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RIBAUCOUR TRANSFORMATIONS OF THE SURFACES WITH CONSTANT POSITIVE GAUSSIAN CURVATURES IN THE 3-DIMENSIONAL EUCLIDEAN SPACE

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ABSTRACT. We associate the surfaces of constant Gaussian curvature $K = 1$ with no umbilics to a subclass of the solutions of $O(4,1)/O(3) \times O(1,1)$-system. From this correspondence, we can construct new $K = 1$ surfaces from a known $K = 1$ surface by using a kind of dressing actions on the solutions of this system.

1. Introduction

Recently, the theory of surfaces has been studied extensively in differential geometry. Classically, many mathematicians had interests in the surfaces in the 3-dimensional Euclidean space $\mathbb{R}^3$ which have special geometric properties that they have good coordinate systems so that the corresponding Gauss-Codazzi equations have nice forms and they have geometric transformations to construct such surfaces from known ones. The well-known examples are surfaces with constant Gaussian curvatures, surfaces with constant mean curvatures and so on, and there are transformations for such surfaces studied by Bianchi, Bäcklund, Ribaucour and Darboux, etc. Nowadays, the Gauss equations of such surfaces are known as soliton equations and those transformations can be explained as the loop group actions on the solution spaces of the corresponding equations.

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A surface with the Gaussian curvature $K = 1$ which has no umbilic points is characterized by the so-called sinh-Gordon equation

$$\Delta\phi + \sinh\phi\cosh\phi = 0,$$

which turns out to be a soliton equation.

L. Bianchi was the first one who found a geometric method to construct a new $K = 1$ surface from a known $K = 1$ surface [1]. The so-called Bianchi-Bäcklund transformation is obtained by taking line congruences twice. Recently, U. Hetrich-Jeromin and F. Pedit [5], using quaternionic calculation, showed that Bianchi- Bäcklund transformations are Darboux transformations for isothermic surfaces, and F. Burstall [3] showed that Darboux transformations are equivalent to some loop group actions of the simple type on the quaternion algebra. Finally, S. Kobayashi and J. Inoguchi [4] proved that all the above three transformations are equivalent for $K = 1$ or $H = 1$ surface.

In this paper, we introduce the $O(4,1)/O(3)\times O(1,1)$-system, which is a G/K-system defined by C. L. Terng [8], and we relate a subclass of the solutions of this system to the $K = 1$ surfaces with no umbilics. Also, we construct Ribaucour transformations on these surfaces, which are sphere congruences, using some actions on the solutions of this system.

2. Surfaces in $\mathbb{R}^3$ with positive constant Gaussian curvature

In this section, we investigate on the characterization of the surface in $\mathbb{R}^3$ with $K = 1$ which has no umbilic points.

Let $X : M \longrightarrow \mathbb{R}^3$ be an immersed surface. Take a local orthonormal frame e_1, e_2, e_3 on M so that e_3 is normal to M. Let ω_1 and ω_2 be the dual coframe field on M. For the local invariants of M, we use the following convention. Let ω_{ij} be the connection 1-form, that is,

$$de_i = \sum_{i=1}^{3} e_j \otimes \omega_{ji}.$$

The shape operator A is

$$A = de_3 = e_1 \otimes \omega_{13} + e_2 \otimes \omega_{23},$$

and the first and the second fundamental forms are

$$I = \omega_1 \otimes \omega_1 + \omega_2 \otimes \omega_2,$$
$$II = \omega_1 \otimes \omega_{13} + \omega_2 \otimes \omega_{23}.$$

The structure equations, the Gauss and the Codazzi equations are

$$d\omega_1 + \omega_2 \wedge \omega_{21} = 0, \qquad d\omega_2 + \omega_1 \wedge \omega_{12} = 0, \tag{2.1}$$

$$d\omega_{12} = K\omega_1 \wedge \omega_2 = \omega_{13} \wedge \omega_{23}, \tag{2.2}$$

$$d\omega_{13} = -\omega_{12} \wedge \omega_{23}, \qquad d\omega_{23} = -\omega_{21} \wedge \omega_{13}. \tag{2.3}$$

The $K = 1$ surface with no umbilics is characterized by the sinh-Gordon equation

$$\Delta\phi + \sinh\phi\cosh\phi = 0.$$

The following theorem for the $K = 1$ surfaces is well-known, to which we give an elementary proof below.

THEOREM 2.1. *Suppose $X : M \longrightarrow \mathbb{R}^3$ is an immersed surface which has the Gaussian curvature $K = 1$ with no umbilic points. Then there exists an isothermal and line of curvature coordinate system (x, y) on M and a function ϕ such that*

$$\begin{cases} I &= \cosh^2\phi \; dx^2 + \sinh^2\phi \; dy^2, \\ I\!I &= \sinh\phi \;\; \cosh\phi \; (dx^2 + dy^2). \end{cases}$$

Moreover, the Gauss equation for M is

$$\Delta\phi + \sinh\phi\cosh\phi = 0. \tag{2.4}$$

Conversely, if ϕ is a solution of (2.4), then there exists a $K = 1$ surface in $\mathbb{R}^3$ whose fundamental forms are given as above.

PROOF. Take a local orthonormal frame e_1, e_2, e_3 on M so that

$$A(e_1) = \tanh\phi \; e_1, \qquad A(e_2) = \coth\phi \; e_2, \tag{2.5}$$

where $\tanh\phi$ and $\coth\phi$ are the principal curvatures and $K = \tanh\phi$ $\coth\phi = 1$.

By the Frobenius theorem, there exists (u, v) such that $e_1 = \frac{1}{E}\frac{\partial}{\partial u}$ and $e_2 = \frac{1}{G}\frac{\partial}{\partial v}$ for some E and G. So $\omega_1 = Edu$ and $\omega_2 = Gdv$.

By the structure equations (2.1) and from (2.5), we have

$$\omega_{12} = \frac{E_v}{G}du - \frac{G_u}{E}dv, \tag{2.6}$$

$$\omega_{13} = \tanh\phi\, \omega_1 = E \tanh\phi\, du, \tag{2.7}$$

$$\omega_{23} = \coth\phi\, \omega_2 = G \coth\phi\, dv. \tag{2.8}$$

Substituting (2.6), (2.7) and (2.8) to the Codazzi equations (2.3), we obtain

$$E_v(\tanh\phi - \coth\phi) + E\phi_v \mathrm{sech}^2\phi = 0,$$
$$G_u(\tanh\phi - \coth\phi) + G\phi_u \mathrm{csch}^2\phi = 0.$$

This implies that

$$(E\mathrm{sech}\phi)_v = 0, \ (G\cosh\phi)_u = 0.$$

Hence, there exist functions $a(u)$ and $b(v)$ such that

$$E\mathrm{sech}\phi = a(u), \quad G\cosh\phi = b(v).$$

Choose x and y such that $dx = a(u)du$ and $dy = b(v)dv$. Then

$$\begin{aligned} \omega_1 &= Edu = \cosh\phi\, dx, \qquad \omega_2 = Gdv = \sinh\phi\, dy, \\ \omega_{13} &= \tanh\phi\, \omega_1 = \sinh\phi\, dx, \quad \omega_{23} = \coth\phi\, \omega_2 = \cosh\phi\, dy. \end{aligned} \tag{2.9}$$

Therefore,

$$\begin{cases} I &= \cosh^2\phi\, dx^2 + \sinh^2\phi\, dy^2, \\ I\!I &= \sinh\phi\ \cosh\phi\, (dx^2 + dy^2). \end{cases}$$

On the other hand, by differentiating (2.9), we obtain $\omega_{12} = \phi_y dx - \phi_x dy$. Thus by the Gauss equation (2.2), we have $d\omega_{12} = -\Delta\phi\, dx \wedge dy = 1 \cdot \omega_1 \wedge \omega_2$ so that $\Delta\phi + \sinh\phi\cosh\phi = 0$.

Conversely, suppose ϕ is a solution of the sinh-Gordon equation (2.4). It follows from the fundamental theorem of surface theory that there exists a surface M such that I and $I\!I$ are as above and it is easy to check that M has $K = 1$. □

3. $O(4,1)/O(3) \times O(1,1)$-system and $K = 1$ surfaces

Let G/K be a symmetric space . G/K-system was defined by C. L. Terng [8], which is very useful in the study of submanifold geometry. In particular, when G/K is a Grassmannian manifold, G/K-system is related with some nice submanifolds in the space forms [2], [6], [7]. Here, we briefly introduce the $O(4,1)/O(3) \times O(1,1)$-system whose details are referred to [2], and explain how this system is related to the $K = 1$ surfaces.

Denote by $\mathcal{M}_{m\times n}$ the set of $m \times n$ matrices, $\mathcal{O}$ a simply connected open subset in $\mathbb{R}^3$ containing $(0,0)$, $\delta = \mathrm{diag}(dx, dy) \in \mathcal{M}_{2\times 2}$, $J = \mathrm{diag}(1,-1) \in \mathcal{M}_{2\times 2}$ and let $gl_*(n) = \{F \in \mathcal{M}_{n\times n} \mid f_{ii} = 0 \text{ for any } i\}$.

DEFINITION 3.1. ([2]) *The* $O(4,1)/O(3) \times O(1,1)$*-system* is a partial differential equation for $(F,G) : \mathcal{O} \subset \mathbb{R}^2 \longrightarrow gl_*(2) \times \mathcal{M}_{1\times 2}$ such that

$$\theta_\lambda = \begin{pmatrix} \delta F^t - F\delta & \delta G^t & -\lambda\delta J \\ -G\delta & 0 & 0 \\ \lambda\delta & 0 & \delta F - JF^t\delta J \end{pmatrix} \tag{3.1}$$

is a family of flat connections on $\mathcal{O} \subset \mathbb{R}^2$ for all $\lambda \in \mathbb{C}$. The map E such that $E^{-1}dE = \theta_\lambda$ and $E(0,0,\lambda) = 0$ is called a frame for the solution (F,G).

Here, the matrix θ_λ is partitioned into blocks so that

$$\omega = \begin{pmatrix} \delta F^t - F\delta & \delta G^t \\ -G\delta & 0 \end{pmatrix} \in o(3), \quad \eta = \delta F - JF^t\delta J \in o(1,1). \tag{3.2}$$

Since θ_λ in (3.1) is flat for any λ, so are ω and η. Thus there exist maps $A : \mathcal{O} \longrightarrow O(3)$ and $B : \mathcal{O} \longrightarrow O(1,1)$ such that

$$A^{-1}dA = \omega, \qquad B^{-1}dB = \eta. \tag{3.3}$$

Since B is $O(1,1)$-valued, we may assume

$$B = \begin{pmatrix} \cosh\phi & \sinh\phi \\ \sinh\phi & \cosh\phi \end{pmatrix}.$$

Then by (3.3), we have

$$\eta = \begin{pmatrix} 0 & d\phi \\ d\phi & 0 \end{pmatrix} = \begin{pmatrix} 0 & f_{12}dx + f_{21}dy \\ f_{12}dx + f_{21}dy & 0 \end{pmatrix}$$

and thus $f_{12} = \phi_x$ and $f_{21} = \phi_y$. Hence we say (ϕ, g_1, g_2) is a solution or not instead of (F,G) being so, where $G = (g_1, g_2)$.

Taking a gauge transformation on θ_λ by $h = \left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right)$, we obtain a flat connection

$$h * \theta_\lambda = \begin{pmatrix} 0 & -\lambda A_1\delta B^t J \\ \lambda B\delta A_1^t & 0 \end{pmatrix}, \tag{3.4}$$

where $A = (A_1, A_2) = \mathcal{M}_{3\times 2} \times \mathcal{M}_{3\times 1}$.

PROPOSITION 3.2 ([2]). *Suppose* (ϕ, g_1, g_2) *is a solution of the* $O(4,1)/O(3) \times O(1,1)$*-system and let* θ_λ *be the corresponding flat connection. Then*

(i)

$$\begin{cases} \Delta\phi + g_1 g_2 = 0, \\ (g_1)_y = \phi_y g_2, \\ (g_2)_x = \phi_x g_1. \end{cases}$$

(ii) *There exists a map* $X = (X_1, X_2) : \mathcal{O} \longrightarrow \mathcal{M}_{3\times 2}$ *such that* $dX = A_1 \delta B^t J$, *where* A *and* B *are given by* (3.3),

(iii) X *can be obtained by the Sym's formula*

$$\frac{\partial E}{\partial \lambda} \cdot E^{-1}\bigg|_{\lambda=0} = \begin{pmatrix} 0 & -X \\ JX^t & 0 \end{pmatrix},$$

where E *is the frame for* θ_λ, *that is,* $E^{-1}dE = \theta_\lambda$ *and* $E(0,0,\lambda) = 0$.

(iv) *Put* $A = (e_1, e_2, e_3) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3$. *Then* e_1, e_2, e_3 *are a local orthonormal frame for* X_1 *and* X_2, *and*

$$\begin{cases} dX_1 = e_1 \otimes \cosh\phi \; dx + e_2 \otimes \sinh\phi \; dy, \\ dX_2 = -(e_1 \otimes \sinh\phi \; dx + e_2 \otimes \cosh\phi \; dy), \\ de_3 = e_1 \otimes g_1 \; dx + e_2 \otimes g_2 \; dy. \end{cases}$$

(v) X_1 *and* X_2 *have the same Gaussian curvatures*

$$K_1 = K_2 = \frac{g_1 g_2}{\sinh\phi \cosh\phi}.$$

Now, we explain the relation between nonumbilic $K = 1$ surfaces and solutions of the $O(4,1)/O(3) \times O(1,1)$-system.

THEOREM 3.3. ϕ *is a solution of the sine-Gordon equation* (2.4) *if and only if* $(\phi, \sinh\phi, \cosh\phi)$ *is a solution of the* $O(4,1)/O(3) \times O(1,1)$*-system. In this case, the immersion* X_1 *in the Proposition 3.2 is a* $K = 1$ *surface with no umbilics such that the fundamental forms are*

$$\begin{cases} I_1 = \cosh^2\phi \; dx^2 + \sinh^2\phi \; dy^2, \\ II_1 = \sinh\phi \; \cosh\phi \; (dx^2 + dy^2). \end{cases}$$

PROOF. Substitute $g_1 = \sinh\phi$ and $g_2 = \cosh\phi$, then the result follows from the Proposition 3.2 by a direct calculation. □

REMARK 3.4. In the above theorem, X_2 becomes a totally umbilic surface in $\mathbb{R}^3$.

4. Transformations

Now, we construct an action on the solutions of the form $(\phi, \sinh\phi, \cosh\phi)$ of the $O(4,1)/O(3)\times O(1,1)$-system (*cf.* [2]).

Consider the bilinear form $<\ ,\ >_1$ on $\mathbb{C}^5$ given by

$$< u, v >_1 = \sum_{i=1}^{4} \bar{u_i} v_i - \bar{u_5} v_5.$$

Let $W = (w_1, w_2, w_3)^t$ and $Z = (z_1, z_2)^t$ be unit vectors in $\mathbb{R}^3$ and the Lorentzian space $\mathbb{R}^{1,1}$, respectively, and let π be the orthogonal projection of $= \mathbb{C}^5$ onto $\mathbb{C}\binom{W}{iZ}$ with respect to $<\ ,\ >_1$. So, $\pi = \frac{1}{2}\begin{pmatrix} WW^t & -iWZ^tJ \\ iZW^t & ZZ^tJ \end{pmatrix}$.

For $0 \neq s \in \mathbb{R}$, define

$$q_{s,\pi}(\lambda) = \big(\pi + \frac{\lambda - is}{\lambda + is}(I - \pi)\big)\big(\bar{\pi} + \frac{\lambda + is}{\lambda - is}(I - \bar{\pi})\big). \tag{4.1}$$

Let (F, G) be the solution of the $O(4,1)/O(3)\times O(1,1)$-system, θ_λ the corresponding flat connection as in (3.1), and E be the frame of (F, G), i.,e., $E^{-1}dE = \theta_\lambda$ and $E(0,0,\lambda) = 0$.

Put $\binom{\widetilde{W}}{i\widetilde{Z}}(x,y) = E(x,y,-is)^{-1}\binom{W}{iZ}$ and let $\tilde{\pi}(x,y)$ be the orthogonal projection onto $\mathbb{C}\binom{\widetilde{W}}{i\widetilde{Z}}$. Denote by

$$\widetilde{E}(x,y,\lambda) = E(x,y,\lambda)q_{s,\tilde{\pi}}(\lambda)^{-1}, \quad \widehat{W} = \frac{\widetilde{W}}{||\widetilde{W}||_3}, \quad \widehat{Z} = \frac{\widetilde{Z}}{||\widetilde{Z}||_{1,1}}.$$

LEMMA 4.1. *The followings hold* [2].
(i) $\widetilde{\theta}_\lambda = \widetilde{E}^{-1}d\widetilde{E}$ *gives a new solution* $(\widetilde{F}, \widetilde{G})$ *of the* $O(4,1)/O(3)\times O(1,1)$*-system,*
(ii)

$$E(x,y,0) = \begin{pmatrix} A(x,y) & 0 \\ 0 & B(x,y) \end{pmatrix},$$

$$\widetilde{E}(x,y,0) = \begin{pmatrix} \widetilde{A}(x,y) & 0 \\ 0 & \widetilde{B}(x,y) \end{pmatrix},$$

for some $A, \widetilde{A} \in O(3)$ *and* $B, \widetilde{B} \in O(1,1)$, *and*

$$\widetilde{A} = A(I - 2\widehat{W}\widehat{W}^t), \qquad \widetilde{B} = B(I - 2\widehat{Z}\widehat{Z}^tJ). \tag{4.2}$$

(iii)

$$\begin{pmatrix} \widetilde{F} \\ \widetilde{G} \end{pmatrix} = \begin{pmatrix} F \\ G \end{pmatrix} - 2s(\widehat{W}\widehat{Z}^t J)_*, \tag{4.3}$$

where $(c_{ij})_*$ *means* $c_{ii} = 0$.

(iv) $(\widetilde{W}, \widetilde{Z})$ *is a solution of*

$$d\begin{pmatrix} \widetilde{W} \\ i\widetilde{Z} \end{pmatrix} = -\theta_{-is}\begin{pmatrix} \widetilde{W} \\ i\widetilde{Z} \end{pmatrix}, \tag{4.4}$$

(v) $X,\ \widetilde{X} : \mathcal{O} \longrightarrow \mathcal{M}_{3\times 2}$ *given by*

$$\frac{\partial E}{\partial \lambda} \cdot E^{-1}\big|_{\lambda=0} = \begin{pmatrix} 0 & -X \\ JX^t & 0 \end{pmatrix} \text{ and } \frac{\partial \widetilde{E}}{\partial \lambda} \cdot \widetilde{E}^{-1}\big|_{\lambda=0} = \begin{pmatrix} 0 & -\widetilde{X} \\ J\widetilde{X}^t & 0 \end{pmatrix}$$

satisfy

$$\widetilde{X} = X - \frac{2}{s} A\widehat{W}\widehat{Z}^t B^t J. \tag{4.5}$$

This action by $q_{s,\pi}(\lambda)$ of the form (4.1) on the immersions $X = (X_1, X_2)$ can be interpreted as a geometric transformation.

Definition 4.2. Let M and $\widetilde{M}$ be surfaces in $\mathbb{R}^3$.

(1) *A sphere congruence* is a diffeomorphism $l : M \longrightarrow \widetilde{M}$ such that the lines normal to M at p and $\widetilde{M}$ at $l(p)$ intersect at a point equidistant to p and $l(p)$.

(2) A sphere congruence $l : M \longrightarrow \widetilde{M}$ is called *a Ribaucour transformation* if any principal vector e_p to M is sent to a principal vector $l_*(e_p)$ to $\widetilde{M}$ and the lines in these directions at p and $l(p)$ intersect at a point equidistant to p and $l(p)$.

Proposition 4.3. *Let* $X = (X_1, X_2)$ *and* $\widetilde{X} = (\widetilde{X}_1, \widetilde{X}_2)$ *be given by* (v) *in the Lemma 4.1. Then* X_i *and* $\widetilde{X}_i$ *are in a Ribaucour transformation. In particular,*

$$X_j + \psi_{ij} e_i = \widetilde{X}_j + \psi_{ij}\widetilde{e}_i,$$

where $\psi_{ij} = -\frac{1}{s\widehat{w}_i}(\widehat{Z}^t B^t J)_{1j}$ *and* $\widetilde{A} = (\widetilde{e}_1, \widetilde{e}_2, \widetilde{e}_3)$.

PROOF. The result follows from (4.2) and (4.5). □

Now, we will show that the action by $q_{s,\pi}(\lambda)$ on the solution $(\phi, \sinh\phi, \cosh\phi)$ of the $O(4,1)/O(3)\times O(1,1)$-system gives a new solution of the same form.

THEOREM 4.4. *Suppose $(\phi, \sinh\phi, \cosh\phi)$ is a solution corresponding to θ_λ in (3.1). Let $X = (X_1, X_2)$ be given as in the Lemma 4.1 (v). Let $q_{s,\pi}(\lambda)$, E, W, Z, $\widetilde{E}, \widetilde{W}, \widetilde{Z}$, $\widetilde{\theta}_\lambda$, $\widetilde{X}$ be defined as in the Lemma 4.1.*

If $sw_3 = z_1 \sinh\phi(0,0) + z_2\cosh\phi(0,0)$, then $(\widetilde{\phi}, \sinh\widetilde{\phi}, \cosh\widetilde{\phi})$ is a solution corresponding to $\widetilde{\theta}_\lambda$, where $\widetilde{\phi} = 2\alpha - \phi$ and $(\widehat{z}_1, \widehat{z}_2) = (\cosh\alpha, -\sinh\alpha)$. In this case,

$$\widetilde{X}_1 = X_1 - \frac{2}{s}(\widehat{z}_1 \cosh\phi + \widehat{z}_2 \sinh\phi) \sum_{i=1}^{3} \widehat{w}_i e_i,$$

and the immersions X_1 and $\widetilde{X}_1$ are $K = 1$ surfaces with no umbilics and they are in a Ribaucour transformation.

PROOF. From (4.4),

$$\begin{cases} d\widetilde{w}_1 &= -\widetilde{w}_2\, \omega_{12} - \widetilde{w}_3 \sinh\phi\ dx + s\widetilde{z}_1 dx,\\ d\widetilde{w}_2 &= -\widetilde{w}_1\, \omega_{21} - \widetilde{w}_3 \cosh\phi\ dy - s\widetilde{z}_2\ dy,\\ d\widetilde{w}_3 &= \widetilde{w}_1 \sinh\phi\ dx + \widetilde{w}_2 \cosh\phi\ dy,\\ d\widetilde{z}_1 &= s\widetilde{w}_1\ dx - \widetilde{z}_2\ d\phi,\\ d\widetilde{z}_2 &= s\widetilde{w}_2\ dy - \widetilde{z}_1\ d\phi. \end{cases}$$

Thus $d(s\widetilde{w}_3 - \widetilde{z}_1 \sinh\phi - \widetilde{z}_2\cosh\phi) = 0$ and thus by the initial condition,

$$s\widetilde{w}_3 = \sinh\phi\ \widetilde{z}_1 + \cosh\phi\ \widetilde{z}_2,$$

or, $s\widehat{w}_3 = \sinh\phi\ \widehat{z}_1 + \cosh\phi\ \widehat{z}_2 = \sinh(\phi - \alpha)$, where $(\widehat{z}_1, \widehat{z}_2) = (\cosh\alpha, -\sinh\alpha)$. Hence by (4.3),

$$\begin{cases} \widetilde{g}_1 = \sinh\phi - 2s\widehat{w}_3\widehat{z}_1 = \sinh\phi - 2\sinh(\phi-\alpha)\cosh\alpha = \sinh\widetilde{\phi},\\ \widetilde{g}_2 = \cosh\phi + 2s\widehat{w}_3\widehat{z}_2 = \cosh\phi - 2\sinh(\phi-\alpha)\sinh\alpha = \cosh\widetilde{\phi}, \end{cases}$$

and thus $(\widetilde{\phi}, \sinh\widetilde{\phi}, \cosh\widetilde{\phi})$ is a new solution. Therefore X_1 and $\widetilde{X}_1$ are nonumbilic $K = 1$ surfaces. From (4.2), we have

$$\begin{aligned} \widetilde{B} &= B(I - \widehat{Z}\widehat{Z}^t J)\\ &= \begin{pmatrix} -\cosh(2\alpha-\phi) & -\sinh(2\alpha-\phi)\\ \sinh(2\alpha-\phi) & \cosh(2\alpha-\phi) \end{pmatrix} = \begin{pmatrix} -\cosh\widetilde{\phi} & -\sinh\widetilde{\phi}\\ \sinh\widetilde{\phi} & \cosh\widetilde{\phi} \end{pmatrix}, \end{aligned}$$

and thus $\widetilde{\phi} = 2\alpha - \phi$. The formula for $\widetilde{X}_1$ comes from (4.5) and the fact that X_1 and $\widetilde{X}_1$ are in a Ribaucour transformation follows from the Proposition 4.3. □

EXAMPLE 4.5. $\phi = 0$ is the trivial solution of the sinh-Gordon equation. By acting $q_{s,\pi}(\lambda)$ on $\phi = 0$ and for $s = \sinh c$, we have

$$\begin{pmatrix} \widetilde{w}_1 \\ \widetilde{w}_2 \\ \widetilde{w}_3 \\ \widetilde{z}_1 \\ \widetilde{z}_2 \end{pmatrix} = \begin{pmatrix} w_1 \cosh(x \sinh c) + z_1 \sinh(x \sinh c) \\ w_2 \cos(y \cosh c) - w_3 \cosh c \sin(y \cosh c) \\ w_3 \cos(y \cosh c) + w_2 \ \mathrm{sech} c \ \sin(y \cosh c) \\ z_1 \cosh(x \sinh c) + w_1 \sinh(x \sinh c) \\ w_3 \sinh c \cos(y \cosh c) + \ w_2 \tanh c \ \sin(y \cosh c) \end{pmatrix}.$$

The Ribaucour transformation for the degenerate surface $X_1 = (x, 0, 0)$ is

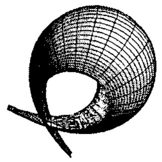
$$\widetilde{X}_1 = \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix} + r \begin{pmatrix} \sinh(x \sinh c) \\ \cos y \cos(y \cosh c) + \mathrm{sech} c \sin y \sin(y \cosh c) \\ -\sin y \cos(y \cosh c) + \mathrm{sech} c \cos y \sin(y \cosh c) \end{pmatrix},$$

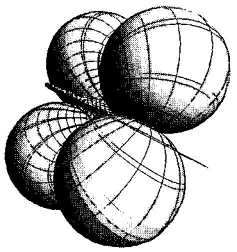
where

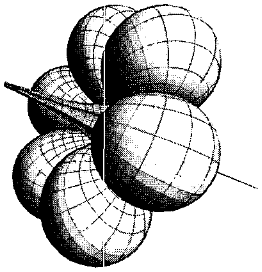
$$r = \frac{-2 \cosh(x \sinh c)}{\sinh c \{\cosh^2(x \sinh c) - \tanh^2 c \ \sin^2(y \cosh c)\}}.$$

The new solution $\widetilde{\phi}$ of the sinh-Gordon equation is

$$\widetilde{\phi} = \log \frac{\cosh(x \sinh c) - \tanh c \ \sin(y \cosh c)}{\cosh(x \sinh c) + \tanh c \ \sin(y \cosh c)}.$$







The above pictures are the immersion $\widetilde{X}_1$ $(0 < y < \pi/2)$, and the full images of $\widetilde{X}_1$ for $\cosh c = 2$ and $\cosh c = 1.5$, respectively.

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