J. Korean Math. Soc. **43** (2006), No. 1, pp. 1–9

FIXED POINTS OF COUNTABLY CONDENSING MAPPINGS AND ITS APPLICATION TO NONLINEAR EIGENVALUE PROBLEMS

IN-SOOK KIM

ABSTRACT. Based on the Schauder fixed point theorem, we give a Leray-Schauder type fixed point theorem for countably condensing mappings in a more general setting and apply it to obtain eigenvalue results on condensing mappings in a simple proof. Moreover, we present a generalization of Sadovskii's fixed point theorem for countably condensing self-mappings due to S. J. Daher.

0. Introduction

The study of condensing operators has been one of the main objects of research in nonlinear functional analysis; see [1, 2]. The celebrated fixed point principle of B. N. Sadovskii [11] states that a condensing operator of a closed, bounded and convex subset of a Banach space into itself has a fixed point. S. J. Daher [3] showed that the Sadovskii result remains valid for countably condensing operators, that is, condensing only on countable subsets. This approach is useful for finding solutions of nonlinear differential equations in Banach spaces; see [9]. In this point of view, M. Väth [13] established a fixed point index theory for countably condensing operators and gave a generalization of the Fredholm alternative as an application.

In the present paper, motivated by H. Mönch [9], we first give a fixed point theorem of the Leray-Schauder type for countably condensing operators in a more general setting, which is closely related to eigenvalue problems for nonlinear operators. Using the main result, we prove the

Received May 11, 2004.
2000 Mathematics Subject Classification: 47H10, 47J10, 47H09.
Key words and phrases: fixed points, eigenvalues, countably condensing mappings, measures of noncompactness.

existence of a positive eigenvalue of condensing mappings. It is remarkable that this proof is much simpler than that in the usual method depending on a fixed point theorem for compact mappings, as we will present two different proofs below. Finally, from the Schauder fixed point theorem, we deduce a new fixed point theorem for countably condensing self-mappings, which generalizes some known results on condensing mappings. An earlier version of this theorem is known due to S. J. Daher [3]; see also [9].

For a subset K of a topological vector space E, the interior, the closure, the convex hull, and the closed convex hull of K in E are denoted by $\operatorname{int} K$, $\overline{K}$, $\operatorname{co} K$, and $\overline{\operatorname{co}}\, K$, respectively. If $K \subset U \subset E$, the boundary and the interior of K in the relative topology of U are denoted by $\partial_U K$ and $\operatorname{int}_U K$, respectively. If $U = E$, we write ∂K for $\partial_E K$. A set K in E is called a *wedge* if $ax + by \in K$ whenever $a, b \in [0, \infty)$ and $x, y \in K$.

Let E be a topological vector space and $\mathcal{M}$ a collection of nonempty subsets of E with the property that for any $M \in \mathcal{M}$, the sets $\overline{\operatorname{co}}\, M$, $M \cup \{x\}$ $(x \in E)$, and every subset of M belong to $\mathcal{M}$. A nonnegative real-valued function $\alpha : \mathcal{M} \to \mathbb{R}^+$ is said to be *a measure of noncompactness* on E provided that the following conditions hold for any $M \in \mathcal{M}$:

(1) $\alpha(\overline{\operatorname{co}}\, M) = \alpha(M)$;
(2) if $x \in E$, then $\alpha(M \cup \{x\}) = \alpha(M)$; and
(3) if $N \subset M$, then $\alpha(N) \leq \alpha(M)$.

The measure α of noncompactness on E is said to be *regular* provided $\alpha(M) = 0$ if and only if M is precompact; *positive homogeneous* provided $t\mathcal{M} \subset \mathcal{M}$ and $\alpha(tM) = t\alpha(M)$ for all $t > 0$ and $M \in \mathcal{M}$.

Let Y be a nonempty subset of a Banach space E and α a measure of noncompactness on E. A mapping $f : Y \to E$ is said to be *countably condensing* provided that if A is any countable subset of Y such that $\alpha(A) \leq \alpha(f(A))$, then $f(A)$ is relatively compact. $f : Y \to E$ is said to be *condensing* provided that if A is any subset of Y such that $\alpha(A) \leq \alpha(f(A))$, then $f(A)$ is relatively compact. Given $k \geq 0$, $f : Y \to E$ is said to be (α, k)*-condensing* if $\alpha(f(A)) \leq k\alpha(A)$ for each set A in Y; see [3, 6].

Recall that if α is regular and $k < 1$ then every (α, k)-condensing mapping is condensing.

1. A Leray-Schauder type fixed point theorem for countably condensing mappings

Motivated by H. Mönch, we give a Leray-Schauder type fixed point theorem for countably condensing mappings in a more general setting whose proof is based on the Schauder fixed point theorem; see [9, Theorem 2.2].

THEOREM 1.1. *Let E be a Banach space, D a subset of E, and U a closed convex subset of E with $0 \in D \subset U$ such that D is a closed neighborhood of 0 in the relative topology of U. Let $f : D \to U$ be a countably condensing continuous mapping that satisfies the Leray-Schauder boundary condition:*

$$x \neq \lambda f(x) \qquad \text{for every } x \in \partial_U D \text{ and } \lambda \in (0,1).$$

Then f has a fixed point in D.

Proof. Let

$$D_0 = \{0\} \quad \text{and} \quad D_{n+1} = \operatorname{co}(\{0\} \cup f(D_n \cap D)) \quad \text{for each } n \in \mathbb{N} \cup \{0\}.$$

Then the sequence $\{D_n\}_{n \in \mathbb{N}\cup\{0\}}$ is increasing with respect to inclusion and the sets D_n are relatively compact because f is continuous. For each $n \in \mathbb{N} \cup \{0\}$, there exists a countable set C_n such that $\overline{D_n \cap D} = \overline{C_n}$.

Set $V = \bigcup_{n \geq 0} D_n$ and $C = \bigcup_{n\geq 0} C_n$. Since each D_n is convex and $\{D_n\}$ is increasing, we obtain that V is convex and

$$V = \bigcup_{n\geq 0} D_{n+1} = \bigcup_{n \geq 0} \operatorname{co}(\{0\} \cup f(D_n \cap D)) = \operatorname{co}(\{0\} \cup f(V \cap D)).$$

For the countable set C, we have

$$\begin{aligned} C \subset \overline{\bigcup_{n\geq 0} C_n} \subset \overline{\bigcup_{n\geq 0} (D_n \cap D)} \subset \overline{V} = \overline{\operatorname{co}}(\{0\} \cup f(V \cap D)) \\ = \overline{\operatorname{co}}(\{0\} \cup f(\bigcup_{n \geq 0} (D_n \cap D))) \subset \overline{\operatorname{co}}(\{0\} \cup f(C)), \end{aligned}$$

where for the last inclusion we use the fact that f is continuous on the compact set $\overline{D_n \cap D}$. Since α is a measure of noncompactness on E and f is countably condensing, it follows that

$$\alpha(C) \leq \alpha(\overline{\operatorname{co}}(\{0\} \cup f(C))) = \alpha(f(C))$$

and hence $f(C)$ is relatively compact. Since $\overline{\text{co}}\,(\{0\} \cup f(C))$ is compact, we see that $\overline{V}$ is also compact.

Now we may suppose without loss of generality that f has no fixed point in $\partial_U D$. Then $x \neq \lambda f(x)$ for all $x \in \partial_U D$ and $\lambda \in [0,1]$. Let $M = \bigcup_{\lambda \in [0,1]} \text{Fix}\,(\lambda f)$, where Fix denotes the fixed point set of an operator. Then M is closed in E and $M \cap \partial_U D = \emptyset$. Hence it follows from $\overline{V} \subset U$ that

$$(M \cap \overline{V}) \cap \partial_{\overline{V}}(D \cap \overline{V}) \subset (M \cap \overline{V}) \cap (\partial_U D \cap \overline{V}) = \emptyset.$$

Since the sets $M \cap \overline{V}$ and $\partial_{\overline{V}}(D \cap \overline{V})$ are disjoint and closed relative to the compact set $\overline{V}$, there exists a continuous function $\mu : \overline{V} \to [0,1]$ such that $\mu(M \cap \overline{V}) = \{1\}$ and $\mu(\partial_{\overline{V}}(D \cap \overline{V})) = \{0\}$. Set $A = D \cap \overline{V}$ and define a mapping $g : \overline{V} \to \overline{V}$ by

$$g(x) = \begin{cases} \mu(x)f(x) & \text{if } x \in A \\ 0 & \text{if } x \in \overline{V} \setminus \text{int}_{\overline{V}} A. \end{cases}$$

Since g is continuous on the compact convex set $\overline{V}$, by the Schauder fixed point theorem [12], g has a fixed point $x_0 \in \overline{V}$. From $0 \in (\text{int}_U D) \cap \overline{V} \subset \text{int}_{\overline{V}} A$ it follows that $x_0 = g(x_0) = \mu(x_0)f(x_0)$ which implies $x_0 \in M \cap \overline{V}$ and so $\mu(x_0) = 1$. Thus, x_0 is a fixed point of f. This completes the proof. □

As a special case, we obtain a fixed point theorem of the Rothe type [10] for countably condensing mappings.

COROLLARY 1.2. *Let K be a closed neighborhood of 0 in a Banach space E such that K is starshaped with respect to 0. If $f : K \to E$ is a countably condensing continuous mapping with $f(\partial K) \subset K$, then f has a fixed point in K.*

Proof. From $f(\partial K) \subset K$ and the starshapedness of K it follows that $f(x) \neq \lambda^{-1}x$ for every $x \in \partial K$ and $\lambda \in (0,1)$. Applying Theorem 1.1 with $U = E$, f has a fixed point. □

COROLLARY 1.3. *Let K be a closed neighborhood of 0 in a Banach space E. Let $f : K \to E$ be a compact continuous mapping that has no fixed point in K. Then there exist an $x \in \partial K$ and a $\lambda \in (0,1)$ such that $x = \lambda f(x)$.*

Proof. This is an immediate consequence of Theorem 1.1 since f is clearly countably condensing. □

COROLLARY 1.4. *Let $K = \{x \in E : \|x\| \leq 1\}$ be the closed unit ball in an infinite-dimensional Banach space E. Let $f : \partial K \to E$ be a compact continuous mapping such that*

$$cf(\partial K) \cap K = \emptyset \qquad \text{for some } c > 0.$$

Then there exist an $x \in \partial K$ and a $\lambda > 0$ such that $f(x) = \lambda x$.

Proof. Note that ∂K is a retract of K; see [5]. Let $r : K \to \partial K$ be a retraction of K on ∂K. Define a mapping $g : K \to E$ by

$$g(x) := f(r(x)) \qquad \text{for } x \in K.$$

Then g is a compact continuous mapping and $cg(K) \cap K = \emptyset$. Hence the compact continuous mapping $g_0 : K \to E$ given by $g_0(x) = cg(x)$ has no fixed point in K. By Corollary 1.3, there exist an $x \in \partial K$ and a $t \in (0, 1)$ such that $x = tg_0(x)$. Setting $\lambda = (ct)^{-1}(> 0)$, we conclude that $f(x) = \lambda x$. This completes the proof. □

2. Positive eigenvalues of condensing mappings

In this section, we show the existence of a positive eigenvalue of condensing mappings in two different approaches. For the case of set-valued maps, we refer to [6, Theorem 2].

THEOREM 2.1. *Let E be a Banach space, K a closed neighborhood of 0 in E, and U a closed wedge in E. Let α be a regular positive homogeneous measure of noncompactness on E, $k \geq 0$, and $f : K \cap U \to U$ an (α, k)-condensing continuous mapping. Suppose that there is a real number $c > k$ such that*

$$f(K \cap U) \cap c(\operatorname{int} K) = \emptyset.$$

Then there exist an $x \in \partial K \cap U$ and a $\lambda \geq c$ such that $f(x) = \lambda x$.

Proof. Let a mapping $g : K \cap U \to U$ be defined by

$$g(x) := \frac{1}{c} f(x) \quad \text{for } x \in K \cap U.$$

Then g is $(\alpha, k/c)$-condensing because α is positive homogeneous and f is (α, k)-condensing. Since α is regular and $k/c < 1$, it follows that

g is condensing. Hence g is trivially countably condensing. Without loss of generality we may suppose that g has no fixed point in $K \cap U$. Otherwise, there exists an $x \in \partial K \cap U$ such that $x = g(x)$ because $g(K \cap U) \cap \operatorname{int} K = \emptyset$. Theorem 1.1 implies that there are an $x \in \partial K \cap U$ and a $t \in (0, 1)$ such that $x = tg(x)$, because of $\partial_U(K \cap U) \subset \partial K \cap U$. Setting $\lambda = c/t(> c)$, we conclude that $f(x) = \lambda x$. This completes the proof. □

In fact, Theorem 2.1 was easily obtained from Theorem 1.1 concerning countably condensing mappings. But if we use a fixed point theorem for compact mappings in the usual method we often do about condensing mappings, the proof is not so simple as follows:

Another Proof. Define a mapping $g : K \cap U \to U$ by

$$g(x) := \frac{1}{c} f(x) \quad \text{for } x \in K \cap U.$$

Then g is condensing. Observe that there exists a closed convex subset S of E with $0 \in S$ such that $g(K \cap U \cap S)$ is a relatively compact subset of $U \cap S$; see [7]. Without loss of generality we may suppose that $g_0 = g|_{K \cap U \cap S}$ has no fixed point in $K \cap U \cap S$. Set $R := U \cap S$ and

$$X_1 := \{x \in K \cap R : x = tg_0(x) \quad \text{for some } t \in [0, 1]\}.$$

Then X_1 is a compact subset of E. Now we claim that $X_1 \cap (\partial K \cap R) \neq \emptyset$. We suppose to the contrary that $X_1 \cap (\partial K \cap R) = \emptyset$. Since $\partial K \cap R$ is closed in E, there exists a continuous function $\mu : E \to [0, 1]$ such that $\mu(x) = 1$ for all $x \in \partial K \cap R$ and $\mu(x) = 0$ for all $x \in X_1$. Let a mapping $h : R \to R$ be defined by

$$h(x) = \begin{cases} (1 - \mu(x))g_0(x) & \text{if } x \in K \cap R \\ 0 & \text{if } x \in R \setminus \operatorname{int} K. \end{cases}$$

Then h is a compact continuous mapping on R. By Himmelberg's fixed point theorem [8], h has a fixed point $x_0 \in R$; that is, $x_0 = h(x_0)$. Hence we have $x_0 \in X_1$ and so $\mu(x_0) = 0$ and hence $x_0 = g_0(x_0)$ which contradicts our assumption that g_0 has no fixed point in $K \cap R$.

Consequently, we have shown that $X_1 \cap (\partial K \cap R) \neq \emptyset$. This means that there exist an $x \in \partial K \cap R$ and a $t \in [0, 1]$ such that $x = tg(x)$. From $x \neq 0$ it follows that $t \neq 0$. Setting $\lambda = c/t(\geq c)$, the conclusion follows. This completes the proof. □

The following particular form of Theorem 2.1 is given in [6, Folgerung 6].

COROLLARY 2.2. *Let K be the closed unit ball in a Banach space E, α the Kuratowski measure of noncompactness on E, and $k \geq 0$. Let $f : K \to E$ be an (α, k)-condensing continuous mapping and $c > k$ a real number such that $\|f(x)\| \geq c$ for all $x \in K$. Then there exist an $x \in \partial K$ and a $\lambda \geq c$ such that $f(x) = \lambda x$.*

Proof. Note that the Kuratowski measure α of noncompactness on E is regular and positive homogeneous; see [1, 2]. Theorem 2.1 is applicable. □

3. A fixed point theorem for countably condensing self-mappings

The following fixed point theorem for countably condensing mappings is due to S. J. Daher in the special case of the Hausdorff measure of noncompactness; see [3, Theorem]. Here we follow the method of proof that H. Mönch suggests in [9, Theorem 2.1].

THEOREM 3.1. *Let K be a nonempty closed convex subset of a Banach space E. If $f : K \to K$ is a countably condensing continuous mapping, then f has a fixed point in K.*

Proof. Fix $x_0 \in K$. Let

$$D_0 = \{x_0\} \quad \text{and} \quad D_{n+1} = \operatorname{co}(\{x_0\} \cup f(D_n)) \quad \text{for each } n \geq 0.$$

Then $\{D_n\}_{n \geq 0}$ is increasing and the sets D_n are relatively compact. For each $n \geq 0$, there exists a countable set C_n such that $\overline{D_n} = \overline{C_n}$. Set $V = \bigcup_{n \geq 0} D_n$ and $C = \bigcup_{n \geq 0} C_n$. As in the proof of Theorem 1.1, we conclude that $\overline{V} = \overline{\operatorname{co}}(\{x_0\} \cup f(V))$ and the convex set $\overline{V}$ is compact. From the continuity of f it follows that $f(\overline{V}) \subset \overline{V}$. By the Schauder fixed point theorem, the restriction $f|_{\overline{V}}$ has a fixed point and so does f. This completes the proof. □

The following result is just Sadovskii's fixed point theorem [11] if α is the Hausdorff measure of noncompactness on E and K has an additional requirement that K is bounded.

COROLLARY 3.2. *Let K be a nonempty closed convex subset of a Banach space E. If $f : K \to K$ is a condensing continuous mapping, then f has a fixed point in K.*

The following is Darbo's fixed point theorem for the so called k-set contractions; see [4].

COROLLARY 3.3. *Let K be a nonempty, closed, bounded, and convex subset of a Banach space E and α the Kuratowski measure of noncompactness on E. If $f : K \to K$ is an (α, k)-condensing continuous mapping with $k < 1$, then f has a fixed point in K.*

Proof. This is an immediate consequence of Corollary 3.2 because f is condensing. □

ACKNOWLEDGMENT. This work was supported by Grant No. R05-2002-000-00320-0 from the Basic Research Program of the Korea Science & Engineering Foundation.

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Department of Mathematics
Sungkyunkwan University
Suwon 440-746, Korea
E-mail: iskim@math.skku.ac.kr