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ON A FINSLER SPACE WITH (α, β)-METRIC AND CERTAIN METRICAL NON-LINEAR CONNECTION

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Abstract. The purpose of this paper is to introduce an L-metrical non-linear connection N^{*i}_j and investigate a conformal change in the Finsler space with (α, β)-metric. The $(v)h$-torsion and $(v)hv$-torsion in the Finsler space with L-metrical connection $F\Gamma^*$ are obtained. The conformal invariant connection and conformal invariant curvature are found in the above Finsler space.

1. Introduction

A Finsler connection $F\Gamma$ ([1]) is determined by the Finsler metric L and a non-linear connection $N^i{}_j$ on an n-dimensional differentiable manifold M^n. If the Finsler space admits an (α, β)-metric, where $\alpha = (a_{ij}(x)y^i y^j)^{1/2}, \beta = b_i(x)y^i$, then can be defined the Christoffel symbol constructed from $a_{ij}(x)$.

In the present paper, we consider a Finsler space with (α, β)-metric and a new non-linear connection $N^{*i}{}_j$ which is constructed from the given non-linear connection $N^i{}_j$, the Finsler metric L and the covariant differentiation of L with respect to the Levi-Civita connection. We find the torsion tensors, curvature tensor of the new Finsler structure and some conformal invariants in a Finsler space with the new connection $F\Gamma^*$.

2. An L-metrical non-linear connection

Let (M^n, L) be a Finsler space with (α, β)-metric $L = L(\alpha, \beta)$ on an

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n-dimensional differentiable manifold M, and let $\{{}_j{}^i{}_k\}$ be the Christoffel symbols constructed from $a_{ij}(x)$ and let ∇_k denote the h-covariant differentiation with respect to $(\{{}_k{}^i{}_j\}, \{{}_0{}^i{}_j\}, 0)$, where the index 0 denotes the transvection by y^k. The Finsler fundamental metric tensor $g_{ij}(x,y) = \dot\partial_j \dot\partial_i L^2/2$ is given in ([1]).

For the given non-linear connection $N^i{}_j(x,y)$ on (M^n, L) we denote

$$X_k = \partial_k - N^r{}_k \dot\partial_r,$$

where $\partial_k = \partial/\partial x^k, \dot\partial_k = \partial/\partial y^k$.

Now we consider a new non-linear connection $N^{*i}{}_j(x,y)$ which is given by

$$N^{*i}{}_j = N^i{}_j + \frac{y^i}{L}\nabla_j L. \tag{2.1}$$

If we put

$$\begin{aligned} X^*_k &= \partial_k - N^{*r}{}_k \dot\partial_r, \\ \Gamma^{*i}_j{}_k &= g^{ir}(X^*_j g_{rk} + X^*_k g_{jr} - X^*_r g_{jk})/2, \\ C_j{}^i{}_k &= g^{ir}(\dot\partial_j g_{rk} + \dot\partial_k g_{jr} - \dot\partial_r g_{jk})/2, \end{aligned} \tag{2.2}$$

then we have

$$\begin{aligned} X^*_k &= X_k - \frac{y^r}{L}(\nabla_k L)\dot\partial_r, \\ \Gamma^{*i}_j{}_k &= \Gamma_j{}^i{}_k, \\ C_j{}^i{}_k &= g^{ir}\dot\partial_j g_{rk}/2, \end{aligned} \tag{2.3}$$

where $\Gamma_j{}^i{}_k = g^{ir}(X_j g_{rk} + X_k g_{jr} - X_r g_{jk})/2$.

Here we define a symmetric Finsler connection $F\Gamma^* = (\Gamma^{*i}_j{}_k, N^{*i}{}_j, C_j{}^i{}_k)$ on (M^n, L) from the given Finsler connection $F\Gamma = (\Gamma_j{}^i{}_k, N^i{}_j, C_j{}^i{}_k)$. We denote by ∇^*_k the h-covariant differentiation with respect to $F\Gamma^*$.

For any p-homogeneous scalar $\rho(x,y)$ of degree r in y^i,

$$\nabla^*_j \rho = \partial_j \rho - (N^s{}_j + \frac{y^s}{L}\nabla_j L)\dot\partial_s \rho = \rho_{|j} - \frac{r\rho}{L}\nabla_j L,$$

where ${}_{|j}$ is the h-covariant differentiation with respect to $F\Gamma$. Thus we have proved the following theorem:

THEOREM 2.1. *A Finsler connection $F\Gamma^*$ is L-metrical in case $N^i{}_j = \{{}_0{}^i{}_j\}$.*

Next we put $N^i{}_j = \{{}_0{}^i{}_j\}$ in (M^n, L), then

$$N^{*i}{}_j = \{{}_0{}^i{}_j\} + \frac{y^i}{L}(\partial_j L - \{{}_0{}^r{}_j\}\dot{\partial}_r L). \tag{2.4}$$

The $(v)h$-torsion for $F\Gamma^*$ is given by

$$R^{*i}{}_{jk} = X^*_k N^{*i}{}_j - (j/k), \tag{2.5}$$

where (j/k) means the interchange of indices (j,k) of the preceding terms.

From (2.4)

$$\begin{aligned}\partial_k N^{*i}{}_j = \partial_k\{{}_0{}^i{}_j\} &+ \frac{y^i}{L}(\partial_k\partial_j L - \partial_k\{{}_0{}^r{}_j\}\dot{\partial}_r L - \{{}_0{}^r{}_j\}\partial_k\dot{\partial}_r L)\\ &- \frac{y^i}{L^2}(\partial_j L - \{{}_0{}^r{}_j\}\dot{\partial}_r L)\partial_k L,\end{aligned} \tag{2.6}$$

$$\begin{aligned}\dot{\partial}_r N^{*i}{}_j = \{{}_r{}^i{}_j\} &+ \frac{\delta^i_r}{L}(\partial_j L - \{{}_0{}^s{}_j\}\dot{\partial}_s L)\\ &+ \frac{y^i}{L}(\dot{\partial}_r\partial_j L - \{{}_r{}^s{}_j\}\dot{\partial}_s L - \{{}_0{}^s{}_j\}\dot{\partial}_r\dot{\partial}_s L)\\ &- \frac{y^i}{L^2}(\partial_j L\dot{\partial}_r L - \{{}_0{}^s{}_j\}\dot{\partial}_s L\dot{\partial}_r L).\end{aligned} \tag{2.7}$$

By the homogeneity of L, we have

$$(\dot{\partial}_r\dot{\partial}_j L)y^r = 0, \quad (\dot{\partial}_r\partial_i L)y^r = \partial_i L, \quad (\dot{\partial}_r L)y^r = L. \tag{2.8}$$

Using (2.4) and (2.8), we get

$$\begin{aligned}N^{*r}{}_k\dot{\partial}_r N^{*i}{}_j = \{{}_0{}^r{}_k\}\{{}_r{}^i{}_j\} &+ \frac{1}{L}\{{}_0{}^i{}_k\}(\partial_j L - \{{}_0{}^m{}_j\}\dot{\partial}_m L)\\ &+ \frac{y^i}{L}\{{}_0{}^r{}_k\}(\dot{\partial}_r\partial_j L - \{{}_r{}^m{}_j\}\dot{\partial}_m L) - \frac{y^i}{L}\{{}_0{}^r{}_k\}\{{}_0{}^s{}_j\}\dot{\partial}_r\dot{\partial}_s L\\ &+ \frac{1}{L}\{{}_0{}^i{}_j\}(\partial_k L - \{{}_0{}^s{}_k\}\dot{\partial}_s L)\\ &+ \frac{y^i}{L^2}(\partial_j L - \{{}_0{}^m{}_j\}\dot{\partial}_m L)\partial_k L\\ &- \frac{2y^i}{L^2}(\partial_j L - \{{}_0{}^m{}_j\}\dot{\partial}_m L)\{{}_0{}^s{}_k\}\dot{\partial}_s L.\end{aligned}$$

Therefore we get

$$N^{*r}{}_k\dot{\partial}_r N^{*i}{}_j - (j/k) = \{{}_0{}^r{}_k\}[\{{}_r{}^i{}_j\} + \frac{y^i}{L}\{\dot{\partial}_r\partial_j L - \{{}_r{}^m{}_j\}\dot{\partial}_m L - \frac{1}{L}(\dot{\partial}_r L)(\partial_j L)\}] - (j/k). \tag{2.9}$$

Substituting (2.6) and (2.9) in (2.5) we have

$$R^{*i}{}_{jk} = R_0{}^h{}_{jk}(\delta^i_h - \frac{y^i}{L}\dot{\partial}_h L), \tag{2.10}$$

where $R_t{}^h{}_{jk}$ is the Riemannian curvature tensor constructed from $a_{ij}(x)$.

Next we consider the $(v)hv$-torsion:

$$P^{*i}{}_{jk} = \dot{\partial}_k N^{*i}{}_j - \Gamma^{*i}_{k\ j} \tag{2.11}$$

of $F\Gamma^*$ on (M^n, L). Substituting (2.11) and (2.7) in (2.11), we have

$$\begin{aligned} P^{*i}{}_{kj} = \{{}_k{}^i{}_j\} - \Gamma_k{}^i{}_j &+ \frac{\delta^i_k}{L}(\partial_j L - \{{}_0{}^s{}_j\}\dot{\partial}_s L) \\ &+ \frac{y^i}{L}(\dot{\partial}_k\partial_j L - \{{}_k{}^s{}_j\}\dot{\partial}_s L - \dot{\partial}_k\dot{\partial}_s L\{{}_0{}^s{}_j\}) \\ &- \frac{y^i}{L^2}(\partial_j L\dot{\partial}_k L - \dot{\partial}_s L\dot{\partial}_k L\{{}_0{}^s{}_j\}). \end{aligned} \tag{2.12}$$

Thus we have

THEOREM 2.2. *In the case $N^i{}_j = \{{}_0{}^i{}_j\}$, the $(v)h$-torsion $R^{*i}{}_{jk}$ and $(v)hv$-torsion $P^{*i}{}_{kj}$ of a Finsler space (M^n, L) with connection $F\Gamma^*$ are given by (2.10) and (2.12) respectively.*

3. Conformal invariants

In a Finsler space (M, L) with connection $F\Gamma^*$, let's consider the following conformal change ([2]):

$$L = L(\alpha, \beta) \longrightarrow \overline{L}(\alpha, \beta) = e^{\sigma(x)}L(\alpha, \beta). \tag{3.1}$$

We have also $\overline{L}(\alpha, \beta) = L(\overline{\alpha}, \overline{\beta})$, where $\overline{\alpha} = e^{\sigma(x)}\alpha, \overline{\beta} = e^{\sigma(x)}\beta$.

Putting $\overline{\alpha} = (\overline{a}_{ij}(x)y^i y^j)^{1/2}, \overline{\beta} = \overline{b}_i(x)y^i$, we have $\overline{a}_{ij} = e^{2\sigma(x)}a_{ij}, \overline{b}_i = e^{\sigma(x)}b_i$. The Christoffel symbols $\overline{\{{_j}{^i}{_k}\}}$ constructed from $\overline{a}_{ij}$ are written

$$\overline{\{{_j}{^i}{_k}\}} = \{{_j}{^i}{_k}\} + \delta^i_j \sigma_k + \delta^i_k \sigma_j - \sigma^i a_{jk}, \tag{3.2}$$

where $\sigma_k = \partial_k \sigma, \sigma^i = a^{ir}\sigma_r$. Thus we have ([4])

$$\overline{\nabla}_k \overline{b}_j = e^{\sigma}(\nabla_k b_j - b_k \sigma_j + b_r \sigma^r a_{jk}), \tag{3.3}$$

from which

$$\sigma_j = M_j - \overline{M}_j, \tag{3.4}$$

where

$$M_j = \frac{1}{b^2}(b^r \nabla_r b_j - \frac{1}{n-1} b_j \nabla_r b^r). \tag{3.5}$$

Now we consider $N^i{_j} = \{{_0}{^i}{_j}\}$. From (2.1) and (3.2) we have

$$\begin{aligned} \overline{N}^{*i}{_j} &= N^{*i}{_j} + y^i \sigma_j + \delta^i_j \sigma_0 - y^r a_{rj} \sigma^i \\ &\quad - \frac{y^i}{L}(\sigma_0 \dot{\partial}_j L - y^s a_{js} \sigma^r \dot{\partial}_r L). \end{aligned} \tag{3.6}$$

Substituting (3.4) in (3.6) and paying attention to

$$\frac{1}{L}\dot{\partial}_j L = \frac{1}{\overline{L}}\dot{\partial}_j \overline{L},$$

then we have

$$\begin{aligned} &\overline{\{{_0}{^i}{_j}\}}^* + y^i \overline{M}_j + \delta^i_j \overline{M}_0 - y^r \overline{a}_{rj} \overline{M}^i - \frac{y^i}{\overline{L}}(\overline{M}_0 \dot{\partial}_j \overline{L} - y^s \overline{a}_{js} \overline{M}^r \dot{\partial}_r \overline{L}) \\ &= \{{_0}{^i}{_j}\}^* + y^i M_j + \delta^i_j M_0 - y^r a_{rj} M^i - \frac{y^i}{L}(M_0 \dot{\partial}_j L - y^s a_{js} M^r \dot{\partial}_r L), \end{aligned}$$

where $M^i = a^{ir}M_r$. Putting

$$\begin{aligned} M^{*i}{_j} &= \{{_0}{^i}{_j}\}^* + y^i M_j + \delta^i_j M_0 - y^r a_{rj} M^i \\ &\quad - \frac{y^i}{L}(M_0 \dot{\partial}_j L - y^s a_{js} M^r \dot{\partial}_r L), \end{aligned} \tag{3.7}$$

it is a conformal invariant non-linear connection; that is, $\overline{M}^{*i}{}_j = M^{*i}{}_j$.

Next from (2.2), (3.4) and (3.6) we have

$$\overline{\Gamma}^{*i}_{jk} + \overline{M}_0\overline{C}_j{}^i{}_k + \overline{P}_j{}^i{}_k + \overline{Q}_j{}^i{}_k = \Gamma^{*i}_{jk} + M_0 C_j{}^i{}_k + P_j{}^i{}_k + Q_j{}^i{}_k,$$

where $P_j{}^i{}_k = \delta^i_j M_k + \delta^i_k M_j - g^{ir} M_r g_{jk}$, $Q_j{}^i{}_k = y^r(a_{rj}C_s{}^i{}_k + a_{rk}C_s{}^i{}_j - a_{rt}g^{it}C_{sjk})M^s$.

Putting

$$M^{*i}_j{}_k = \Gamma^{*i}_j{}_k + M_0 C_j{}^i{}_k + P_j{}^i{}_k + Q_j{}^i{}_k, \tag{3.8}$$

it is a symmetric conformal invariant connection, that is,

$$\overline{M}^{*i}_j{}_k = M^{*i}_j{}_k. \tag{3.9}$$

If we denote $M^{*i}_h{}_{jk}$ the curvature tensor constructed from $M^{*i}_j{}_k$, then we have from (3.9),

$$\overline{M}^{*i}_h{}_{jk} = M^{*i}_h{}_{jk}, \tag{3.10}$$

where

$$M^{*i}_h{}_{jk} = (\delta^*_k M^{*i}_h{}_j + M^{*r}_h{}_j M^{*i}_r{}_k) - (j/k), \tag{3.11}$$

where $\delta^*_k = \partial_k - M^{*r}{}_k \dot{\partial}_r$.

Thus we have

THEOREM 3.1. *In a Finsler space (M^n, L) with connection $F\Gamma^*$, there exists a conformal invariant connection $(M^{*i}_j{}_k, M^{*i}{}_j)$ and conformal invariant curvature tensor $M^{*i}_h{}_{jk}$ given by (3.7), (3.8) and (3.11) respectively.*

Especially, if $\nabla_k b_j = 0$, then $M_j = 0$. Therefore from (3.7) and (3.9) we have

THEOREM 3.2. *If b_i is parallel in the Riemannian space $R^n = (M^n, \alpha)$, then we have $M^{*i}{}_j = \{{}_0{}^i{}_j\}^*$, $M^{*i}_j{}_k = \Gamma^{*i}_j{}_k$ and $M^{*i}_h{}_{jk} = R^{*i}_h{}_{jk}$, where $R^{*i}_h{}_{jk}$ is the curvature tensor constructed from $\Gamma^{*i}_j{}_k$.*

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