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SOME APPLICATIONS OF EXTREMAL LENGTH TO ANALYTIC FUNCTIONS

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ABSTRACT. We consider some applications of extremal length to the boundary behavior of analytic functions and derive theorems in connection with the conformal mappings. It shows us the usefulness of the method of extremal length. And we present some geometric applications of extremal length. The method of extremal length lead to simple proofs of theorems.

0. Introduction

The method of extremal length is a useful tool in a wide variety of areas. Especially, it has been successfully applied to conformal mappings and analytic functions of a complex variable. Extremal length was introduced as a conformally invariant measure of curve families. This development appeared in Ahlfors and Beurling [3].

The purpose of this paper is to apply the extremal length of a curve family in the complex plane to the boundary behavior of analytic functions of a complex variable. And we consider some geometric applications of extremal length. This method lead to simple proofs of theorems.

Throughout this paper, $\mathbb{C}$ denote the finite complex plane, D is a domain (open and connected set) in $\mathbb{C}$, g is an arbitrary function defined on D (Def. 2.1), ∂D is a boundary of D, and $cl(D)$ is a closure of D.

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1. Extremal length

Let Γ be a family whose elements γ are locally rectifiable curves (sim-

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ply, curves or arcs) in D, and let $\rho(z)$ be a non-negative Borel measurable function defined on $\mathbb{C}$. Every curve γ has a well-defined

$$(1) \qquad L(\gamma, \rho) = \int_\gamma \rho(z)\,|dz|, \quad z = x + iy$$

which may be infinite, and D has a

$$(2) \qquad A(D, \rho) = \iint_D [\rho(z)]^2\,dx\,dy \neq 0, \infty.$$

In order to define an invariant which depends on the whole set Γ, we introduce

$$(3) \qquad L(\Gamma, \rho) = \inf_{\gamma\in\Gamma} L(\gamma, \rho),$$

where we agree that $L(\Gamma, \rho) = \infty$ in case Γ is empty.

To obtain a quantity that does not change when the weight function ρ is multiplied by a constant, we form the homogeneous expression $[L(\Gamma, \rho)]^2/A(D, \rho)$.

DEFINITION 1.1. ([1]) The *extremal length* of Γ in D is defined by

$$(4) \qquad \lambda(\Gamma) = \lambda_D(\Gamma) = \sup_\rho\,[L(\Gamma, \rho)]^2/A(D, \rho),$$

where ρ is subject to the condition $0 < A(D, \rho) < \infty$, obviously $0 \leq \lambda(\Gamma) \leq \infty$.

REMARK 1.2. (i) $\lambda_D(\Gamma)$ depends only on Γ and not on D. Accordingly, we shall simplify the notation to $\lambda(\Gamma)$ [1].

(ii) Since almost every curve in $\mathbb{C}$ is rectifiable, the non-rectifiable curves of a family Γ have no influence on the extremal length of Γ. Accordingly, we shall simplify the terminology to curve or arc [14].

There are two special cases in which the extremal length is very easy to determine explicitly.

PROPOSITION 1.3. (a) ([2]) *Let B be a rectangle of sides a and b. Let Γ be the family of arcs in B which join the sides of length b. Then*

$$\lambda(\Gamma) = a/b.$$

(b) ([7]) *Let* Δ *be the annulus* $\Delta = \{z \mid a < |z| < b\}$. *Let* Γ *be the family of arcs in* Δ *which join the two contours. Then*

$$\lambda(\Gamma) = (1/2\pi)\log(b/a).$$

In fact, for any $\rho(z)$, we have

$$\int_a^b \rho\, dr \geq L(\Gamma, \rho), \qquad \iint_\Delta \rho\, dr\, d\theta \geq 2\pi L(\Gamma, \rho).$$

Then, by the Schwarz inequality ([8]),

$$\begin{aligned} 4\pi^2[L(\Gamma,\rho)]^2 &\leq [\iint_\Delta \rho\, dr\, d\theta]^2 \\ &\leq [\iint_\Delta \rho^2(1/r)\, dr\, d\theta][\iint_\Delta r\, dr\, d\theta] \\ &= [2\pi \log(b/a)][\iint_\Delta \rho^2 r\, dr\, d\theta]. \end{aligned}$$

This proves $\lambda(\Gamma) \leq (1/2\pi)\log(b/a)$.
Equality for $\rho = 1/r$, we have

$$L(\Gamma, 1/r) = \log(b/a), \quad A(\Delta, 1/r) = 2\pi \log(b/a).$$

Thus $\lambda(\Gamma) \geq (1/2\pi)\log(b/a)$.

The conformal invariance of extremal length is an immediate consequence of the definition.

PROPOSITION 1.4. ([12], Conformal invariance) *Let* $z^* = f(z)$ *be a 1-1 conformal mapping on* D *upon a domain* D^* *and* Γ *a family of curves on* D. *Then*

$$\lambda(\Gamma) = \lambda[f(\Gamma)].$$

PROPOSITION 1.5. ([1], Comparison principle) *For two curve families* Γ_1, Γ_2, *if every* $\gamma_2 \in \Gamma_2$ *contains a* $\gamma_1 \in \Gamma_1$, *then*

$$\lambda(\Gamma_1) \leq \lambda(\Gamma_2).$$

Indeed, both extremal lengths can be evaluated with respect to the same D. For any ρ in D it is clear that $L(\Gamma_2,\rho) \geq L(\Gamma_1,\rho)$. These minimum lengths are compared with the same $A(D,\rho)$.

REMARK 1.6. (i) Briefly, the set Γ_2 of fewer or longer curves has the larger extremal length [1].

PROPOSITION 1.7. ([12]) *Suppose that there exist disjoint open sets G_n containing the curves in Γ_n. If $\cup_n \Gamma_n \subset \Gamma$, then*

$$\sum_n 1/\lambda(\Gamma_n) \leq 1/\lambda(\Gamma).$$

2. Some boundary behavior of conformal mappings

DEFINITION 2.1. ([10]) By an *arbitrary function* g, we mean a (single-valued) function whose domain is a subset of $\mathbb{C}$ and whose range is on the Riemann sphere Ω.

DEFINITION 2.2. ([5]) Let Λ be a curve at $z_0 \in cl(D)$. Then the *cluster set* of g at z_0 along Λ, denoted by $C_\Lambda(g,z_0)$, is defined to be the set of all points $\omega \in \Omega$ with the property that, for some sequence of points $\{z_n\}$ on Λ converging to z_0, we have

$$\lim_{n\to\infty} g(z_n) = \omega.$$

A value ω is called a *cluster value* of g at z_0 along Λ. It follows readily that $C_\Lambda(g,z_0)$ is a nonempty closed subset of Ω.

We consider some applications of extremal length to conformal mappings. A purely function-theoretic proof of the following theorem is difficult. The use of extremal length, however, makes the proof trivial.

THEOREM 2.3. *Let D be a Jordan domain in $\mathbb{C}$, ∂D its boundary. Let z_0, z_1 be two distinct points of ∂D, and denote by C_1 and C_2 the two curves between z_0 and z_1, where $C_1, C_2 \subset \partial D$. Let $f(z)$ be a 1-1 conformal mapping defined on D satisfying*

$$\iint_D |f'(z)|^2\,dx\,dy < \infty.$$

If $f(z)$ has the cluster values ω_1 and ω_2 ($\omega_1,\omega_2 \neq \infty$) for some sequence of points $\{z_n\}$ on C_1 and C_2 converging to z_0 respectively, then

$$\omega_1 = \omega_2.$$

In our discussion we will need the followings.

DEFINITION 2.4. ([1]) A non-negative Borel measurable function $\rho(z)$ will be called *allowable* if it satisfies the condition (2).

DEFINITION 2.5. ([1]) Let D be a simply connected domain in $\mathbb{C}$. A *crosscut* of D is a Jordan curve γ in D which in both directions tends to a boundary point.

LEMMA 2.6. ([13]) *Let R be a ring domain in $\mathbb{C}$ and let R_0 and R_1 denote the bounded component and unbounded component of R^c the complement of R, respectively. Let ∂R_0 and ∂R_1 denote the two components of the boundary of R, and let Γ_R be the family of all curves in R connecting ∂R_0 and ∂R_1. Then*

$$\lambda(\Gamma_R) = \infty$$

if and only if R_0 consists of a single point.

LEMMA 2.7. ([4, ch.4]) *Let R, R_0, R_1 and Γ_R be as in Lemma 2.6. We say that the closed curve γ in R separates R_0 and R_1 if γ has non-zero winding number about the points of R_0. Let Γ_S be the family of all closed curves in R which separate R_0 and R_1. Then*

$$\lambda(\Gamma_R) \cdot \lambda(\Gamma_S) = 1.$$

We say that $\lambda(\Gamma_S)$ is the *conjugate extremal length* of $\lambda(\Gamma_R)$.

PROOF OF THEOREM 2.3. Let Γ be the family of all crosscuts γ in R_D connecting points of C_1 and points of C_2, where

$$R_D = \{z \,|\, 0 < |z - z_0| < r_0\} \cap D.$$

Here r_0 is a sufficiently small positive real number. Then

$$|\omega_1 - \omega_2| \le \inf_{\gamma \in \Gamma} \int_\gamma |f'(z)||dz|.$$

There remains to show that

$$\inf_{\gamma \in \Gamma} \int_\gamma |f'(z)||dz| = 0. \tag{5}$$

Since

$$R = \{z \mid 0 < |z - z_0| < r_0\}$$

is a ring domain, by Lemma 2.6, we see that

$$\lambda(\Gamma_R) = \infty,$$

where Γ_R is as in Lemma 2.6.

Suppose now that we are considering Γ_S, the family of all simple closed curves in R separating z_0 from $\{z \mid |z - z_0| = r_0\}$. Then $\lambda(\Gamma_S)$ is the conjugate extremal length of $\lambda(\Gamma_R)$. Hence by Lemma 2.6,

$$\lambda(\Gamma_S) = 0.$$

And clearly,

$$\Gamma < \Gamma_S.$$

Thus by the comparison principle of extremal length (Proposition 1.5), we see that

$$\lambda(\Gamma) = 0. \tag{6}$$

On the other hand, if we choose the allowable function $\rho(z) = |f'(z)|$ on D, then

$$\iint_D [\rho(z)]^2 \, dx \, dy < \infty. \tag{7}$$

Hence by (6), (7), we have (5).

This completes the proof of the theorem. □

3. Geometric applications of extremal length

There are a number of purely geometric applications of extremal length. The simplest example concerns the ring domain.

THEOREM 3.1. *Let $R, \partial R_0$ and ∂R_1 be as in Lemma 2.6. Let a be the length of the shortest arc in D connecting ∂R_0 and ∂R_1. Let b be the length of the Jordan curve, ∂R_0. Then*

$$a \cdot b < S.$$

where S is the area of R.

PROOF. The purely geometric proof of this theorem is difficult. The use of extremal length, however, makes the proof trivial.

Let Γ_R and Γ_S be as in Lemmas 2.6 and 2.7 respectively. Then by Lemma 2.7,

$$\lambda(\Gamma_R)\cdot\lambda(\Gamma_S)=1.$$

On the other hand, if we choose the non-negative Borel measurable function $\rho = 1$, then $\lambda(\Gamma_R)$ and $\lambda(\Gamma_S)$ has the following lower bounds respectively. That is,

$$\begin{aligned}(a^2/S)\cdot(b^2/S) &= [\{L(\Gamma_R,1)\}^2\,/\,A(D,1)]\cdot[\{L(\Gamma_S,1)\}^2\,/\,A(D,1)]\\ &\le \lambda(\Gamma_R)\cdot\lambda(\Gamma_S)\\ &=1,\end{aligned}$$

and the theorem follows at once. □

THEOREM 3.2. *Suppose that we have a set of n disjoint general quadrilaterals Q_k, for $k = 1,2,\ldots,n$, contained in the annulus $\Delta = \{z\,|\,r<|z|<R\}$, $(0<r<R,\ R\neq\infty)$ and bounded by Jordan curves, each of which has an arc, in common with each of the circles $\{z\,|\,|z|=r\}$ and $\{z\,|\,|z|=R\}$ (the Q_k can be regarded as strips extending from the inner to the outer circle). If these domains Q_k are mapped onto rectangles B_k with sides equal respectively to a_k and b_k in such a way that the arcs referred to are mapped into sides of lengths a_k, then*

$$\sum_{k=1}^{n} a_k/b_k \le 2\pi/\log(R/r), \tag{8}$$

where the equality holds only if the Q_k are domains of the form $\{z\,|\,r<|z|<R,\ \phi_k<\arg z<\phi_{k+1}\}$ completely filling the annulus.

PROOF. The method of extremal length considered leads to a simple proof of the inequality (8).

We can map an arbitrary general quadrilateral conformally onto a rectangle ([9, p.15]). Let $w = f_k(z)$ be 1-1 conformal mappings on Q_k upon B_k respectively. Let Γ be the family of arcs in Δ which join the two boundary circles, and let Γ_k be the family of arcs in Q_k which join the two sides of $Q_k \subset \partial\Delta$. Then by the conformal invariance of extremal length (Proposition 1.5) and Proposition 1.3(a),

$$\lambda(\Gamma_k)=\lambda[f_k(\Gamma_k)]=b_k/a_k. \tag{9}$$

By the hypothesis, there exist disjoint open sets $Q_k (k = 1, 2, \dots, n)$ containing Γ_k and $\cup_k \Gamma_k \subset \Gamma$. Hence by Proposition 1.7,

$$\sum_{k=1}^{n} 1/\lambda(\Gamma_k) \leq 1/\lambda(\Gamma). \tag{10}$$

Therefore by Proposition 1.3(b), (9) and (10), we obtain (8).
The proof is complete. □

Now, we will prove alternatively the well-known result by making use of extremal length. In particular, this method shortens the length of the proof significantly as we shall see by comparing the following proof with that of Theorem 14.22 in [11].

THEOREM 3.3. ([11]) *Let* $\Delta(r, R) = \{z \,|\, r < |z| < R\}$, $(0 < r < R, R \neq \infty)$. *Then* $\Delta_1(r_1, R_1)$ *and* $\Delta_2(r_2, R_2)$ *are conformally equivalent if and only if*

$$R_1/r_1 = R_2/r_2. \tag{11}$$

PROOF. (Method of extremal length) Since the proof of sufficient condition is trivial, we discuss the proof of necessary condition. Let Γ_Δ be the family of arcs in $\Delta(r, R)$ which join the two contours. Then by Proposition 1.3(b),

$$\lambda(\Gamma_\Delta) = (1/2\pi)\log(R/r). \tag{12}$$

Suppose that $\Delta_1(r_1, R_1)$ and $\Delta_2(r_2, R_2)$ are conformally equivalent and let f be a 1-1 conformal mapping on $\Delta_1(r_1, R_1)$ upon $\Delta_2(r_2, R_2)$. Then by the conformal invariance of extremal length,

$$\lambda(\Gamma_{\Delta_1}) = \lambda[f(\Gamma_{\Delta_1})] = \lambda(\Gamma_{\Delta_2}). \tag{13}$$

Hence by (12), (13), we obtain (11).
The proof is now complete. □

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