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ALMOST KÄHLER DEFORMATION OF SCALAR CURVATURE

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ABSTRACT. We demonstrate a novel almost Kähler deformation on $\mathbb{R}^4$ which diffuses the scalar curvature.

1. Introduction

In Riemannian geometry an interesting question is whether every smooth manifold of dimension bigger than 2 admits a metric with its curvature negative. The Ricci curvature case was recently studied in [4], while the relatively simpler case of scalar curvature had been understood earlier, see [1, Theorem 4.32]. In either case, one first finds a metric which has negative (Ricci or scalar) curvature somewhere in the manifold, and next they used a conformal deformation to diffuse the negative part all across the manifold.

One may ask a similar question on symplectic manifolds, i.e. whether every symplectic manifold of real dimension bigger than or equal to four admits a compatible Riemannian metric, so-called an almost Kähler metric, of negative (scalar or Ricci) curvature. Along this line, we found recently an almost Kähler scalar-island metric [3], i.e., a metric on $\mathbb{R}^{2n}$ which has the scalar curvature negative on a pre-compact open subset and is Euclidean outside of its closure.

This raises a hope that one may obtain an almost Kähler metric of negative scalar curvature by following the Lohakamp's argument in [4] where he plugged many copies of a Ricci-island metric [4, Proposition 3.1] into a (enlarged) given almost-Kähler manifold and then apply a Ricci-curvature-diffusing deformation. In almost Kähler case, however,

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we cannot apply his *conformal* deformation because it does not preserve a symplectic structure. So in almost Kähler case even the scalar curvature question does not seem trivial.

Therefore it is intriguing to find a non-conformal and effective almost Kähler deformation which diffuses the scalar curvature. In this article we propose, at least on $\mathbb{R}^4$, such a deformation inspired by that of Lohkamp's. Then, we present some speculation on how this novel deformation be used to attack the problem of finding an almost Kähler metric on any 4-dimensional symplectic manifold.

In section 2 after explaining the definition of almost Kähler metrics, we describe the almost Kähler deformation of the Euclidean metric on $\mathbb{R}^4$ and show some of its properties. In section 3 we gives an outline of how the deformation of section 2 can be applied in studying the existence of almost Kähler metrics of negative scalar curvature.

2. Almost Kähler metric and scalar curvature diffusion

An *almost-Kähler* metric is a Riemannian metric g compatible with a symplectic structure ω on a smooth manifold, i.e., $\omega(x,y) = g(Jx,y)$ for an almost complex structure J, where x, y are tangent vectors at a point of the manifold. We often denote it by the triple (g,ω,J) or the couple (g,ω) to specify what ω or J is. Notice that any one of the pairs (g,ω), (ω, J), or (g, J) determines the other two. For a fixed ω, we shall call a metric g ω-almost Kähler if g is compatible with ω. An almost-Kähler metric (g,ω,J) is Kähler if and only if J is integrable.

Consider the symplectic structure $\omega = dx_1 \wedge dx_2 + dx_3 \wedge dx_4$ on $\mathbb{R}^4 = \{(x_1,x_2,x_3,x_4) | \, x_i \in \mathbb{R}\}$ together with the compatible Euclidean metric $g = \sum_{i=1}^4 dx_i \otimes dx_i$ and the (almost) complex structure J. The Euclidean metric $(\mathbb{R}^4, g, \omega, J)$ has an orthonormal co-frame field

$$(1) \qquad \omega_1 = dr, \quad \omega_2 = J(dr) = r\sigma_1, \quad \omega_3 = r\sigma_2, \quad \omega_4 = r\sigma_3 = J(\omega_3),$$

where $r(\cdot) = d_g(p,\cdot)$ and $\sigma_1, \sigma_2, \sigma_3$ is the canonical orthonormal co-frame field of the standard metric on S^3 which satisfy $d\sigma_1 = 2\sigma_2 \wedge \sigma_3, d\sigma_2 = 2\sigma_3 \wedge \sigma_1$ and $d\sigma_3 = 2\sigma_1 \wedge \sigma_2$. Now we write $g = \sum_{i=1}^4 \omega_i \otimes \omega_i$ and consider the curvature 1-forms in the co-frame ω_i: $d\omega_i = \sum_{j=1}^4 \omega_{ij} \wedge \omega_j$, with $\omega_{ij} = -\omega_{ji}$, and define a_{ijk} in the formula $\omega_{ij} = \sum_{k=1}^4 a_{ijk} \cdot \omega_k$. Then, the a_{ijk}'s are all zero except

$$(2) \qquad \begin{cases} a_{122} &= a_{133} = a_{144} = -a_{212} = -a_{313} = -a_{414} = \frac{1}{r}, \\ a_{234} &= a_{342} = a_{423} = -a_{324} = -a_{432} = -a_{243} = -\frac{1}{r}. \end{cases}$$

We shall perturb $g = \sum_{i=1}^{4} \omega_i \otimes \omega_i$ on $\mathbb{R}^4$.

We use the functions of [4]; $f_{d,s}(t) \in C^\infty(\mathbb{R}, \mathbb{R}^{\geq 0})$ for $d, s > 0$ by $f_{d,s}(t) = s \cdot \exp(-\frac{d}{t})$ on $\mathbb{R}^{>0}$ and $f_{d,s} = 0$ on $\mathbb{R}^{\leq 0}$. Also choose an $h \in C^\infty(\mathbb{R}, [0,1])$ with $h = 0$ on $\mathbb{R}^{\geq 1}$, $h = 1$ on $\mathbb{R}^{\leq 0}$ and $h^b_\epsilon(t) = h(\frac{1}{\epsilon}(t-b))$, $b > 0$, $\epsilon > 0$. Now we define a perturbed metric for $b + \epsilon < c$;

$$
\begin{aligned}
g^{b,\epsilon,c}_{d,s} = & \frac{\omega_1 \otimes \omega_1}{\{1 + h^b_\epsilon(c-r) \cdot f_{d,s}(c-r)\}} \\
& + \{1 + h^b_\epsilon(c-r) \cdot f_{d,s}(c-r)\}\omega_2 \otimes \omega_2 + \omega_3 \otimes \omega_3 + \omega_4 \otimes \omega_4.
\end{aligned} \tag{3}
$$

Then $g^{b,\epsilon,c}_{d,s}$ is a smooth ω-almost Kähler metric. For convenience we set $\alpha = \frac{1}{\sqrt{1+h^b_\epsilon(c-r)\cdot f_{d,s}(c-r)}}$. We compute the scalar curvature using the orthonormal co-frame field $\tilde{\omega_1} = \alpha\omega_1$, $\tilde{\omega_2} = \frac{1}{\alpha}\omega_2$, $\tilde{\omega_3} = \omega_3$, $\tilde{\omega_4} = \omega_4$ and the formula $d\tilde{\omega}_i = \sum_{j=1}^{4} \tilde{\omega}_{ij} \wedge \tilde{\omega}_j$ with $\tilde{\omega}_{ij} = -\tilde{\omega}_{ji}$. And set $\tilde{\omega}_{ij} = \sum_{k=1}^{4} \tilde{a}_{ijk} \cdot \tilde{\omega}_k$.
Using (2) we compute that all $\tilde{a}_{ijk}$'s are zero except the following;

$$
\begin{aligned}
&\tilde{a}_{212} = -\tilde{a}_{122} = \frac{\alpha'}{\alpha^2} - \frac{1}{\alpha r}, \quad \tilde{a}_{313} = -\tilde{a}_{133} = -\frac{1}{\alpha r}, \quad \tilde{a}_{414} = -\tilde{a}_{144} = -\frac{1}{\alpha r}, \\
&\tilde{a}_{342} = -\tilde{a}_{432} = \frac{1}{\alpha r} - \frac{2\alpha}{r}, \quad \tilde{a}_{234} = -\tilde{a}_{324} = \frac{-1}{\alpha r}, \quad \tilde{a}_{243} = -\tilde{a}_{423} = \frac{1}{\alpha r},
\end{aligned}
$$

Curvature components R_{ijij}'s of $g^{b,\epsilon,c}_{d,s}$ are as follows;

$$
\begin{aligned}
&R_{1212} = -\frac{\alpha''}{\alpha^3} + 3\frac{(\alpha')^2}{\alpha^4} - \frac{3\alpha'}{\alpha^3 r}, \qquad R_{1313} = R_{1414} = -\frac{\alpha'}{\alpha^3 r}, \\
&R_{2323} = R_{2424} = -\frac{\alpha'}{\alpha^3 r}, \qquad R_{3434} = \frac{4}{\alpha^2 r^2} - \frac{4}{r^2}.
\end{aligned} \tag{4}
$$

From above we get

$$
-\frac{s(g^{b,\epsilon,c}_{d,s})}{2} = -\frac{\alpha''}{\alpha^3} + 3\frac{(\alpha')^2}{\alpha^4} - \frac{7\alpha'}{\alpha^3 r} + \frac{4}{\alpha^2 r^2} - \frac{4}{r^2}.
$$

Setting

$$
Y = \frac{1}{\alpha^2} - 1, \quad s(g^{b,\epsilon,c}_{d,s}) = -Y'' - \frac{7}{r}Y' - \frac{8}{r^2}Y. \tag{5}
$$

This is a favorable form, because we can prove the following property, analogously to [4, Proposition 2.1].

PROPOSITION 1. *For each $a, b, \epsilon, c > 0$ such that $c > b+\epsilon > b > a > 0$, define $g^{b,\epsilon,c}_{d,s}$ on $\mathbb{R}^4$ as in (3). There are constants $\gamma = \gamma(b,c) > 0$ such that for $d \geq \gamma$, $s \in [0,1]$ the following statements hold:*

(i) $g_{d,s}^{b,\epsilon,c} \equiv g$ *on* $B^g_{c-b-\epsilon}(p)$,
(ii) $s(g_{d,s}^{b,\epsilon,c}) \leq 0$ *on* $\mathbb{R}^4 \setminus B^g_{c-b}(p)$,
(iii) $s(g_{d,s}^{b,\epsilon,c}) \leq -s \cdot e^{-\frac{d}{a}}$ *on* $B^g_{c-a}(p) \setminus B^g_{c-b}(p)$.

PROOF. (i) is clear from the definition. As $f''_{d,s}(t) = e^{-\frac{d}{t}}(\frac{d^2}{t^4} - \frac{2d}{t^3})$, if $d \geq 2b$ then $f^{(k)}_{d,s} \geq 0$ on $(0,b]$ for $k = 0,1,2$. On $(0,b]$ we get

$$\begin{aligned} f''_{d,s} - f'_{d,s} \cdot \frac{7}{(c-t)} &\geq f''_{d,s} - f'_{d,s} \cdot \frac{7}{(c-b)} \\ &= se^{-\frac{d}{t}}\{\frac{d^2}{t^4} - \frac{2d}{t^3} - \frac{7}{t^2(c-b)}\}. \end{aligned}$$

If $d \geq \gamma(b,c) := 2b + \frac{7b^2}{c-b} + \sqrt{4b^4 + (2b + \frac{7b^2}{c-b})^2}$, then $\frac{d^2}{t^4} - \frac{2d}{t^3} - \frac{7d}{t^2(c-b)} \geq 0$ for $t \in (0,b]$.

Therefore if $d \geq \gamma(b,c)$, on $B^g_c(p) \setminus B^g_{c-b}(p)$ where $Y = Y(r) = f_{d,s}(c-r)$,

$$\begin{aligned} Y'' + \frac{7}{r}Y' + \frac{8}{r^2}Y &= f''_{d,s}(c-r) - f'_{d,s}(c-r)\frac{7}{r} + f_{d,s}(c-r) \cdot \frac{8}{r^2} \\ &\geq f''_{d,s}(t) - f'_{d,s}(t) \cdot \frac{7}{(c-t)} \\ &\geq 0, \end{aligned}$$

where we set $t = c - r$.

And if $d \geq \gamma(b,c)$, on $B^g_{c-a}(p) \setminus B^g_{c-b}(p)$ we have that $Y'' + \frac{7}{r}Y' + \frac{8}{r^2}Y \geq se^{-\frac{d}{a}}$, using $e^{-\frac{d}{t}} \geq e^{-\frac{d}{a}}$ for $t \in (a,b]$. The proof is finished. □

3. Towards almost Kähler metrics of negative scalar curvature

In this section we discuss on how the above deformation on $\mathbb{R}^4$ in Proposition 1 can be applied to a general almost Kähler manifold, following the scheme of Lohkamp.

First, one can diffuse an almost Kähler *island* metric g on $\mathbb{R}^4$, such as the one in [3], which has the scalar curvature negative on a pre-compact open subset V and is Euclidean outside the closure $\overline{V}$. Assume that g is close to the Euclidean metric in some C^k-norm, as is possible with the metric in [3]. Then one can construct a g-orthonormal frame field $\{\omega_i\}$ which are close to the frame field in (1) by a canonical process, say the Gram-Schmidt process. Then deform g as in the formula (3). The deformed metric $g_{d,s}^{b,\epsilon,c}$ may diffuse the scalar curvature, i.e., it decreases the scalar curvature away from V by Proposition 1 (actually a modified version of this proposition for *almost Euclidean* metrics) and keeps the scalar curvature negative even inside V by absorbing the counter effect of scalar-curvature increase by choosing the parameter s small.

Next, one adapts it to any almost Kähler metric on a compact symplectic manifold. Namely, one enlarges the metric by rescaling and then embeds many copies of the island metric into the enlarged manifold by some surgery preserving the symplectic structure. Finally one uses the deformation of Proposition 1 near the copies of the island metric and diffuse them to get a negatively scalar-curved almost Kähler metric everywhere on the symplectic manifold.

The implementation of this argument could involve a long analysis and author hopes to work out the detail in a forthcoming article [2].

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