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A CHANGE OF SCALE FORMULA FOR WIENER INTEGRALS OF UNBOUNDED FUNCTIONS II

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ABSTRACT. Cameron and Storvick discovered change of scale formulas for Wiener integrals of bounded functions in a Banach algebra S of analytic Feynman integrable functions on classical Wiener space. Yoo and Skoug extended these results to abstract Wiener space for a generalized Fresnel class $\mathcal{F}_{A_1,A_2}$ containing the Fresnel class $\mathcal{F}(B)$ which corresponds to the Banach algebra S on classical Wiener space. In this paper, we present a change of scale formula for Wiener integrals of various functions on B^2 which need not be bounded or continuous.

1. Introduction

It has long been known that Wiener measure and Wiener measurability behave badly under the change of scale transformation [3] and under translations [2]. Cameron and Storvick [6] expressed the analytic Feynman integral for a rather large class of functionals as a limit of Wiener integrals. In doing so, they discovered nice change of scale formulas for Wiener integrals of bounded functions in a Banach algebra S of analytic Feynman integrable fucntions on classical Wiener space $(C_0[0,1], m_w)$ [5]. In [23, 24, 25], Yoo, Yoon and Skoug extended these results to classical Yeh-Wiener space and to an abstract Wiener space (H, B, ν). In particular, Yoo and Skoug [22] established a change of scale formula for Wiener integrals of functions in the Fresnel class $\mathcal{F}(B)$ on abstract Wiener space, which corresponds to Cameron and Storvick's Banach algebra S and then they [25] developed this formula for a more generalized Fresnel class $\mathcal{F}_{A_1,A_2}$ than the Fresnel class $\mathcal{F}(B)$. Also, Yoo,

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Song, Kim and Chang [24] investigated a change of scale formula for Wiener integrals of unbounded functions on abstract Wiener space.

Let X_{n_1,n_2} be an $\mathbb{R}^{n_1+n_2}$ valued random variable on B^2 such that

$$X_{n_1,n_2}(x_1,x_2) = (X_{1,n_1}(x_1), X_{2,n_2}(x_2)) \tag{1.1}$$

where $X_{j,n_j}(x_j) = ((e_{j,1},x_j)^\sim, \dots, (e_{j,n_j},x_j)^\sim)$ and $\{e_{j,1},\dots,e_{j,n_j}\}$ is an orthonormal set in H for $j=1,2$, and $(\cdot,\cdot)^\sim$ is a stochastic inner product which will be defined in Section 2.

In this paper, we establish a change of scale formula for Wiener integrals of functions of the form

$$F(x_1,x_2) = G(x_1,x_2)\Psi(X_{n_1,n_2}(x_1,x_2)) \tag{1.2}$$

for $G \in \mathcal{F}_{A_1,A_2}$ and $\Psi = \psi + \phi$ where $\psi \in L_p(\mathbb{R}^{n_1+n_2})$, $1 \le p < \infty$, and ϕ is a Fourier transform of a measure of bounded variation over $\mathbb{R}^{n_1+n_2}$. Note that the functions of the form (1.2) need not be bounded or continuous.

2. Definitions and preliminaries

Let H be a real separable infinite dimensional Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. Let $|||\cdot|||$ be a measurable norm on H with respect to the Gaussian cylinder set measure σ on H. Let B denote the completion of H with respect to $|||\cdot|||$. Let ι denote the natural injection from H to B. The adjoint operator ι^* of ι is one-to-one and maps B^* continuously onto a dense subset of H^* where B^* and H^* are the topological dual of B and H respectively. By identifying H with H^* and B^* with ι^*B^*, we have a triple $B^* \subset H^* \equiv H \subset B$ and $\langle h,x\rangle = (h,x)$ for all h in H and x in B^* where $(\cdot,\cdot)$ denotes the natural dual pairing between B and B^*. By a well-known result of Gross [15], $\sigma\circ\iota^{-1}$ has a unique countably additive extension ν to the Borel σ-algebra $\mathcal{B}(B)$ of B. The triple (H,B,ν) is called an abstract Wiener space. For more details, see [15, 18, 19, 20].

Let $\mathbb{C}$ denote the set of complex numbers, and let

$$\Omega = \{\vec{z} = (z_1,z_2) \in \mathbb{C}^2 : \text{ Re } z_k > 0 \text{ for } k=1,2\}$$

and

$$\tilde{\Omega} = \{\vec{z} = (z_1,z_2) \in \mathbb{C}^2 : z_k \neq 0, \text{ Re } z_k \ge 0 \text{ for } k=1,2\}.$$

DEFINITION 2.1. Let F be a functional on B^2 such that the integral

$$J_F(z_1, z_2) = \int_{B^2} F(z_1^{-1/2} x_1, z_2^{-1/2} x_2)\, d(\nu \times \nu)(x_1, x_2) \tag{2.1}$$

exists for all real numbers $z_1 > 0$ and $z_2 > 0$. If there exists an analytic function $J_F^*(z_1, z_2)$ on Ω such that $J_F^*(z_1, z_2) = J_F(z_1, z_2)$ for all $z_1, z_2 > 0$, then we call $J_F^*(z_1, z_2)$ the analytic Wiener integral of F over B^2 with parameter $\vec{z} = (z_1, z_2)$, and for $\vec{z} = (z_1, z_2) \in \Omega$, we write

$$I_a^{\vec{z}}[F] = J_F^*(z_1, z_2). \tag{2.2}$$

Let q_1 and q_2 be non-zero real numbers. If the following limit (2.3) exists, we define it to be the analytic Feynman integral of F over B^2 with parameter $\vec{q} = (q_1, q_2)$ and we write

$$I_a^{\vec{q}}[F] = \lim_{\vec{z} \to (-iq_1, -iq_2)} I_a^{\vec{z}}[F], \tag{2.3}$$

where $\vec{z} = (z_1, z_2)$ approaches $(-iq_1, -iq_2)$ through values in Ω.

Let $\{e_n\}$ denote a complete orthonormal system in H such that the e_n's are in B^*. For each $h \in H$ and $x \in B$, we introduce a stochastic inner product $(\cdot, \cdot)^\sim$ on $H \times B$ defined by

$$(h, x)^\sim = \begin{cases} \lim_{n\to\infty} \sum_{j=1}^{n} \langle h, e_j \rangle (x, e_j), & \text{if the limit exists} \\ 0, & \text{otherwise.} \end{cases} \tag{2.4}$$

Then, for every $h \in H$, $(h, x)^\sim$ is a Borel measurable function on B having a Gaussian distribution with mean 0 and variance $\|h\|^2$. Also if both h and x are in H, then $(h, x)^\sim = \langle h, x \rangle$.

A subset E of a product abstract Wiener space B^2 is said to be scale-invariant measurable provided $\{(\alpha x_1, \beta x_2) : (x_1, x_2) \in E\}$ is abstract Wiener measurable for every $\alpha > 0$ and $\beta > 0$, and a scale-invariant measurable set N is said to be scale-invariant null provided $(\nu \times \nu)(\{(\alpha x_1, \beta x_2) : (x_1, x_2) \in N\}) = 0$ for every $\alpha > 0$ and $\beta > 0$. A property that holds except on a scale-invariant null set is said to hold scale-invariant almost everywhere (s-almost everywhere). Given two complex-valued function F and G on B^2, we say that $F = G$ s-almost everywhere, and write $F \approx G$, if $F(\alpha x_1, \beta x_2)$ for $\nu \times \nu$ almost every $(x_1, x_2) \in B^2$ for all $\alpha > 0$ and $\beta > 0$. For a functionals F on B^2, we will denote by $[F]$ the equivalence class of functionals which are equal to F s-almost everywhere.

DEFINITION 2.2. Let A_1 and A_2 be bounded, nonnegative self-adjoint operators on H. Let $\mathcal{F}_{A_1,A_2}$ be the space of all functions G on B^2 which have the form

$$G(x_1,x_2)=\int_H \exp\Big\{i\sum_{j=1}^{2}(A_j^{1/2}h,x_j)^\sim\Big\}\,d\mu(h) \tag{2.5}$$

for $\mu\in M(H)$.

As is customary, we will identify a functional with its s-equivalence class and think of $\mathcal{F}_{A_1,A_2}$ as a collection of functionals on B^2 rather than as a collection of equivalence classes.

Let $M(H)$ denote the space of complex Borel measures μ on H. Then $M(H)$ is a Banach algebra under convolution as multiplication with the norm $\|\mu\|$ where $\|\mu\|$ is the total variation of μ. In addition the map $\sigma\to[F]$ defined by (2.5) sets up an algebra isomorphism between $M(H)$ and $\mathcal{F}_{A_1,A_2}$ if the range of A_1+A_2 is dense in H. In this case $\mathcal{F}_{A_1,A_2}$ becomes a Banach algebra under the norm $\|F\|=\|\sigma\|$.

REMARK 2.3. Let $\mathcal{F}(B)$ denote the class of all functions F on B of the form

$$F(x)=\int_H \exp\{i(h,x)^\sim\}\,d\mu(h)$$

for some $\mu\in M(H)$. Then we know that if A_1 is the identity operator on H and $A_2=0$, then $\mathcal{F}_{A_1,A_2}$ is essentially the Fresnel class $\mathcal{F}(B)$.

THEOREM 2.4 ([13]). *Let $G\in\mathcal{F}_{A_1,A_2}$ be given by (2.5). Then the analytic Feynman integral of G over B^2 exists and*

$$I_a^{\vec q}[G]=\int_H \exp\Big\{-\sum_{j=1}^{2}\frac{i}{2q_j}\|A_j^{1/2}h\|\Big\}\,d\mu(h) \tag{2.6}$$

for all nonzero real numbers q_1,q_2.

To simplify the expressions, we use the following notations: For $j=1,2$, we write

$$\begin{aligned}\vec e_j &= (e_{j,1},\dots,e_{j,n_j})\in H^{n_j},\\ (\vec e_j,x_j)^\sim &= ((e_{j,1},x_j)^\sim,\dots,(e_{j,n_j},x_j)^\sim),\\ \langle\vec e_j,h\rangle &= (\langle e_{j,1},h\rangle,\dots,\langle e_{j,n_j},h\rangle),\quad h\in H\end{aligned}$$

and

$$\begin{aligned} \vec{v}_{j;n} &= (v_{j,1}, \ldots, v_{j,n}) \in \mathbb{R}^n, \\ d\vec{v}_{j;n} &= dv_{j,1} \cdots dv_{j,n}, \quad |\vec{v}_{j;n}|^2 = \sum_{k=1}^{n} v_{j,k}^2. \end{aligned}$$

In particular if $n = n_j$, we denote $\vec{v}_{j;n}$ as $\vec{v}_j$. That is, $\vec{v}_j = \vec{v}_{j;n_j}$, $d\vec{v}_j = d\vec{v}_{j;n_j}$ and $|\vec{v}_j|^2 = |\vec{v}_{j;n_j}|^2$ for $j = 1, 2$. Also for $\vec{u}_j, \vec{v}_j \in \mathbb{R}^{n_j}$,

$$\vec{u}_j \cdot \vec{v}_j = \sum_{k=1}^{n_j} u_{j,k} v_{j,k}.$$

Hence (1.1) can be expressed alternatively as

$$X_{n_1,n_2}(x_1, x_2) = ((\vec{e}_1, x_1)^\sim, (\vec{e}_2, x_2)^\sim). \tag{2.7}$$

The followings are some examples of the simplified expressions used in this paper,

$$\begin{aligned} |iz_j\vec{v}_j + \langle \vec{e}_j, A_j^{1/2}h\rangle|^2 &= \sum_{k=1}^{n_j} (iz_j v_{j,k} + \langle e_{j,k}, A_j^{1/2}h\rangle)^2, \\ \langle \vec{e}_j, A_j^{1/2}h\rangle \cdot (\vec{e}_j, x_j)^\sim &= \sum_{k=1}^{n_j} \langle e_{j,k}, A_j^{1/2}h\rangle (e_{j,k}, x_j) \end{aligned}$$

for $j = 1, 2$.

3. Change of scale formulas

We begin this section by giving some existence theorems of the analytic Wiener integral and the analytic Feynman integral of functions on abstract Wiener space which need not be bounded or continuous.

THEOREM 3.1. *Let $F(x_1, x_2) = G(x_1, x_2)\psi(X_{n_1,n_2}(x_1, x_2))$ where $G \in \mathcal{F}_{A_1,A_2}$ is given by (2.5), $\psi \in L_p(\mathbb{R}^{n_1+n_2})$ for $1 \le p < \infty$ and X_{n_1,n_2} is given by (1.1) or (2.7). Then for each $(z_1, z_2) \in \Omega$, F is analytic Wiener integrable and*

(3.1)

$$\begin{aligned} I_a^{\vec{z}}(F) = \prod_{j=1}^{2} \Big(\frac{z_j}{2\pi}\Big)^{n_j/2} \int_H \int_{\mathbb{R}^{n_1+n_2}} \exp\Big\{\sum_{j=1}^{2} \frac{1}{2z_j} (|iz_j\vec{v}_j + \langle \vec{e}_j, A_j^{1/2}h\rangle|^2 \\ - \|A_j^{1/2}h\|^2)\Big\} \psi(\vec{v}_1, \vec{v}_2)\, d\vec{v}_1\, d\vec{v}_2\, d\mu(h). \end{aligned}$$

PROOF. Let z_j, $j = 1, 2$, be positive real numbers. We begin by evaluating the Wiener integral

$$\begin{aligned} I(\vec{z}, F) &\equiv \int_{B^2} F(z_1^{-1/2}x_1, z_2^{-1/2}x_2)\, d(\nu \times \nu)(x_1, x_2) \\ &= \int_{B^2}\int_H \exp\Big\{\sum_{j=1}^{2} iz_j^{-1/2}(A_j^{1/2}h, x_j)^\sim\Big\} \\ &\quad \psi(z_1^{-1/2}(\vec{e}_1, x_1)^\sim, z_2^{-1/2}(\vec{e}_2, x_2)^\sim)\, d\mu(h)\, d(\nu \times \nu)(x_1, x_2). \end{aligned}$$

Using the Fubini theorem, we change the order of integration in the above equation. In fact, we have

$$\begin{aligned} &\int_H \int_{B^2} |\psi(z_1^{-1/2}(\vec{e}_1, x_1)^\sim, z_2^{-1/2}(\vec{e}_2, x_2)^\sim)|\, d(\nu \times \nu)(x_1, x_2)\, d\mu(h) \\ &= \prod_{j=1}^{2}\Big(\frac{z_j}{2\pi}\Big)^{n_j/2} \int_H \int_{\mathbb{R}^{n_1+n_2}} |\psi(\vec{v}_1, \vec{v}_2)| \exp\Big\{-\sum_{j=1}^{2}\frac{z_j}{2}|\vec{v}_j|^2\Big\}\, d\vec{v}_1\, d\vec{v}_2\, d\mu(h) \end{aligned}$$

which is finite since $\psi \in L_p(\mathbb{R}^{n_1+n_2})$ and $\mu \in M(H)$. For a given $h \in H$, using the Gram-Schmidt process, we obtain e_{j,n_j+1} in H such that $\{e_{j,1}, \ldots, e_{j,n_j}, e_{j,n_j+1}\}$ forms an orthonormal set in H and

$$A_j^{1/2}h = \sum_{k=1}^{n_j}\langle e_{j,k}, A_j^{1/2}h\rangle e_{j,k} + c_j e_{j,n_j+1},$$

where

$$c_j = (\|A_j^{1/2}h\|^2 - |\langle \vec{e}_j, A_j^{1/2}h\rangle|^2)^{1/2}$$

for each $j = 1, 2$. Hence we have

$$\begin{aligned} I(\vec{z}, F) = \int_H \int_{B^2} &\exp\Big\{\sum_{j=1}^{2} iz_j^{-1/2}(\langle \vec{e}_j, A_j^{1/2}h\rangle \cdot (\vec{e}_j, x_j)^\sim \\ &+ c_j(e_{j,n_j+1}, x_j)^\sim)\Big\}\psi(z_1^{-1/2}(\vec{e}_1, x_1)^\sim, z_2^{-1/2}(\vec{e}_2, x_2)^\sim) \\ &d(\nu \times \nu)(x_1, x_2)\, d\mu(h). \end{aligned}$$

By the Wiener integration formula, we have

$$\begin{aligned} I(\vec{z}, F) = \prod_{j=1}^{2}\Big(\frac{z_j}{2\pi}\Big)^{(n_j+1)/2} &\int_H \int_{\mathbb{R}^{n_1+n_2+2}} \exp\Big\{\sum_{j=1}^{2}\Big[i(\langle \vec{e}_j, A_j^{1/2}h\rangle \cdot \vec{v}_j \\ &+ c_j v_{j,n_j+1}) - \frac{z_j}{2}|\vec{v}_{j;n_j+1}|^2\Big]\Big\}\psi(\vec{v}_1, \vec{v}_2)\, d\vec{v}_{1;n_1+1}\, d\vec{v}_{2;n_2+1}\, d\mu(h). \end{aligned}$$

Evaluating the above integral with respect to v_{j,n_j+1} for $j = 1, 2$ and using the expression for c_j we obtain

$$I(\vec{z}, F) = \prod_{j=1}^{2} \Big(\frac{z_j}{2\pi}\Big)^{n_j/2} \int_H \int_{\mathbb{R}^{n_1+n_2}} \exp\Big\{\sum_{j=1}^{2} \frac{1}{2z_j}(|iz_j\vec{v}_j + \langle \vec{e}_j, A_j^{1/2}h\rangle|^2 - \|A_j^{1/2}h\|^2)\Big\}\psi(\vec{v}_1, \vec{v}_2)\, d\vec{v}_1\, d\vec{v}_2\, d\mu(h).$$

Now we will show that the right hand side of the above expression is an analytic function of $(z_1, z_2) \in \Omega$. Let $(z_{1,l}, z_{2,l}) \to (z_1, z_2)$ in Ω. Then there exists $\alpha_j > 0$ such that $\operatorname{Re} z_{j,l} \geq \alpha_j$, $j = 1, 2$, for all sufficiently large l and by the Bessel inequality, we have

$$\Big|\exp\Big\{\sum_{j=1}^{2} \frac{1}{2z_j}(|iz_j\vec{v}_j + \langle \vec{e}_j, A_j^{1/2}h\rangle|^2 - \|A_j^{1/2}h\|^2)\Big\}\psi(\vec{v}_1, \vec{v}_2)\Big| \leq \exp\Big\{-\sum_{j=1}^{2} \frac{\alpha_j}{2}|\vec{v}_j|^2\Big\}|\psi(\vec{v}_1, \vec{v}_2)|$$

which is integrable on $H \times \mathbb{R}^{n_1+n_2}$ since $\psi \in L_p(\mathbb{R}^{n_1+n_2})$ and $\mu \in M(H)$. Hence we can apply the dominated convergence theorem to the last expression for $I(\vec{z}, F)$ to conclude that it is a continuous function of $(z_1, z_2) \in \Omega$. Moreover by using the Morera theorem, we easily show that it is an analytic function of (z_1, z_2) throughout Ω and this completes the proof. □

If we restrict our attention to the case $p = 1$, by applying the dominated convergence theorem to the expression (3.1), we obtain the following existence theorem of the analytic Feynman integral. But if $p > 1$, we are not able to justify the application of the dominated convergence theorem and so we could not claim the existence of the analytic Feynman integral.

COROLLARY 3.2. *Let $F(x_1, x_2)$ be given as in Theorem 3.1. Then for each nonzero real numbers q_1, q_2, F is analytic Feynman integrable and*

(3.2)

$$I_a^{\vec{q}}(F) = \prod_{j=1}^{2} \Big(-\frac{iq_j}{2\pi}\Big)^{n_j/2} \int_H \int_{\mathbb{R}^{n_1+n_2}} \exp\Big\{\sum_{j=1}^{2} \frac{i}{2q_j}(|iq_j\vec{v}_j + \langle \vec{e}_j, A_j^{1/2}h\rangle|^2 - \|A_j^{1/2}h\|^2)\Big\}\psi(\vec{v}_1, \vec{v}_2)\, d\vec{v}_1\, d\vec{v}_2\, d\mu(h).$$

Let $\hat{M}(\mathbb{R}^{n_1+n_2})$ be the set of functions ϕ defined on $\mathbb{R}^{n_1+n_2}$ by

$$\phi(\vec{r}_1,\vec{r}_2)=\int_{\mathbb{R}^{n_1+n_2}}\exp\Big\{i\sum_{j=1}^{2}\vec{r}_j\cdot\vec{v}_j\Big\}\,d\rho(\vec{v}_1,\vec{v}_2) \tag{3.3}$$

where ρ is a complex Borel measure of bounded variation on $\mathbb{R}^{n_1+n_2}$.

THEOREM 3.3. *Let $F(x_1,x_2)=G(x_1,x_2)\phi(X_{n_1,n_2}(x_1,x_2))$ where $G\in\mathcal{F}_{A_1,A_2}$, $\phi\in\hat{M}(\mathbb{R}^{n_1+n_2})$ and X_{n_1,n_2} are given by (2.5), (3.3) and (1.1), respectively. Then for each $\vec{z}\in\Omega$, F is analytic Wiener integrable and*

$$\begin{aligned}I_a^{\vec{z}}[F]=\int_H\int_{\mathbb{R}^{n_1+n_2}}&\exp\Big\{-\sum_{j=1}^{2}\frac{1}{2z_j}(\|A_j^{1/2}h\|^2\\&+2\vec{v}_j\cdot\langle\vec{e}_j,A_j^{1/2}h\rangle+|\vec{v}_j|^2)\Big\}\,d\rho(\vec{v}_1,\vec{v}_2)\,d\mu(h).\end{aligned} \tag{3.4}$$

Moreover for each nonzero real numbers q_1,q_2, F is analytic Feynman integrable and

$$\begin{aligned}I_a^{\vec{q}}[F]=\int_H\int_{\mathbb{R}^{n_1+n_2}}&\exp\Big\{-\sum_{j=1}^{2}\frac{i}{2q_j}(\|A_j^{1/2}h\|^2\\&+2\vec{v}_j\cdot\langle\vec{e}_j,A_j^{1/2}h\rangle+|\vec{v}_j|^2)\Big\}\,d\rho(\vec{v}_1,\vec{v}_2)\,d\mu(h).\end{aligned} \tag{3.5}$$

PROOF. Using the expressions (2.5), (3.3) and Fubini theorem, we have for positive real numbers z_1,z_2

$$\begin{aligned}I(\vec{z},F)&\equiv\int_{B^2}F(z_1^{-1/2}x_1,z_2^{-1/2}x_2)\,d(\nu\times\nu)(x_1,x_2)\\&=\int_H\int_{\mathbb{R}^{n_1+n_2}}\int_{B^2}\exp\Big\{\sum_{j=1}^{2}iz_j^{-1/2}[(A_j^{1/2}h,x_j)^\sim+\vec{v}_j\cdot(\vec{e}_j,x_j)^\sim]\Big\}\\&\quad d(\nu\times\nu)(x_1,x_2)\,d\rho(\vec{v}_1,\vec{v}_2)\,d\mu(h).\end{aligned}$$

By the same method as in the proof of Theorem 3.1, we obtain

$$\begin{aligned}I(\vec{z},F)=\int_H\int_{\mathbb{R}^{n_1+n_2}}\int_{B^2}&\exp\Big\{\sum_{j=1}^{2}iz_j^{-1/2}[\langle\vec{e}_j,A_j^{1/2}h\rangle\cdot(\vec{e}_j,x_j)^\sim\\&+c_j(e_{j,n_j+1},x_j)^\sim+\vec{v}_j\cdot(\vec{e}_j,x_j)^\sim]\Big\}\\&d(\nu\times\nu)(x_1,x_2)\,d\rho(\vec{v}_1,\vec{v}_2)\,d\mu(h),\end{aligned}$$

where c_j is given as in the proof of Theorem 3.1. Evaluating the Wiener integral in the last expression, we have

$$I(\vec{z},F)=\int_H\int_{\mathbb{R}^{n_1+n_2}}\exp\Big\{-\sum_{j=1}^{2}\frac{1}{2z_j}[c_j^2+|\langle\vec{e}_j,A_j^{1/2}h\rangle+\vec{v}_j|^2]\Big\}$$
$$d\rho(\vec{v}_1,\vec{v}_2)\,d\mu(h).$$

Finally using the expression for c_j, we obtain

$$I(\vec{z},F)=\int_H\int_{\mathbb{R}^{n_1+n_2}}\exp\Big\{-\sum_{j=1}^{2}\frac{1}{2z_j}[\|A_j^{1/2}h\|^2$$
$$+2\vec{v}_j\cdot\langle\vec{e}_j,A_j^{1/2}h\rangle+|\vec{v}_j|^2]\Big\}\,d\rho(\vec{v}_1,\vec{v}_2)\,d\mu(h).$$

The exponential in the last expression is bounded in absolute value by 1 for $\vec{z}\in\tilde{\Omega}$. Since ρ is a complex Borel measure of bounded variation on $\mathbb{R}^{n_1+n_2}$, it follows that the last expression above is analytic in $\vec{z}$ for $\vec{z}\in\Omega$ and is continuous in $\vec{z}$ for $\vec{z}\in\tilde{\Omega}$. This completes the proof. □

Using the linearity of the analytic Wiener integral and the analytic Feynman integral on abstract Wiener space, we have the following corollary.

COROLLARY 3.4. *Let $F(x_1,x_2)=G(x_1,x_2)\Psi(X_{n_1,n_2}(x_1,x_2))$ where $G\in\mathcal{F}_{A_1,A_2}$ is given by (2.5), $\Psi=\psi+\phi\in L_p(\mathbb{R}^{n_1+n_2})+\hat{M}(\mathbb{R}^{n_1+n_2})$ for $1\le p<\infty$ and X_{n_1,n_2} is given by (1.1). Then for each $\vec{z}\in\Omega$, F is analytic Wiener integrable and $I_a^{\vec{z}}[F]$ is given by the sum of the right hand sides of (3.1) and (3.4). Moreover if we restrict our attention to the case $p=1$, for each nonzero real numbers q_1,q_2, F is analytic Feynman integrable and $I_a^{\vec{q}}[F]$ is given by the sum of the right hand sides of (3.2) and (3.5).*

Now we give a relationship between Wiener integral and analytic Wiener integral on abstract Wiener space.

THEOREM 3.5. *Let $\{e_{j,n}:n=1,2,\ldots\}$, $j=1,2$, be complete orthonormal sets in H. Let $F(x_1,x_2)=G(x_1,x_2)\Psi(X_{n_1,n_2}(x_1,x_2))$ where $G\in\mathcal{F}_{A_1,A_2}$ is given by (2.5), $\psi\in L_p(\mathbb{R}^{n_1+n_2})$ for $1\le p<\infty$ and X_{n_1,n_2} is given by (1.1). Then for each $\vec{z}\in\Omega$, we have*

$$I_a^{\vec{z}}[F]=\lim_{m_1,m_2\to\infty}z_1^{m_1/2}z_2^{m_2/2}\int_{B^2}\exp\Big\{\sum_{j=1}^{2}\frac{1-z_j}{2}|(\vec{e}_{j;m_j},x_j)^{\sim}|^2\Big\}\tag{3.6}$$
$$F(x_1,x_2)\,d(\nu\times\nu)(x_1,x_2).$$

PROOF. Let m_j be natural numbers with $m_j > n_j$ for $j = 1, 2$ and let $\Gamma(m_1, m_2)$ be the Wiener integral on the right hand side of (3.6). By (2.5) and the Fubini theorem, we have

$$\Gamma(m_1,m_2) = \int_H \int_{B^2} \exp\Big\{\sum_{j=1}^2 \Big[\frac{1-z_j}{2}|(e_{j;m_j}, x_j)^\sim|^2 + i(A_j^{1/2}h, x_j)^\sim\Big]\Big\} \\ \psi((\vec{e}_1, x_1)^\sim, (\vec{e}_2, x_2)^\sim)\, d(\nu\times\nu)(x_1,x_2)\, d\mu(h).$$

We can evaluate the above Wiener integral either by direct calculations as in the proof of Theorem 3.1 or by using Lemma 3.6 of [24] and obtain

$$\Gamma(m_1,m_2) = \prod_{j=1}^2 \Big(\frac{z_j}{2\pi}\Big)^{n_j/2}\Big(\frac{1}{z_j}\Big)^{m_j/2} \int_H \int_{\mathbb{R}^{n_1+n_2}} \\ \exp\Big\{\sum_{j=1}^2 \Big[\frac{z_j-1}{2z_j}|\langle \vec{e}_{j;m_j}, A_j^{1/2}h\rangle|^2 - \frac{1}{2}\|A_j^{1/2}h\|^2 \\ + \frac{1}{2z_j}|iz_j\vec{v}_j + \langle \vec{e}_j, A_j^{1/2}h\rangle|^2\Big]\Big\}\psi(\vec{v}_1,\vec{v}_2)\, d\vec{v}_1\, d\vec{v}_2\, d\mu(h).$$

By the Bessel inequality and the fact that $m_j > n_j$, the absolute value of the integrand in the last expression above is bounded by

$$\exp\Big\{-\sum_{j=1}^2 \frac{\operatorname{Re} z_j}{2}|\vec{v}_j|^2\Big\}|\psi(\vec{v}_1,\vec{v}_2)|$$

which is integrable on $H \times \mathbb{R}^{n_1+n_2}$, since $\psi \in L_p(\mathbb{R}^{n_1+n_2})$ and $\mu \in M(H)$. Hence by the dominated convergence theorem and the Parseval's relation, we obtain

$$\lim_{m_1,m_2\to\infty} z_1^{m_1/2} z_2^{m_2/2}\Gamma(m_1,m_2) \\ = \prod_{j=1}^2 \Big(\frac{z_j}{2\pi}\Big)^{n_j/2} \int_H \int_{\mathbb{R}^{n_1+n_2}} \exp\Big\{\sum_{j=1}^2 \frac{1}{2z_j}\Big[|iz_j\vec{v}_j + \langle \vec{e}_j, A_j^{1/2}h\rangle|^2 \\ - \|A_j^{1/2}h\|^2\Big]\Big\}\psi(\vec{v}_1,\vec{v}_2)\, d\vec{v}_1\, d\vec{v}_2\, d\mu(h).$$

By equation (3.1) in Theorem 3.1, the proof is completed. □

Moreover if $p = 1$, we obtain the following relationship between Wiener integral and analytic Feynman integral on abstract Wiener space.

THEOREM 3.6. *Let $\{e_{j,n} : n = 1, 2, \ldots\}$ be given as in Theorem 3.5. Let $F(x_1, x_2) = G(x_1, x_2)\Psi(X_{n_1,n_2}(x_1, x_2))$ where $G \in \mathcal{F}_{A_1,A_2}$ is given* *by (2.5), $\psi \in L_1(\mathbb{R}^{n_1+n_2})$ for $1 \le p < \infty$ and X_{n_1,n_2} is given by (1.1). Let $\{z_{j,n} : n = 1, 2, \ldots\}$, $j = 1, 2$, be sequences of complex numbers with positive real part and $z_{j,n} \to -iq_j$ as $n \to \infty$. Then we have*

(3.7)

$$I_a^{\vec{q}}[F] = \lim_{m_1,m_2\to\infty} z_{1,m_1}^{m_1/2} z_{2,m_2}^{m_2/2} \int_{B^2} \exp\Big\{\sum_{j=1}^{2} \frac{1 - z_{j,m_j}}{2} |(\vec{e}_{j;m_j}, x_j)^{\sim}|^2\Big\} F(x_1, x_2)\, d(\nu \times \nu)(x_1, x_2).$$

PROOF. The proof of this theorem is similar to that of Theorem 3.5. Let m_j be natural numbers with $m_j > n_j$ for $j = 1, 2$ and let $\Gamma(z_{1,m_1}, z_{2,m_2})$ be the Wiener integral on the right hand side of (3.7). By the same method as in the proof of Theorem 3.5, we have

$$\Gamma(z_{1,m_1}, z_{2,m_2}) = \prod_{j=1}^{2} \Big(\frac{z_{j,m_j}}{2\pi}\Big)^{n_j/2} \Big(\frac{1}{z_{j,m_j}}\Big)^{m_j/2} \int_H \int_{\mathbb{R}^{n_1+n_2}} \exp\Big\{\sum_{j=1}^{2} \Big[\frac{z_{j,m_j} - 1}{2z_{j,m_j}} |\langle \vec{e}_{j;m_j}, A_j^{1/2} h\rangle|^2 - \frac{1}{2}\|A_j^{1/2} h\|^2 + \frac{1}{2z_{j,m_j}} |i z_{j,m_j} \vec{v}_j + \langle \vec{e}_j, A_j^{1/2} h\rangle|^2\Big]\Big\} \psi(\vec{v}_1, \vec{v}_2)\, d\vec{v}_1\, d\vec{v}_2\, d\mu(h).$$

By the Bessel inequality we know that the absolute value of the integrand in the last expression above is bounded by $|\psi(\vec{v}_1, \vec{v}_2)|$, which is integrable on $H \times \mathbb{R}^{n_1+n_2}$, since $\psi \in L_1(\mathbb{R}^{n_1+n_2})$ and $\mu \in M(H)$. Hence by the dominated convergence theorem and the Parseval's relation, we obtain

$$\lim_{m_1,m_2\to\infty} z_{1,m_1}^{m_1/2} z_{2,m_2}^{m_2/2} \Gamma(z_{1,m_1}, z_{2,m_2}) = \prod_{j=1}^{2} \Big(\frac{z_{j,m_j}}{2\pi}\Big)^{n_j/2} \int_H \int_{\mathbb{R}^{n_1+n_2}} \exp\Big\{\sum_{j=1}^{2} \frac{i}{2q_j} \Big[|q_j \vec{v}_j + \langle \vec{e}_j, A_j^{1/2} h\rangle|^2 - \|A_j^{1/2} h\|^2\Big]\Big\} \psi(\vec{v}_1, \vec{v}_2)\, d\vec{v}_1\, d\vec{v}_2\, d\mu(h).$$

By equation (3.2) in Corollary 3.2, the proof is completed. □

THEOREM 3.7. *Let $\{e_{j,n} : n = 1, 2, \ldots\}$ be given as in Theorem 3.5. Let $F(x_1, x_2) = G(x_1, x_2)\phi(X_{n_1,n_2}(x_1, x_2))$ where $G \in \mathcal{F}_{A_1,A_2}(B)$, $\phi \in \hat{M}(\mathbb{R}^{n_1+n_2})$ and X_{n_1,n_2} be given by (2.5), (3.3) and (1.1), respectively. Then equation (3.6) holds.*

PROOF. Let m_j be natural numbers with $m_j > n_j$ for $j = 1, 2$ and let $\Gamma(m_1, m_2)$ be given as in the proof of Theorem 3.5. By (2.5), (3.3) and Fubini theorem, we have

$$\Gamma(m_1, m_2) = \int_H \int_{\mathbb{R}^{n_1+n_2}} \int_{B^2} \exp\Big\{\sum_{j=1}^{2}\Big[\frac{1-z_j}{2}|(\vec{e}_{j,m_j}, x_j)^\sim| + i(A_j^{1/2}h, x_j)^\sim + iv_j \cdot (\vec{e}_j, x_j)^\sim\Big]\Big\} d(\nu \times \nu)(x_1, x_2)\, d\rho(\vec{v}_1, \vec{v}_2)\, d\mu(h).$$

Evaluating the above Wiener integral by using Lemma 3.7 of [24], we obtain

$$\Gamma(m_1, m_2) = \prod_{j=1}^{2} z_j^{-m_j/2} \int_H \int_{\mathbb{R}^{n_1+n_2}} \exp\Big\{\sum_{j=1}^{2}\Big[\frac{1-z_j}{2z_j}|\langle \vec{e}_{j;m_j}, A_j^{1/2}\rangle|^2 - \frac{1}{z_j}\vec{v}_j \cdot \langle \vec{e}_j, A_j^{1/2}h\rangle - \frac{1}{2z_j}|\vec{v}_j|^2 - \frac{1}{2}\|A_j^{1/2}h\|^2\Big]\Big\} d\rho(\vec{v}_1, \vec{v}_2)\, d\mu(h).$$

By the Bessel inequality, we know that the absolute value of the last expression above is bounded by 1 which is integrable on $H \times \mathbb{R}^{n_1+n_2}$ with respect to the measure $\rho \times \mu$. Hence by the dominated convergence theorem and the Parseval's relation, we obtain

$$\lim_{m_1, m_2 \to \infty} z_1^{m_1/2} z_2^{m_2/2} \Gamma(m_1, m_2) = \int_H \int_{\mathbb{R}^{n_1+n_2}} \exp\Big\{-\sum_{j=1}^{2}\frac{1}{2z_j}\Big[\|A_j^{1/2}h\|^2 + 2\vec{v}_j \cdot \langle \vec{e}_j, A_j^{1/2}h\rangle + |v_j|^2\Big]\Big\} d\rho(\vec{v}_1, \vec{v}_2)\, d\mu(h).$$

By equation (3.4) in Theorem 3.3, the proof is completed. □

Modifying the proof of Theorem 3.7, by replacing "z_j" by "$z_{j,n}$", $j = 1, 2$, whenever it occurs, we have the following corollary.

THEOREM 3.8. *Let $\{e_{j,n} : n = 1, 2, \ldots\}$ and $\{z_{j,n} : n = 1, 2, \ldots\}$ be given as in Theorem 3.6 and let F be given as in Theorem 3.7. Then equation (3.7) holds.*

From Theorems 3.5, 3.7 and the linearity of the analytic Wiener integral on abstract Wiener space, we obtain the following corollary.

COROLLARY 3.9. *Let $\{e_{j,n} : n = 1, 2, \ldots\}$ be given as in Theorem 3.6 and let F be given as in Corollary 3.4. Then equation (3.6) holds.*

Similarly, from Theorems 3.6, 3.8 and the linearity of the analytic Feynman integral on abstract Wiener space, we have the following corollary.

COROLLARY 3.10. *Let* $\{e_{j,n} : n = 1, 2, \ldots\}$ *and* $\{z_{j,n} : n = 1, 2, \ldots\}$ *be given as in Theorem 3.6 and let* F *be given as in Corollary 3.4 with* $p = 1$. *Then equation* (3.7) *holds.*

Our main result, namely a change of scale formula for Wiener integrals on abstract Wiener space now follows from Corollary 3.9.

THEOREM 3.11. *Let* $\{e_{j,n} : n = 1, 2, \ldots\}$ *be given as in Theorem 3.6 and let* F *be given as in Corollary 3.4. Then for any* $\rho_j > 0$, $j = 1, 2$,

$$\begin{aligned} &\int_{B^2} F(\rho_1 x_1, \rho_2 x_2)\, d(\nu \times \nu)(x_1, x_2) \\ &= \lim_{m_1, m_2 \to \infty} \rho_1^{-m_1} \rho_2^{-m_2} \int_{B^2} \exp\Big\{ \sum_{j=1}^{2} \Big[\frac{\rho_j^2 - 1}{2\rho_j^2} |(\vec{e}_{j;m_j}, x_j)^{\sim}|^2 \Big] \Big\} \\ &\quad F(x_1, x_2)\, d(\nu \times \nu)(x_1, x_2). \end{aligned} \tag{3.8}$$

PROOF. By letting $z_j = \rho_j^{-2}$ for $j = 1, 2$ in (3.6), we have equation (3.8). □

Obviously the constant function $\phi \equiv 1$ is a member of $\hat{M}(\mathbb{R}^{n_1+n_2})$. Hence we have the following corollary which is a change of scale formula for Wiener integrals on an abstract Wiener space given in [23].

COROLLARY 3.12. *Let* $\{e_{j,n} : n = 1, 2, \ldots\}$ *be given as in Theorem 3.5 and let* $F \in \mathcal{F}_{A_1, A_2}$. *Then for any* $\rho_j > 0$, $j = 1, 2$, *equation* (3.8) *holds.*

4. Corollaries

In this section we give various corollaries which show that our results in Section 3 are indeed very general theorem.

4.1. Abstract Wiener space

As we see in Remark 2.3, if A_1 is the identity operator on H and A_2 is the zero operator, then $\mathcal{F}_{A_1, A_2}$ is essentially the Fresnel class $\mathcal{F}(B)$ and

$$I_a^{\vec{q}}[G(x_1, x_2)] = I_a^{q_1}[G_0(x_1)]$$

where $G_0(x_1) = G(x_1, x_2)$ for all $(x_1, x_2) \in B^2$ and $I_a^{q_1}[G_0(x_1)]$ means the analytic Feynman integal over B.

Hence all of the results discussed in [24] hold as our corollaries. In particular, we obtain the following change of scale formula for Wiener integrals on abstract Wiener space.

COROLLARY 4.1 (Theorem 3.14 in [24]). *Let* $F(x) = G(x)\Psi((e_1, x)^\sim, \ldots, (e_r, x)^\sim)$ *where* $G \in \mathcal{F}(B)$ *and* $\Psi = \psi + \phi \in L_p(\mathbb{R}^r) + \hat{M}(\mathbb{R}^r), 1 \le p < \infty$. *Then for any* $\rho > 0$,

(4.1)
$$\int_B F(\rho x)\, d\nu(x) = \lim_{n\to\infty} \rho^{-n} \int_B \exp\Big\{\frac{\rho^2 - 1}{2\rho^2} \sum_{k=1}^{n} [(e_k, x)^\sim]^2\Big\} F(x)\, d\nu(x).$$

4.2. Classical Wiener space

Let $H_0 = H_0[a, b]$ be the space of real-valued functions f on $[a, b]$ which are absolutely continuous and whose derivative Df is in $L_2[a, b]$. The inner product on H_0 is given by

$$\langle f, g\rangle = \int_a^b (Df)(s)(Dg)(s)\, ds.$$

Then H_0 is a real separable infinite dimensional Hilbert space. Let $B_0 = B_0[a, b]$ be the space $C_0[a, b]$ of all continuous functions x on $[a, b]$ with $x(0) = 0$ and equip B_0 with the sup norm. Let ν_0 be classical Wiener measure. Then (H_0, B_0, ν_0) is an example of an abstract Wiener space. Note that if $\{e_n\}$ is a complete orthonormal set in H_0 then $\{De_n\}$ is also a complete orthonormal set in $L_2[a, b]$ and $(e_n, x)^\sim$ equals the Paley-Wiener-Zygmund stochastic integral $\int_a^b (De_n)(s)\, \tilde{d}x(s)$ for s-a.e. $x \in B_0$.

In [4], Cameron and Storvick introduced a Banach algebra $\mathcal{S}$ of functionals on $C_0[a, b]$ which are expressible in the form

$$F(x) = \int_{L_2[a,b]} \exp\Big\{i \int_a^b v(s)\, \tilde{d}x(s)\Big\}\, d\sigma(v) \tag{4.2}$$

for s-a.e. $x \in C_0[a, b]$, where $\sigma \in M(L_2[a, b])$. Then we know that $F \in \mathcal{F}(B_0)$ if and only if $F \in \mathcal{S}$.

Hence we obtain the following change of scale formula for Wiener integrals on classical Wiener space.

COROLLARY 4.2 (Theorem 2 in [5]). *Let* $\rho > 0$ *and let* $\{\phi_k\}_{k=1}^\infty$ *be a complete orthonormal sequence of functions on* $[a, b]$. *Then if* $F \in \mathcal{S}$ *is* *given by* (4.2),

(4.3)

$$\int_{C_0[a,b]} F(\rho x) d\nu_0(x) = \lim_{n\to\infty} \rho^{-n} \int_{C_0[a,b]} \exp\Big\{\frac{\rho^2-1}{2\rho^2}\sum_{k=1}^{n}\Big[\int_a^b \phi_k(t)\, \tilde{d}x(t)\Big]^2\Big\} F(x)\, d\nu_0(x).$$

4.3. Some other examples

Ahn, Johnson and Skoug [1] established a very general theorem insuring that many functions of interest in Feynman integration theory and quantum mechanics are in $\mathcal{F}(B)$ for various abstract Wiener space (H, B, ν). Below are some examples of functionals in [1].

1. For $\psi \in \hat{M}(\mathbb{R}^d)$,

$$f(x) = \psi((x, h_1)^\sim, \ldots, (x, h_d)^\sim).$$

2. For a Borel measure η on $[0, t]$ and $\theta(\tau, \cdot) = \hat{\mu}_\tau(\cdot)$ where $\mu_\tau \in M(\mathbb{R})$,

$$f(x) = \int_0^t \theta(\tau, x(\tau))\, d\eta(\tau).$$

3. For a measure space Y and θ defined on $Y \times \mathbb{R}^d$ by $\theta(y, \cdot) = \hat{\mu}_y(\cdot)$ where $\mu_y \in M(\mathbb{R}^d)$,

$$f(x) = \int_Y \theta\Big(y, \int_0^t g(s, y) x(s)\, ds\Big)\, d\eta(y)$$

for an appropriate function g.

By Corollary 4.1 we have the following result.

COROLLARY 4.3. *All of the functions discussed in Corollaries 1-11 of* [1] *satisfy the change of scale formula* (4.1) *where* $\{e_k\}_{k=1}^\infty$ *is a complete orthonormal set of functions in the corresponding Hilbert space* H.

Another examples of functionals in $\mathcal{F}(B)$ or $\mathcal{S}$ are given in [4, 6, 9, 10, 11, 12, 17, 18, 19]. Hence we have the following corollary.

COROLLARY 4.4. *All of the functions considered in* [4, 6, 9, 10, 11, 12, 17, 18, 19] *satisfy the change of scale formula* (4.1) *or* (4.3), *where* $\{e_k\}_{k=1}^\infty$ *is a complete orthonormal set of functions in the corresponding Hilbert space* H *or* $\{\phi_k\}_{k=1}^\infty$ *is a complete orthonormal sequence of functions on* $[a, b]$.

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