Commun. Korean Math. Soc. **21** (2006), No. 1, pp. 151–163

POSITIVELY CURVED MANIFOLDS WITH FIXED POINT COHOMOGENEITY ONE

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Abstract. Any simply connected fixed point cohomogeneity one riemannian manifold with positive sectional curvature is diffeomorphic to one of the compact rank one symmetric spaces.

1. Introduction

The general program about classifying positively curved riemannian manifolds with large isometry groups (cf. [9, 10, 17, 25, 26]) has recently enjoyed considerable progress. Here we will provide a solution to one particular aspect of this program.

There is no doubt that one of the most measurements for the group G being large is that the orbit space is small, in particular in the sense that the dimension of its orbit space M/G, cohomogeneity, is small. If the fixed point set M^G is non-empty, this puts a constraint on M/G since we can view M^G it as a proper subset via the projection map π. In particular, the codimension $\dim(M/G) - \dim(M^G)$ is at least one. We define the *fixed point cohomogeneity* of an action by

$$\text{cohomfix}(M, G) = \dim(M/G) - \dim(M^G) - 1,$$

and if $M^G = \emptyset$, then, by convention, $\text{cohomfix}(M, G) = \dim(M/G)$. Thus to analyze manifolds with minimal fixed point cohomogeneity has two parts: homogeneous manifolds and G-manifolds M with $M^G \neq \emptyset$ of codimension one in M/G as called fixed point homogeneous. Homogeneous positively curved manifolds was known, due to their classification carried out by Berger [4], Wallach [23], Aloff-Wallach [1], Berard-Bergery [3], and Wilking [24]. In the cases of fixed point homogeneous manifolds,

Received August 8, 2005. Revised November 10, 2005.

2000 Mathematics Subject Classification: 53C20, 53C30, 53C40.

Key words and phrases: positive curvature, transformation groups, fixed point cohomogeneity.

Grove-Searle [14] classified by using G acts transitively on a normal sphere to maximal dimension connected component of fixed point set.

We are looking for next step of minimal fixed point cohomogeneity it seems natural to consider the class of the manifolds with fixed point cohomogeneity one. In the case of empty fixed point set, Verdiani [22] completely classified the even dimensional cohomogeneity one manifolds with positive sectional curvature. In view of Verdiani's result it remains to do this for odd dimensional manifolds. Recently Grove, Wilking, and Ziller [15] proved that a compact positively curved cohomogeneity one manifold is either diffeomorphic to one of the known positively curved biquotients or it corresponding to one of the 4-dimensional cohomogeneity one selfdual Einstein orbifolds that have been classified by Hitchin.

In the case of non-empty fixed point set, Bredon [5] has determined the structure of the orbit space around the image of fixed point set. We provide a complete classification of non-empty fixed point cohomogeneity one manifolds with positive sectional curvature (cf. (3.4), (3.8), [12]). As a special case, we obtain the simple proof that:

THEOREM 1.1. *Any simply connected non-empty fixed point cohomogeneity one manifold with positive sectional curvature is diffeomorphic to either* S^n, $\mathbb{CP}^m$, $\mathbb{HP}^k$, *or* $\mathbb{C}a\mathbb{P}^2$.

Note that our result in a sense is optimal, since the simply connected positively curved normal homogeneous Aloff-Wallach space $W_{1,1} = \mathrm{SU}(3)/\mathrm{S}^1_{1,1} = (\mathrm{SU}(3)\,\mathrm{SO}(3))/\mathrm{U}^\bullet(2)$ (cf. [24]) has an isometric circle action with non-empty fixed point cohomogeneity two.

The paper was completed while the author was visiting the University of Maryland in the academic year 2001–2003. The author would like to thank University for their hospitality and support. It is a pleasure to thank Karsten Grove for his interest in this project, and for the help he has provided by sharing his most valuable knowledge.

2. Alexandrov geometry on orbit spaces

In this section, we shall recall some well-known fact about Alexandrov geometry on orbit spaces. See [6], [7], [10], [14], for more details.

Since we only consider smooth compact transformation groups throughout, we may as well assume that each transformation is an isometry relative to a fixed auxiliary riemannian metric. Throughout, M will denote a closed, connected riemannian manifold, and G a compact Lie group which acts isometrically and (almost) effectively on M. For $x \in$

M, $G_x = \{g \in G \mid gx = x\}$ is the *isotropy group* of G at x, and $Gx = \{gx \mid g \in G\} \simeq G/G_x$ is the orbit through x. For any closed subgroup $L \subset G$, $M^L = \{x \in M \mid Lx = x\}$ will denote the fixed point set of L in M, and M^L is a finite union of closed totally geodesic submanifolds of M.

We start by recalling the slice theorem that is the most crucial basic result in the theory of compact transformation groups;

LEMMA 2.1. [slice theorem] *For any $x \in M$, a sufficiently small tubular neighborhood $D(Gx)$ of Gx is equivariantly diffeomorphic to $G \times_{G_x} D_x^{\perp}$.*

Here $D(Gx)$ is a suitable r-neighborhood of Gx and $D_x^{\perp}$ is the corresponding r-ball at the origin of the normal space $T_x^{\perp}$ to Gx at x. For minimal geodesics between different orbits, the following simple but very useful fact was found in Kleiner's thesis [18];

LEMMA 2.2. [isotropy lemma] *Let $\gamma : [0,1] \to M$ be a minimal geodesic between the orbits $G\gamma(0)$ and $G\gamma(1)$. Then for any $t \in (0,1)$, $G_{\gamma(t)} = G_\gamma$ is a subgroup of $G_{\gamma(0)}$ and of $G_{\gamma(1)}$.*

Using this one easily proves the following important and well known;

THEOREM 2.3. [principal orbit theorem] *There is a unique maximal orbit type. These so-called principal orbits form an open and dense subset of M.*

The usual distance between compact subsets defines a natural metric on the space of G-orbits in M denoted by M/G. By definition, this metric is a so-called *length metric*, indeed there is even a shortest curve, a geodesic, between any two orbits. We now proceed to describe the geometry of the orbit space M/G in more detail.

Throughout we will consider the orbit space M/G equipped with the above mentioned so-called *orbital metric*, and denote the quotient map by $\pi : M \to M/G$. When we view an orbit $Gx \subset M$ as a point in M/G, we will also use the notation $\pi(x) = [x]$. The following is immediate;

PROPOSITION 2.4. *The orbit map $\pi : M \to M/G$ is a* submetry, *i.e., $\pi(B_x(r)) = B_{[x]}(r)$ for all $x \in M$ and all $r \geq 0$.*

Here $B_x(r)$ denotes the open r-ball centered at x. This has a very important consequence;

THEOREM 2.5. *The orbit space M/G has the structure of an Alexandrov space with locally totally geodesic orbit strata.*

A finite dimensional length space $(X,\ d)$ is called and *Alexandrov space* if it has curvature bounded from below, say curv $X \geq k$. For a riemannian manifold M the property curv $M \geq k$ is equivalent to saying that its *sectional curvature* is bounded below by k, or in short $\sec M \geq k$. For a general metric space $(X,\ d)$, the property curv $X \geq k$ can be expressed by the requirement that any four tuple of points $x = (x_0, x_1, x_2, x_3) \in X^4$ can be isometrically embedded in the simply connected 3-manifold, $\mathrm{S}^3_{k(x)}$ with constant curvature $k(x) \geq k$.

Even general Alexandrov spaces have an amazingly rich structure (see [7], for the basic facts described here, and [20] for deeper developments). An important notion in this context is that of the *space of directions* S_x at $x \in X$. By definition this space is the completion of the space of *geodesic directions* at x, i.e., of germs of unit speed geodesics emanating from $x = x_0$. The curvature bound yields a natural notion of *angle* between geodesic directions represented by γ_1 and γ_2, namely $\angle(\gamma_1, \gamma_2) = \lim_{t\to 0} \angle_{\gamma_1(t),\gamma_2(t)}(k)$. In M/G all directions are geodesic directions and although not completely trivial one has the following natural;

PROPOSITION 2.6. *The space of directions $S_{[x]}$ at $[x] = \pi(x) \in M/G$ is isometric to $S_x^{\perp}/G_x$.*

The Euclidean cone $CS_x = T_xX$ on S_x is called the *tangent space* to X at x. The metric on $CS_x = (S_x \times [0,\infty))/\{(u,0) = (v,0)\}$ is defined so that the distance between (u,s) and (v,t) is the distance in the Euclidean plane between the end points of a hinge with sides of lengths u and v and angle $d(u,v)$. One also has the characterization $T_xX = \lim_{\lambda\to\infty} \lambda(X,x)$, where this limit refers to the so-called Gromov-Hausdorff limit (cf. [7]). In our case, $T_{[x]}M/G \simeq T_x^{\perp}/G_x$.

In a general Alexandrov space X its *boundary* ∂X consists of those points x for which ∂S_x is non-empty. This inductive definition is based on the fact that $\dim S_x = \dim X - 1$, and that the only compact one-dimensional Alexandrov spaces are the circle without boundary and the interval with two boundary points.

Now suppose that $\partial X \neq \emptyset$ and curv $X > 0$. Then the soul theorem adapted to Alexandrov spaces by Perelman [20] asserts that

$$\mathrm{dist}(\partial X, \cdot) : X \to \mathbb{R} \text{ is } \textit{strictly concave}.$$

In particular, this tells us the following; (1) there is a unique point $s_0 \in X$ at maximal distance from ∂X and it is called the *soul point* of X, (2) for any $x \in X - (\partial X \cup \{s_0\})$ and minimal geodesics γ_0, γ_1 from x to ∂X and s_0, respectively, we has $\angle(\gamma_0, \gamma_1) > \pi/2$, (3) X is contractible.

One has the following orbit space analogue of the celebrated *soul theorem* by Cheeger and Gromoll [8];

THEOREM 2.7. [soul theorem] *If* curv $M/G \geq 0$ *and* $\partial M/G \neq \emptyset$, *then there exists a totally convex compact subset* $S \subset M/G$ *with* $\partial S = \emptyset$, *which is a strong deformation retract of* M/G. *If* curv $M/G > 0$, *then* $S = [s_0]$ *is a point, and* $\partial(M/G)$ *is homeomorphic to* $S_{[s_0]}M/G \simeq S_{s_0}^{\perp}/G_{s_0}$.

Note that curv $M/G \geq 0$ or curv $M/G > 0$ are ensured if for example $\sec M \geq 0$ or $\sec M > 0$.

If a compact Lie group G acts smoothly on a manifold M with only one orbit type, then it is well known that the orbit space M/G is a manifold. Moreover, there is an induced (ineffective) action of the normalizer $N(H)$ on the fixed point set M^H which yields the principal $N(H)/H$-bundle, $M^H \to M^H/(N(H)/H) = M/G$. If we write ${}_cM = M^H$ and ${}_cG = N(H)/H$, then we may recover $M = {}_cM \times_{{}_cG} G/H$. In general when the G action has more than one orbit type, the core of M define to be the closure ${}_cM = \mathrm{cl}(M_o^H)$, of the core of the regular part M_o^H in M. Clearly, each of the sets $M_o^H \subset {}_cM \subset M^H$ are invariant under ${}_cG$-action, and in general each inclusion is strict.

The following theorem is very useful fact to study the orbit spaces and can be found in [2];

THEOREM 2.8. [reduction theorem] *The inclusion* ${}_cM \subset M$ *induces an isometry* ${}_cM/{}_cG = M/G$.

The part of essence of the core groups and manifolds is, that it reduces many general questions about group actions to those that have trivial principal isotropy group.

3. Non-empty fixed point cohomogeneity one

In this section we will classify simply connected positively curved riemannian manifolds with non-empty fixed point cohomogeneity one, that is, manifolds, M for which the fixed point cohomogeneity

$$\mathrm{cohomfix}(M, G) = \dim(M/G) - \dim(M^G) - 1$$

is equal to one. We need only consider the case in which $M^G \neq \emptyset$. If F is a connected component of M^G with maximal dimension, then the codimension of F in $X = M/G$ is two more than the cohomogeneity of any normal sphere $S_F^{\perp}$ to F under the induced G-action. This means that if cohomfix$(M, G) = 1$, then the action $G \times S^{\perp}_F \to S^{\perp}_F$ is of cohomogeneity one. Considering a cohomogeneity one action on a sphere, we know that the quotient space S^n/G is necessarily an interval, and its length (and thus its diameter) is π, $\pi/2$, $\pi/3$, $\pi/4$, or $\pi/6$, corresponding to the number of principal curvatures of the action (see [19], [11]) and all compact, connected Lie groups acting on spheres have been classified in [16], [21].

In particular, at each singular orbit the isotropy group acts by cohomogeneity one on the normal space to its orbit. In the case of a vertex orbit, the singular orbits of the cohomoegeneity one action cannot be points and the number of principal curvatures of such actions are 2, 3, 4, or 6 leading to diameters of $\pi/2$, $\pi/3$, $\pi/4$, or $\pi/6$.

g	G	H	K_-	K_+
2	$\mathrm{SO}(n)\,\mathrm{SO}(m)$	$\mathrm{SO}(n-1)\,\mathrm{SO}(m-1)$	$\mathrm{SO}(n-1)\,\mathrm{SO}(m)$	$\mathrm{SO}(n)\,\mathrm{SO}(m-1)$
	$\mathrm{Spin}(8)$	G_2	$\mathrm{Spin}(7)^-$	$\mathrm{Spin}(7)^+$
3	$\mathrm{SO}(3)$	$S(\mathrm{O}(1)^2) = \mathbb{Z}_2 + \mathbb{Z}_2$	$\mathrm{O}(2)$	$\mathrm{O}(2)$
	$\mathrm{SU}(3)$	$S(\mathrm{U}(1)^3) = T^2$	$S(\mathrm{U}(2)\,\mathrm{U}(1))$	$S(\mathrm{U}(1)\,\mathrm{U}(2))$
	$\mathrm{Sp}(3)$	$\mathrm{Sp}(1)^2$	$\mathrm{Sp}(2)\,\mathrm{Sp}(1)$	$\mathrm{Sp}(1)\,\mathrm{Sp}(2)$
	F_4	$\mathrm{Spin}(8)$	$\mathrm{Spin}(9)^-$	$\mathrm{Spin}(9)^+$
4	$\mathrm{SO}(2)\,\mathrm{SO}(m)$	$\mathbb{Z}_2\,\mathrm{SO}(m-2)$	$\mathrm{SO}(2)\,\mathrm{SO}(m-2)$	$\mathbb{Z}_2\,\mathrm{SO}(m-1)$
	$S(\mathrm{U}(2)\,\mathrm{U}(m))$	$T^2\,\mathrm{SU}(m-2)$	$S(\mathrm{U}(2)\,\mathrm{U}(m-2))$	$T^2\,\mathrm{SU}(m-1)$
	$\mathrm{Sp}(2)\,\mathrm{Sp}(m)$	$Sp(1)^2 Sp(m-2)$	$\mathrm{Sp}(2)\,\mathrm{Sp}(m-2)$	$\mathrm{Sp}(1)^2\,\mathrm{Sp}(m-1)$
	$\mathrm{Sp}(2)$	T^2	$S(\mathrm{U}(2)\,\mathrm{U}(1))$	$S(\mathrm{U}(1)\,\mathrm{U}(2))$
	$\mathrm{U}(1)\,\mathrm{Spin}(10)$	$T^1\,\mathrm{SU}(4)$	$T^1\,\mathrm{SU}(5)$	$T^1\,\mathrm{SU}(5)$
	$\mathrm{U}(5)$	$\mathrm{SU}(2)\,\mathrm{SU}(2)T^1$	$\mathrm{SU}(2)\,\mathrm{SU}(3)T^1$	$\mathrm{SU}(3)\,\mathrm{SU}(2)T^1$
6	$\mathrm{SO}(4)$	$\mathbb{Z}_2 + \mathbb{Z}_2$	T^1	T^1
	G_2	T^2	$S(\mathrm{U}(2)\,\mathrm{U}(1))$	$S(\mathrm{U}(1)\,\mathrm{U}(2))$

TABLE 1. Groups on a sphere with cohomogeneity one

When $\mathrm{S}^n/G = I = [-1, 1]$, there are precisely two non-principal G-orbits corresponding to the endpoints ± 1 of I, and S^n is decomposed as the union of two tubular neighborhoods of the non-principal orbits, with common boundary a principal orbit. Specifically, if $x_\pm \in S^n$ realize the distance between the non-principal orbits $B_\pm = \pi^{-1}(\pm 1)$ relative to a G-invariant riemannian metric on S^n, then

$$\mathrm{S}^n = D(B_-) \cup_E D(B_+),$$

and by the slice theorem $D(B_\pm) = G \times_{K_\pm} D^{\ell_\pm + 1}$, where $K_\pm = G_{x_\pm}$. Here $E = \pi^{-1}(0)$, the orbit $Gx_0 = G/H$ through the midpoint of a minimal geodesic γ from x_- to x_+ is canonically identified with the boundaries $\partial D(B_\pm) = G \times_{K_\pm} S^{\ell_\pm}$, via the maps $G \to G \times S^{\ell_\pm}$, $g \to$

$(g, \mp\dot{\gamma}(\pm 1))$. Note also that $\partial D^{\ell_\pm+1} = S^{\ell_\pm} = K_\pm/H$. In the case $K_- \neq K_+$, we are so called that the orbit spaces have *two sided* and the other case $K_- = K_+$, it have *one sided.*

The proof of main theorem is carried out dimension of M^G for ≥ 1 and zero separately.

3.1. $\dim M^G \geq 1$

Under $\dim M^G \geq 1$, the two sided orbit space has not singular orbits except fixed point set and the one sided orbit space rules out.

Let M/G be the two sided orbit space and $\partial(M/G) = F \cup X_+ \cup X_-$. Since $d(X_\pm, \cdot)$ is strictly concave, there is a unique point $[x_\pm] \in X_\pm$ at maximal distance from $X_\mp$ and the two isolated singular orbits $[x_\pm]$ has the following property;

LEMMA 3.1. *The two sided orbit spaces have at most one isolated singular orbit except fixed point set.*

PROOF. Assume that there are at least two isolated singular orbits $[x_+] = [x_1]$, $[x_-] = [x_2]$ in M/G. Then the diameter of the space of direction $S_{[x_i]}, i = 1, 2$, is less than equal to $\pi/2$. Let $x_i = [x_i] \in F$, $i = 3, 4$,

$$C_{ij} = \{\gamma : [0,1] \to M/G \mid \gamma \text{ is a minimal geodesic from } [x_i] \text{ to } [x_j]\},$$

and

$$\angle_{ijk} = \min\{\angle(\dot{\gamma}_j(0), \dot{\gamma}_k(0)) \mid \gamma_j \in C_{ji}, \gamma_k \in C_{jk}\}.$$

Since the directions $\dot{\gamma}_{31}(0)$ and $\dot{\gamma}_{32}(0)$ are perpendicular to F and the normal sphere $S_F^\perp$ has the cohomogenity one structure, we have $\angle_{132}$ is less than equal to $\pi/2$. Thus we obtain

$$\angle_{132} + \angle_{324} + \angle_{241} + \angle_{413} \leq 2\pi,$$

contradicting the hypothesis that curv $M/G > 0$. □

As an application of reduction theorem (2.8) we prove the following lemma which is the singular structure of the two sided orbit spaces.

LEMMA 3.2. *The fixed point sets are only singular orbits in two sided orbit spaces.*

PROOF. By the above lemma (3.1), we can assume that the two sided orbit space has one isolated singular orbit $[x_1] \notin F$ and using the reduction theorem (2.8), we consider that the group action has trivial principal isotropy group. Since the following theorem [6, §4, Theorem 5.3]

$$\operatorname{cohomfix}(M, G) \geq \operatorname{rank} G - \operatorname{rank} H - 1,$$

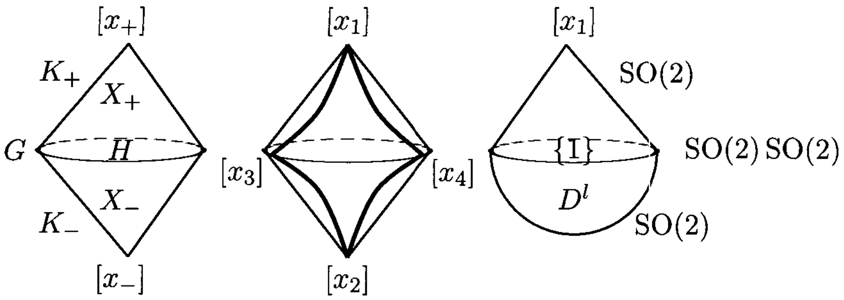

FIGURE 1. $\dim M^G \geq 1$

we shall determine the structure of orbit space to the case in which H is trivial by studying the action G on M and $\operatorname{rank} G \leq 2$. Thus the cohomogeneity one groups diagram on sphere are exactly $G = G_i$ or $G_i \times G_j$, $G_i = \mathrm{SO}(2)$ or $\mathrm{Sp}(1)$ and that G is only *almost* effective so that the effective group is obtained by dividing out by $H \cap Z(G)$.

We consider only case $G = \mathrm{SO}(2)\,\mathrm{SO}(2)$ and suppose there is isolated singular orbit $[x_1]$. By soul theorem (2.7), we have $S_{[x_1]} = S^{\perp}_{x_1}/G_{x_1}$ is homeomorphic to D^l, l-dimensional disk, $l \geq 2$. There are two cases to consider: The isotropy subgroup of isolated orbits are (a) $\mathrm{SO}(1) \times \mathbb{Z}_k$ and (b) $\mathrm{SO}(1)\,\mathrm{SO}(1)$. In case (a) it follows that $S^{\perp}_{x_1}/\mathrm{SO}(1) \times \mathbb{Z}_k$ is not homeomorphic to disk, contradicting the fact that $S^{\perp}_{x_1}/\mathrm{SO}(1) \times \mathbb{Z}_k \simeq D^l$, and thus this case does not occur. In case (b): Since the group $\mathrm{SO}(2)$ fix the normal sphere $S^{\perp}_{x_1}$, there is precisely one orbit type. Thus the group $\mathrm{SO}(2)$ act freely or transitively on $S^{\perp}_{x_1}/\,\mathrm{SO}(2)$. However the group $\mathrm{SO}(2)$ action preserves the soul point, we only consider that the group $\mathrm{SO}(2)$ act transitively on $S^{\perp}_{x_1}/\,\mathrm{SO}(2)$. But it is a contradiction that $D^l \simeq S^{\perp}_{x_1}/\mathrm{SO}(2)\,\mathrm{SO}(2) \simeq (S^{\perp}_{x_1}/\,\mathrm{SO}(2))/\,\mathrm{SO}(2) \not\simeq D^l$. □

We will prove that the isotropy groups $K_+ = K_-$ case can not arise.

LEMMA 3.3. *The one sided orbit space rules out.*

PROOF. Since the connected component F with maximal dimension fixed point set has a nontrivial twofold cover, we have $\pi_1(F) \neq \{1\}$. In the case $\dim M^G = 1$; Since $\partial(M/G) \simeq \mathbb{RP}^2$, we obtain $\pi_1(F) = \{1\}$ and it is a contradiction.

In the cases $\dim M^G \geq 2$; Take $1 \neq \alpha \in \pi_1(F) : S^1 \to F \subset M$, then there are the extension $\bar{\alpha} : D^2 \to M$ of α. Since the soul isotropy group

G_{s_0} is contained the principal isotropy group H, $\dim G/G_{s_0} \leq \dim G/H$, and hence

$$\text{codim } G/G_{s_0} \geq \text{codim } G/H = \dim M/G \geq 3.$$

By the transversality property, we assume that $\bar{\alpha}(D^2) \cup G/G_{s_0} = \emptyset$. Since $M/G = C\partial(M/G)$ is cone of the soul point $[s_0]$, we have that $\pi \circ \bar{\alpha} : D^2 \to \partial(M/G)$ is the homotopy map. We consider the neighborhood $D_\varepsilon(F)$ of F in $\partial(M/G)$ and $Y = \partial(M/G) - \text{int}D_\varepsilon(F)$. Then Y is an Alexandrov space with curvature bounded below by zero with boundary $\partial Y = \partial D_\varepsilon(F)$, hence Y is homeomorphic to $C\partial D_\varepsilon(F)$. Since $\dim Y \geq 3$, we assume that the soul point of Y is not contained in $\pi \circ \bar{\alpha}(D^2)$ and also deform $\pi \circ \bar{\alpha}(D^2) \subset F$. Thus we have $\pi_1(F) = \{1\}$ and this is a contradiction. □

We are ready to classify simply connected positively curved manifolds with fixed point ($\dim M^G \geq 1$) cohomogeneity one;

THEOREM 3.4. *Let M be a simply connected positively curved almost effective G-manifold with fixed point cohomogeneity one. If the dimension of fixed point set M^G is greater than equal to one, the M is diffeomorphic to sphere.*

PROOF. Since curv $M/G > 0$ and $\partial(M/G) \neq \emptyset$, by the soul theorem (2.7), there is a unique soul point $[s_0] \in M/G$ at the maximal distance from $\partial(M/G)$. Because the orbit space M/G has two side, the diameter of space of direction $S_{[x_0]}$ is greater than $\pi/2$ and the principal isotropy group H is normal subgroup of the soul isotropy group. But by isotropy lemma (2.2), in fact, we have $H = G_{s_0}$. By soul theorem (2.7), F is homeomorphic to the boundary of the space of direction $S_{[x_\pm]}$, sphere, and thus we have that $F \subset M$ is diffeomorphic to sphere. Therefore the manifold M is diffeomorphic to the spherical join $F * S_F^\perp$ of F and $S_F^\perp$ and hence sphere. □

3.2. $\dim M^G = 0$

From a purely geometric viewpoint, it is easy to see that there can be no more than three such isolated singular orbits, also known as vertex orbits.

LEMMA 3.5. *In the orbit space M/G there are at most three vertex orbits.*

PROOF. Suppose that there are at least four vertex orbits $[x_1]$, $[x_2]$, $[x_3]$, $[x_4]$ in M/G. Since curv $M/G > 0$, by the Toponogov's comparison theorem, taking the sum for each triangles and then add up over all triangles, we obtain that

$$\sum_{1=1}^{4} \sum_{1\le j<k\le 4, j,k\neq i} \angle_{ijk} > 4\pi.$$

On the other hand, the space of direction at $[x_i]$ $S_{[x_i]} = S^{\perp}_{x_i}/G_{x_i} = S^1(1/2)$ and the estimate the 3-extent, $\mathrm{xt}_3(S^1(1/2)) = \pi/3$ (see [13]), implies that

$$\sum_{1=1}^{4} \sum_{1\le j<k\le 4, j,k\neq i} \angle_{ijk} < 4\pi.$$

This is a contradiction, and hence the orbit space M/G contains at most three vertex orbits. □

The orbit space is a 2-dimensional disk with at most three are vertices. Accordingly, we shall divide the type into three families, reflecting the geometry M/G (Figure 2). The same as $\dim M^G \geq 1$, we have that the principal isotropy group is the soul isotropy group;

LEMMA 3.6. *If* $\dim M^G = 0$ *and fixed point cohomogeneity one, then* $H = G_{s_0}$.

PROOF. By using the distance function from $\partial(M/G)$ is strictly concave, the boundary $\partial(M/G) = S^1$ is homeomorphic to S^l/G_{s_0}. Thus $l = 1$, and $G_{s_0} = H$. □

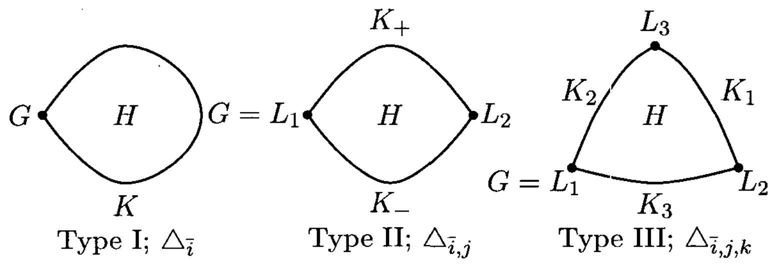

FIGURE 2. Three types of orbit spaces

In fact, all three families exhibit different features which necessitate a separate study for each of them (Figure 2). Since curv $M/G > 0$, by the Toponogov's comparison theorem, the sum of angles at the vertex orbits of triangle is greater than π. We have at most six different isotropy groups, $\triangle_{2,2,2}$, $\triangle_{2,2,3}$, $\triangle_{2,2,4}$, $\triangle_{2,2,6}$, $\triangle_{2,3,3}$, $\triangle_{2,3,4}$, where $\triangle_{i,j,k}$ means that each three angles of the vertex orbits are $\pi/i, \pi/j, \pi/k$. Thus a fixed point cohomogeneity one action on a manifold with zero dimension fixed point set consists of an action with at most six different isotropy groups, in which all the singular groups are related in terms of cohomogeneity one actions on the normal spheres to their orbits with principal isotropy groups that of the original action itself and singular isotropy groups corresponding to those of the corresponding singular orbits.

However using reduction theorem (2.8), we will obtain, as the only possibilities, the orbit spaces are $\triangle_{\bar{2}}$, $\triangle_{\bar{2},\bar{2}}$, or $\triangle_{\bar{2},\bar{2},\bar{2}}$.

LEMMA 3.7. *If* $\dim M^G = 0$ *and fixed point cohomogeneity one, then the orbit spaces are* $\triangle_{\bar{2}}$, $\triangle_{\bar{2},\bar{2}}$, *or* $\triangle_{\bar{2},\bar{2},\bar{2}}$ *and every vertices are fixed.*

PROOF. As the proof of (3.2), we only assume that $G = G_i$ or $G_i \times G_j$, $G_i = \mathrm{SO}(2)$ or $\mathrm{Sp}(1)$. We will discuss only the case $G = \mathrm{SO}(2)\,\mathrm{SO}(2)$ and leave remaining more restrictive cases to the reader.

Note that $\mathrm{S}^n/G = [0, \pi/2]$ if and only if the representation of G on $\mathbb{R}^{n+1}$ has a proper, nontrivial G-invariant subspace of $\mathbb{R}^{n+1}$. If $G = \mathrm{SO}(2)\,\mathrm{SO}(2)$ then there are only three cases to consider:

Types	K	orbit space	M
Type I	SO(2)	$\triangle_{\bar{2}}$	$\mathbb{RP}^4$
Type II	SO(2)	$\triangle_{\bar{2},\bar{2}}$	S^4
Type III	SO(2)	$\triangle_{\bar{2},\bar{2},\bar{2}}$	$\mathbb{CP}^2$, $\mathbb{HP}^1$

TABLE 2. $G = \mathrm{SO}(2)\,\mathrm{SO}(2)$

as was to be shown. □

We now classify simply connected positively curved manifolds with fixed point ($\dim M^G = 0$) cohomogeneity one;

THEOREM 3.8. *Let* M *be as the assumption of* (3.4). *If the dimension of fixed point set* M^G *is equal to zero, then* M *is diffeomorphic to one of the compact rank one symmetric spaces.*

PROOF. In the orbit spaces $\triangle_{\bar{2}}$ and $\triangle_{\bar{2},\bar{2}}$ cases, we easy show that the manifold M is diffeomorphic to sphere or real projective space, so we only consider that the orbit space is $\triangle_{\bar{2},\bar{2},\bar{2}}$. Since all three angles of vertex orbits have $\pi/2$, the action groups of interest then are $\mathrm{SO}(n)^2$, $\mathrm{SU}(n)^2$, or Spin(8) of manifold $\mathbb{CP}^n$, $\mathbb{HP}^n$, or $\mathbb{C}a\mathbb{P}^2$, respectively. □

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