore, the functional J is minimized at $(\vec{u},\vec{g})$. It is a direct application of (2.5) to show that $(\vec{u},\vec{g})$ belongs to $\mathcal{U}$. Thus, $(\vec{u},\vec{g})$ is an optimal solution. Furthermore, since J is convex, the solution is unique. □

2.2. The first order necessary conditions

In this section, we discuss the question of what relations characterizes an optimal solution. We proceed to derive the first order optimality conditions associated with the problem.

To begin with, it will be convenient to introduce an orthogonal projector $\mathcal{P}$, which is called a Leray projector. We note that $\mathbf{L}^2(\Omega)$ can be decomposed by

$$\mathbf{L}^2(\Omega) = \mathbf{H}_0 \oplus \mathbf{H}_0^\perp , \tag{2.6}$$

where

$$\mathbf{H}_0^\perp = \{\vec{v} \in \mathbf{L}^2(\Omega) \,|\, \vec{v} = \nabla\phi \text{ for some } \phi \in H^1(\Omega)\} .$$

If $\vec{u} \in \mathbf{H}_0$, from $\int_\Omega \vec{u}\cdot\nabla\phi \, dx = \int_\Gamma \phi\vec{u}\cdot\vec{n}\, ds - \int_\Omega (\nabla\cdot\vec{u})\phi\, dx$, it follows that $\int_\Omega \vec{u}\cdot\nabla\phi\, dx = 0$ for every $\phi \in H^1(\Omega)$. On the other hand, if $\vec{u} \in \mathbf{L}^2(\Omega)$ satisfies $\vec{u} = \vec{0}$ on Γ and $\int_\Omega \vec{u}\cdot\nabla\phi\, dx = 0$ for all $\phi \in H^1(\Omega)$, then by the Green's formula we have $\nabla\cdot\vec{u} = 0$ in Ω, so that $\vec{u} \in \mathbf{H}_0$. This justifies the orthogonal decomposition of $\mathbf{L}^2(\Omega)$.

Let us define the orthogonal projector $\mathcal{P} : \mathbf{L}^2(\Omega) \to \mathbf{H}_0$. Then, with $\mathcal{A} = \mathcal{P}(-\Delta)$, the variational form (2.1) for the Stokes equations can be equivalently written by

$$\begin{cases} \nu\mathcal{A}\vec{u} &= \mathcal{P}\vec{f} \quad \text{in } \Omega , \\ \vec{u}(s) &= \vec{g}(s) \quad \forall\, s \in \Gamma_c , \\ \vec{u}(s) &= \vec{0} \quad \forall\, s \in \Gamma_0 . \end{cases} \tag{2.7}$$

By Proposition 2.1, the solution $\vec{u}$ for the system (2.7) can be regarded as a function of the control parameter $\vec{g}$. Also, by Proposition 2.2, it is justified to reformulate the cost $J(\cdot,\cdot)$ into the functional of a single parameter $\vec{g}$ as

$$\mathcal{J}(\vec{g}) = J(\vec{u}(\vec{g}), \vec{g}) \quad \text{for } \vec{g} \in \mathbf{W}\,.$$

In order to deduce the necessary conditions for the optimality, we need to examine the first order variation of the functional $\mathcal{J}$ with respect to the control parameter. The rate of the variation $\mathcal{J}$ at $\vec{g}$ can be measured as a directional semi-derivative

$$d\mathcal{J}(\vec{g};\vec{h}) = \frac{d}{dt}\mathcal{J}(\vec{g}+t\vec{h})\Big|_{t=0} \quad \text{for } \vec{h} \in \mathbf{W}\,.$$

This derivation for $\mathcal{J}$ is said to be Gateaux derivative at $\vec{g}$ if

- $d\mathcal{J}(\vec{g};\vec{h})$ exists for every $\vec{h}$,
- $\vec{h} \mapsto d\mathcal{J}(\vec{g};\vec{h})$ is linear and continuous.

Before going ahead, we need to show that the solution of the Stokes equations (2.7) is strictly differentiable.

LEMMA 2.3 *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain of the class $\mathcal{C}^2$. Let $\vec{f} \in \mathbf{L}^2(\Omega)$ be given. Then, the mapping*

$$\vec{u} : \mathbf{W} \to \mathbf{V}\,; \quad \Big(\vec{g} \mapsto \vec{u}(\vec{g})\Big)$$

is differentiable. Furthermore, if we represent the Gateaux derivative of $\vec{u}$ at $\vec{g}$ by $\vec{u}\,'$ with $\vec{u}\,'(\vec{h}) \equiv d\vec{u}(\vec{g};\vec{h})$ in every direction, then for every $\vec{h} \in \mathbf{W}$, $\vec{u}\,' \in \mathbf{V}$ is the solution of the system

$$\text{(2.8)} \qquad \begin{cases} \nu\mathcal{A}\vec{u}\,' &= \vec{0} \quad \text{in } \Omega\,, \\ \vec{u}\,' &= \vec{h} \quad \text{on } \Gamma_c\,. \end{cases}$$

Proof. Since the systems (2.7) and (2.8) are linear, it is obvious that

$$\|\vec{u}(\vec{g}+t\vec{h}) - \vec{u}(\vec{g}) - t\vec{u}\,'(\vec{h})\| = 0\,,$$

and this completes the proof. □

The derivation of the Gateaux derivative for the functional $\mathcal{J}$ essentially depends on the following.

LEMMA 2.4 *We assume the conditions of Lemma 2.3. If $\vec{u}\,'$ is a solution of the system (2.8), then for every $\vec{e} \in \mathbf{L}^2(\Omega)$, we have*

$$\text{(2.9)} \qquad \int_\Omega \vec{e}\cdot\vec{u}\,'(\vec{h})\,dx = \int_{\Gamma_c} -\nu\frac{\partial\widetilde{w}}{\partial\vec{n}}\cdot\vec{h}\,ds\,,$$

where $\widetilde{w} \in \mathbf{V}$ is the solution of the adjoint equations

$$(2.10) \qquad \begin{cases} \nu \mathcal{A}\widetilde{w} &= \mathcal{P}\vec{e} \quad in\ \Omega\,, \\ \widetilde{w} &= \vec{0} \quad on\ \Gamma\,. \end{cases}$$

Proof. For a preliminary arrangement, we provide a noticeable fact. Since the traces $\gamma_c^0(\vec{u}\,') = \vec{h}$ and $\gamma_c^0(\widetilde{w}) = \vec{0}$, the Green's second identity yields that

$$\begin{aligned} & < \mathcal{A}\widetilde{w}, \vec{u}\,' > - < \mathcal{A}\vec{u}\,', \widetilde{w} > \\ = \ & < \gamma_c^1(\vec{u}\,'), \gamma_c^0(\widetilde{w}) >_{\Gamma_c} - < \gamma_c^1(\widetilde{w}), \gamma_c^0(\vec{u}\,') >_{\Gamma_c} \\ = \ & - < \gamma_c^1(\widetilde{w}), \vec{h} >_{\Gamma_c} . \end{aligned}$$

According to the above respects, we obtain

$$\begin{aligned} \int_\Omega \vec{e} \cdot \vec{u}\,'(\vec{h})\, dx &= \int_\Omega \mathcal{P}\vec{e} \cdot \vec{u}\,'(\vec{h})\, dx \\ &= \int_\Omega \nu \mathcal{A}\widetilde{w} \cdot \vec{u}\,'\, dx \\ &= \int_\Omega \nu \mathcal{A}\widetilde{w} \cdot \vec{u}\,'\, dx - \int_\Omega \nu \mathcal{A}\vec{u}\,' \cdot \widetilde{w}\, dx \\ &= \nu < \mathcal{A}\widetilde{w}, \vec{u}\,' > - \nu < \mathcal{A}\vec{u}\,', \widetilde{w} > \\ &= -\nu < \gamma_c^1(\widetilde{w}), \vec{h} >_{\Gamma_c} \\ &= \int_{\Gamma_c} -\nu \frac{\partial \widetilde{w}}{\partial \vec{n}} \cdot \vec{h}\, ds\,. \end{aligned}$$

In this estimation, we used the facts that $\mathcal{A}\vec{u}\,' = \vec{0}$ in Ω and $\vec{u}\,' = \vec{0}$ on Γ_0. □

We are now ready to estimate the first order Gateaux derivative.

PROPOSITION 2.5 *Let Ω be a bounded domain of the class $\mathcal{C}^2$ in $\mathbb{R}^2$. For the body force $\vec{f} \in \mathbf{L}^2(\Omega)$, let $(\vec{u}, \vec{g})$ be an optimal solution for the boundary control problem* (P) *with $\vec{u} = \vec{u}(\vec{g})$. Then, the Gateaux derivative for the functional $\mathcal{J}$ at $\vec{g}$ in the $\vec{h}$-direction is given by*

$$(2.11) \qquad d\mathcal{J}(\vec{g}; \vec{h}) = < \Big(-\nu\gamma_c^1(\vec{w}) + \gamma_c^0(\nabla \times \vec{u})\, \vec{\tau} + \alpha \vec{g} \Big), \vec{h} >_{\Gamma_c} ,$$

where $\vec{w}$ is the solution of the adjoint system

$$
(2.12) \qquad \begin{cases} \nu \mathcal{A}\vec{w} = \mathcal{A}\vec{u} & \text{in } \Omega, \\ \vec{w} = \vec{0} & \text{on } \Gamma. \end{cases}
$$

In this expression, $\vec{\tau}$ denotes the unit tangent vector along Γ_c.

Proof. In $\mathbb{R}^2$, we have the following relation between the unit normal vector $\vec{n} = (n_1, n_2)$ and the unit tangent vector $\vec{\tau} = (\tau_1, \tau_2)$:

$$
\vec{n} = (n_1, n_2) = (\tau_2, -\tau_1).
$$

Hence, if $\phi \in H^1(\Omega)$ and $\vec{u}' = \vec{u}'(\vec{h})$ is the solution of the system (2.8), the Green's formula yields

$$
\int_\Omega \phi \cdot \nabla \times \vec{u}'\, dx = \int_\Omega \vec{\nabla} \times \phi \cdot \vec{u}'\, dx + \int_{\Gamma_c} (\phi\, \vec{\tau}) \cdot \vec{h}\, ds.
$$

It is also noteworthy that the analogous result for the curl operator can be generalized into tangential vector fields on the boundary of a Lipschitz domain in $\mathbb{R}^3$([3]).

If we now evaluate the Gateaux derivative at $\vec{g}$ in the $\vec{h}$-direction, from the above considerations it follows that

$$
\begin{aligned}
d\mathcal{J}(\vec{g}; \vec{h}) &= \frac{d}{dt}\mathcal{J}(\vec{g} + t\vec{h})\Big|_{t=0} \\
&= \int_\Omega \nabla \times \vec{u} \cdot \nabla \times \vec{u}'(\vec{h})\, dx + \alpha \int_{\Gamma_c} \vec{g} \cdot \vec{h}\, ds \\
(2.13) \quad &= \int_\Omega \vec{\nabla} \times (\nabla \times \vec{u}) \cdot \vec{u}'(\vec{h})\, dx + \int_{\Gamma_c} (\nabla \times \vec{u})\, \vec{\tau} \cdot \vec{h}\, ds \\
&\quad + \alpha \int_{\Gamma_c} \vec{g} \cdot \vec{h}\, ds \\
&= \int_\Omega -\Delta \vec{u} \cdot \vec{u}'(\vec{h})\, dx + \int_{\Gamma_c} (\nabla \times \vec{u})\, \vec{\tau} \cdot \vec{h}\, ds + \alpha \int_{\Gamma_c} \vec{g} \cdot \vec{h}\, ds.
\end{aligned}
$$

The relation (1.7) for the curl operator in $\mathbb{R}^2$ has been used. If we replace $\vec{e}$ by $-\Delta\vec{u}$ in (2.9), by Lemma 2.4 the term $\int_\Omega - \Delta\vec{u} \cdot \vec{u}'(\vec{h})\, dx$ can be written in the form

$$
(2.14) \qquad \int_\Omega - \Delta \vec{u} \cdot \vec{u}'(\vec{h})\, dx = \int_{\Gamma_c} - \nu \frac{\partial \vec{w}}{\partial \vec{n}} \cdot \vec{h}\, ds,
$$

where $\vec{w}$ is the solution of the system (2.12). Therefore, from (2.13)-(2.14) the Gateaux derivative for the functional $\mathcal{J}$ can be expressed in the simplified form of the force effected along the control boundary Γ_c as

$$
\begin{aligned}
d\mathcal{J}(\vec{g};\vec{h}) &= \int_\Omega -\Delta\vec{u}\cdot\vec{u}'(\vec{h})\,dx + \int_{\Gamma_c}(\nabla\times\vec{u})\,\vec{\tau}\cdot\vec{h}\,ds \\
&\qquad\qquad\qquad\qquad +\alpha\int_{\Gamma_c}\vec{g}\cdot\vec{h}\,ds \qquad (2.15)\\
&= \int_{\Gamma_c}\Big(-\nu\frac{\partial\vec{w}}{\partial\vec{n}} + (\nabla\times\vec{u})\,\vec{\tau} + \alpha\vec{g}\Big)\cdot\vec{h}\,ds\,.
\end{aligned}
$$

This completes the proof, for (2.11) corresponds to the variational formulation of (2.15). □

REMARK 2.1. In $\mathbb{R}^3$, we encounter some difficulties in evaluating the Gateaux derivative of the functional involving the vorticity term. Naively computing the variation of $\mathcal{J}$ at $\vec{g}$ in the direction of $\vec{h}$ as in (2.16), it is followed

$$
\begin{aligned}
d\mathcal{J}(\vec{g};\vec{h}) &= \int_\Omega \vec{\nabla}\times\vec{u}\cdot\vec{\nabla}\times\vec{u}'(\vec{h})\,dx + \alpha\int_{\Gamma_c}\vec{g}\cdot\vec{h}\,ds \\
&= \int_\Omega \vec{\nabla}\times(\vec{\nabla}\times\vec{u})\cdot\vec{u}'(\vec{h})\,dx + \int_{\Gamma_c}\vec{\nabla}\times\vec{u}\cdot\vec{u}'(\vec{h})\times\vec{n}\,ds \\
&\qquad + \alpha\int_{\Gamma_c}\vec{g}\cdot\vec{h}\,ds\,.
\end{aligned}
$$

In the above estimation, we have no other way to detach the $\vec{h}$ from $\int_{\Gamma_c}\vec{\nabla}\times\vec{u}\cdot\vec{u}'(\vec{h})\times\vec{n}\,ds$ as the two dimensional case. Hence, we meet some difficulty when dealt with the boundary control problem for the vorticity minimization. In this case, one may examine the stream function formulation or stream function-vorticity formulation in accordance with the divergence free condition. This will be studied elsewhere.

In the differential framework, the variational form for the Gateaux derivative can be interpreted to be the gradient of $\mathcal{J}$ under the duality structure between the control space $\mathbf{W}$ and its dual space $\mathbf{W}^*$:

$$
\begin{aligned}
d\mathcal{J}(\vec{g};\vec{h}) &= \;<\nabla\mathcal{J}(\vec{g}),\vec{h}>_{\mathbf{W}^*} \\
&= \;<\Big(-\nu\gamma_c^1(\vec{w}) + \gamma_c^0(\nabla\times\vec{u})\,\vec{\tau} + \alpha\vec{g}\Big),\vec{h}>_{\mathbf{W}^*}\;.
\end{aligned}
$$

Hence, the gradient of $\mathcal{J}$ can be written by

$$\nabla \mathcal{J}(\vec{g}) = -\nu \frac{\partial \vec{w}}{\partial \vec{n}} + (\nabla \times \vec{u})\,\vec{\tau} + \alpha \vec{g}\,. \tag{2.16}$$

This is the key factor for the first order necessary conditions to find an optimal solution for our boundary control problem to minimize the vorticity due to the flow. The candidate for the minimizer necessarily comes from the critical points of the functional $\mathcal{J}$, so that (2.16) provides the conditions for the optimal solution. That is, an appropriate choice for the control parameter will be the one turning the gradient to be vanished.

3. Algorithm based on the conjugate gradient method

Let us summarize the results discussed so far. In order to solve the boundary control problem for the vorticity minimization under the two dimensional Stokes system, we have to solve the two dimensional Stokes system

$$\begin{cases} \nu \mathcal{A}\vec{u} = \mathcal{P}\vec{f} & \text{in } \Omega, \\ \nabla \cdot \vec{u} = 0 & \text{in } \Omega, \\ \vec{u} = \vec{g} & \text{on } \Gamma_c, \\ \vec{u} = \vec{0} & \text{on } \Gamma_0\,, \end{cases} \tag{3.1}$$

the adjoint system

$$\begin{cases} \nu \mathcal{A}\vec{w} = \mathcal{A}\vec{u} & \text{in } \Omega, \\ \nabla \cdot \vec{u} = 0 & \text{in } \Omega, \\ \vec{w} = \vec{0} & \text{on } \Gamma\,, \end{cases} \tag{3.2}$$

and the equation

$$\alpha \vec{g} + (\nabla \times \vec{u})\,\vec{\tau} - \nu \frac{\partial \vec{w}}{\partial \vec{n}} = \vec{0} \ \text{ on } \Gamma_c\,. \tag{3.3}$$

REMARK 3.1. In a distribution sense, the adjoint system (3.2) corresponds to the another Stokes system with the body force is provoked by the velocity of the source system as follows:

$$(3.4) \qquad \begin{cases} -\nu\Delta\vec{w} + \nabla q &= -\Delta\vec{u} \quad \text{in } \Omega, \\ \nabla\cdot\vec{w} &= 0 \quad \text{in } \Omega, \\ \vec{w} &= \vec{0} \quad \text{on } \Gamma, \end{cases}$$

where q is related to the adjoint pressure.

The appearance of this term is naturally identified by the spatial setting for the orthogonal projector in $\mathbf{H}_0$ and the decomposition (2.6) of $\mathbf{L}^2(\Omega)$. Since $\mathcal{P}(-\nu\Delta\vec{w} + \Delta\vec{u}) = \vec{0}$ in Ω, $-\nu\Delta\vec{w} + \Delta\vec{u}$ belongs to $\mathbf{H}_0^\perp$, so that $-\nu\Delta\vec{w} + \Delta\vec{u} = -\nabla q$ for some $q \in H^1(\Omega)$. Note that the pressure and the adjoint pressure are uniquely determined up to additive constants.

REMARK 3.2. For the convex functional as the case of ours, it may be possible to evaluate the Hessian for the functional, so that one can establish an algorithm to get a relatively stable global minimizer.

Since the optimality system is composed of a pair of elliptic systems and an equation, it may be desirable to implement the solver by the conjugate gradient method. For the sake of completeness, let us prescribe a brief description of a conjugate gradient algorithm. The conjugate gradient method is originated as a solver to the finite dimensional problem represented by the positive definite symmetric matrices. However, its sphere has been extended to the infinite dimensional elliptic systems by lots of theoretical and numerical experiments. It also has a variety of variants in accordance with the raised situations(cf. [4], [5]). The basic principal of the conjugate gradient methods lies in the enforcing of the search directions as well as the step sizes to supplement the weakness of the steepest gradient method. The basic scheme of the conjugate gradient method contains the following steps:

To get an approximate solution for the system $\mathcal{A}\vec{x} = \vec{f}$, where $\mathcal{A}\vec{x}$ is X-continuous and X-coercive, one can proceed in the following manner.

- Choose an initial guess $\vec{x}_0$ for the system $\mathcal{A}\vec{x} = \vec{f}$.
- Compute the initial residual $\vec{r}_0 = \vec{f} - \mathcal{A}\vec{x}_0$, and set $\vec{\sigma}_0 = \vec{r}_0$.
- for $k = 0, 1, 2, \ldots$, evaluate the followings recursively until satisfactory
  - ▷ $\rho_k = \dfrac{\|\vec{r}_k\|^2}{< \mathcal{A}\vec{\sigma}_k, \vec{\sigma}_k >}$ (step size)
  - ▷ $\vec{x}_{k+1} = \vec{x}_k + \rho_k\vec{\sigma}_k$ (upgrade the solution)

▷ $\vec{r}_{k+1} = \vec{r}_k - \rho_k \mathcal{A}\vec{\sigma}_k$ (upgrade the residual)
▷ Checking the stopping criteria; Return.
▷ $\beta_k = \dfrac{\|\vec{r}_{k+1}\|^2}{\|\vec{r}_k\|^2}$ (evaluate the conjugate parameter)
▷ $\vec{\sigma}_{k+1} = \vec{r}_k + \beta_k \vec{\sigma}_k$ (upgrade the search direction)

For a stopping criterion, on may set $\|\vec{r}_k\| < TOL$ for a given tolerance TOL. For a special application of the conjugate gradient method to the Stokes system, one may check on [7].

To seek an optimal pair $(\vec{u}, \vec{g})$ for the boundary control, the conjugate gradient algorithm can be employed in two-fold; that is, to solve the sequence of systems (3.1)-(3.2) in the one hand, and to evaluate the minimizer of the gradient (3.3) in the other hand. Let us briefly describe the algorithm proposed for a vorticity minimizer. In the evaluation of the minimizer, one can use the fixed step size instead of the variable step size. This reduces the numerical tasks considerably.

- **(step I: initialization)**
  Given initial guess $\vec{g}^{(0)} \in \mathbf{W}$, seek $\vec{u}^{(0)} = \vec{u}(\vec{g}^{(0)})$ and $\vec{w}^{(0)} = \vec{w}(\vec{u}^{(0)})$ by solving (3.1)-(3.2) consecutively.
  And then, initiate the residual by estimating $\vec{r}^{(0)} = \nabla\mathcal{J}(\vec{g}^{(0)})$ and set the descent direction by $\vec{\sigma}^{(0)} = \vec{r}^{(0)}$.
- **(step II: update)**
  For a given fixed step size $\rho > o$, upgrade the control parameter and the residual recursively, by
  $$\vec{g}^{(n+1)} = \vec{g}^{(n)} + \rho\vec{\sigma}^{(n)}$$
  and
  $$\vec{r}^{(n+1)} = \vec{r}^{(n)} - \rho\nabla\mathcal{J}(\vec{g}^{(n+1)}).$$
- **(step III: stopping criteria; conjugate descent direction)**
  For a given tolerance TOL, check if $\|\vec{r}^{(n+1)}\|_{\Gamma_c} < TOL$ is satisfied. If so, admit $\vec{g} \sim \vec{g}^{(n+1)}$ as a desired control value. Otherwise, compute the conjugate parameter by
  $$\beta^{(n)} = \frac{\|\vec{r}^{(n+1)}\|^2_{\Gamma_c}}{\|\vec{r}^{(n)}\|^2_{\Gamma_c}},$$
  and construct the conjugate search direction by
  $$\vec{\sigma}^{(n+1)} = \vec{r}^{(n)} + \beta^{(n)}\vec{\sigma}^{(n)}.$$
  Set $n = n + 1$ and Goto step II

It is evident that we have to solve the system (3.1) and its adjoint system (3.2) to evaluate $\nabla \mathcal{J}(\vec{g}^{(n)})$ at each iterations. That is, given $\vec{g}^{(n)}$, we need to evaluate $\vec{u}^{(n)} = \vec{u}(\vec{g}^{(n)})$ and $\vec{w}^{(n)} = \vec{w}(\vec{u}^{(n)})$ by solving (3.1) and (3.2) consecutively. Furthermore, once an approximation for the control is determined, then in order to check how the system is affected by the control, it is necessary to evaluate the velocity of the system (3.1) with the input of the estimated control at the final stage.

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References

[1] F. Abergel and R. Temam, *On some control problems in fluid mechanics*, Theoretical and Computational Fluid Dynamics **1** (1990), 303–325.

[2] R. A. Adams, *Sobolev spaces*, Academic Press, New York, 1975.

[3] A. Buffa, M. Costabel, and D. Sheen, *On traces for $H(curl, \Omega)$ in Lipschitz domains*, J. Math. Anal. Appl. **276** (2002), no. 2, 845–867.

[4] Jean Cea, *Lectures on optimization–theory and algorithms*, Tata Institute of Fundamental Research Lectures on Mathematics and Physics, 53, Tata Institute of Fundamental Research, Bombay, 1978.

[5] Phillip G. Ciarlet, *Introduction to numerical linear algebra and optimization*, Cambridge University Press, Cambridge, 1991.

[6] R. Dautray and J. -L. Lions, *Mathematical analysis and numerical methods for science and technology*, Vol. 6. Evolution problems. II, Springer-Verlag, New York, 1993.

[7] Edward J. Dean and R. Glowinsky, *On Some Finite Element Methods for the Numerical Simulation of Incompressible Viscous Flow*, In: Incompressible Computational Fluid Dynamics Trends and Advances, M. D. Gunzburger and R. A. Nocolaides(Eds.), Cambridge University Press, 1993, 17–66.

[8] V. Girault and P. Raviart, *Finite element methods for Navier-Stokes equations*, Theory and algorithms. Springer-Verlag, New York, 1986

[9] O. Ladyzhenskaya, *The mathematical theory of viscous incompressible flow*, Gordon and Breach Science Publishers, New York, 1963.

[10] J. -L. Lions, *Quelques methodes de resolution des problemes aux limites non lineaires*, Dunod; Gauthier-Villars, Paris 1969.

[11] R. E. Showalter, *Hilbert space methods for partial differential equations*, Electronic reprint of the 1977 original, Electronic Monographs in Differential Equations, San Marcos, TX, 1994.

[12] R. Temam, *Navier-Stokes equations. Theory and numerical analysis*, North-Holland Publishing Co., Amsterdam, 1984.

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