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INJECTIVE REPRESENTATIONS OF QUIVERS

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ABSTRACT. We prove that $M_1 \xrightarrow{f} M_2$ is an injective representation of a quiver $Q = \bullet \to \bullet$ if and only if M_1 and M_2 are injective left R-modules, $M_1 \xrightarrow{f} M_2$ is isomorphic to a direct sum of representation of the types $E_1 \to 0$ and $E_2 \xrightarrow{id} E_2$ where E_1 and E_2 are injective left R-modules. Then, we generalize the result so that a representation $M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} M_n$ of a quiver $Q = \bullet \to \bullet \to \cdots \to \bullet$ is an injective representation if and only if each M_i is an injective left R-module and the representation is a direct sum of injective representations.

1. Introduction

A quiver is just a directed graph. We allow multiple edges and edges going from a vertex back to the same vertex. Originally a representation of quiver assigned a vector space to each vertex - and a linear map to each edge (or arrow) - with the linear map going from the vector space assigned to the initial vertex to the one assigned to the terminal vertex. For example, a representation of the quiver $Q = \bullet \to \bullet$ is $M_1 \xrightarrow{f} M_2$. Then we can define a morphism of two representations of the same quiver. Now, instead of vector spaces we can use left R-modules and also instead of linear maps we can use R-linear maps. Representations of quivers were studied in ([1], [2]) and recently in [3] noetherian quivers were studied and in [4] projective representations of quivers were studied.

A left R-module E is injective if, for every left R-module B and every submodule A of B, every R-linear map $f : A \to E$ can be extended to an R-linear map $g : B \to E$. The diagram is

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$$\begin{array}{ccccc} 0 & \longrightarrow & A & \longrightarrow & B \\ & & {\scriptstyle f}\downarrow & \swarrow {\scriptstyle g} & \\ & & E & & \end{array}$$

DEFINITION 1.1. A representation $M_1 \xrightarrow{f} M_2$ of a quiver $Q = \bullet \to \bullet$ is called an injective representation if for any representation $N_1 \xrightarrow{g} N_2$ with a subrepresentation $S_1 \xrightarrow{S_2|g|S_1} S_2$ and morphisms

$$\begin{array}{ccc} S_1 & \xrightarrow{S_2|g|S_1} & S_2 \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle k} \\ M_1 & \xrightarrow{f} & M_2 \end{array}$$

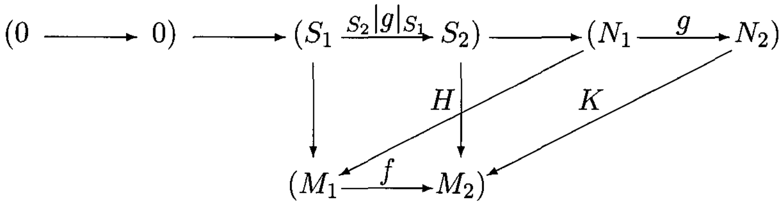
there exist $H \in Hom_R(N_1, M_1)$ and $K \in Hom_R(N_2, M_2)$ such that the following diagram

$$\begin{array}{ccc} N_1 & \xrightarrow{g} & N_2 \\ {\scriptstyle H}\downarrow & & \downarrow{\scriptstyle K} \\ M_1 & \xrightarrow{f} & M_2 \end{array}$$

commutes and $H|_{S_1} =$ h and $K|_{S_2} = k$.
In other words, every diagram of representations

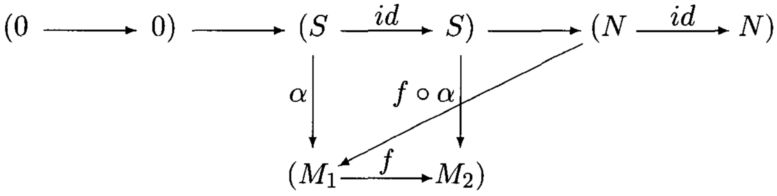
$$\begin{array}{ccccccc} (0 \longrightarrow 0) & \longrightarrow & (S_1 & \xrightarrow{S_2|g|S_1} & S_2) & \longrightarrow & (N_1 \xrightarrow{g} N_2) \\ & & {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle k} & & \\ & & (M_1 & \xrightarrow{f} & M_2) & & \end{array}$$

can be completed to a commutative diagram as follows :



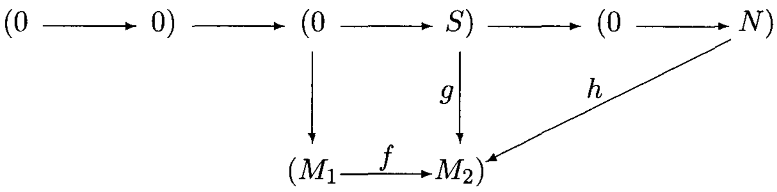
LEMMA 1.2. *If* $M_1 \xrightarrow{f} M_2$ *is an injective representation of a quiver* $Q = \bullet \to \bullet$, *then* M_1 *and* M_2 *are injective left R-modules.*

PROOF. Let N be a left R-module, S be a submodule of N and $\alpha : S \to M_1$ be an R-linear map. Then, since $M_1 \xrightarrow{f} M_2$ is an injective representation, we can complete the diagram



as a commutative diagram. Thus, M_1 is an injective left R-module.

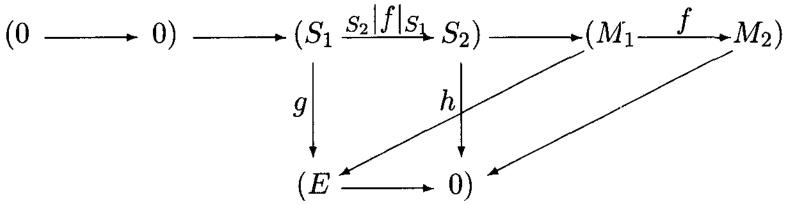
Let $g : S \to M_2$ be an R-linear map. Then, since $M_1 \xrightarrow{f} M_2$ is an injective representation, we can complete the diagram



as a commutative diagram. Thus, M_2 is an injective left R-module. □

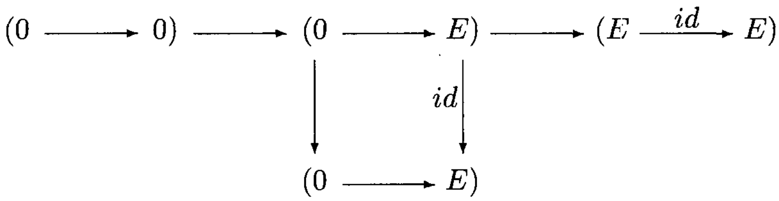
LEMMA 1.3. *If E is an injective left R-module, then a representation* $E \to 0$ *of a quiver* $Q = \bullet \to \bullet$ *is an injective representation.*

PROOF. The lemma follows by completing the diagram



as a commutative diagram. □

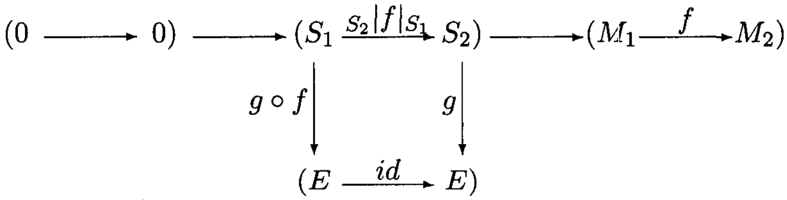
REMARK 1.4. A representation $0 \to E$ of a quiver $Q = \bullet \to \bullet$ is not an injective representation if $E \neq 0$, because we cannot complete the diagram



as a commutative diagram.

LEMMA 1.5. *If E is an injective left R-module, then a representation $E \xrightarrow{id} E$ of a quiver $Q = \bullet \to \bullet$ is an injective representation.*

PROOF. Let M_1, M_2, E, $S_1(\subset M_1)$ and $S_2(\subset M_2)$ be an left R-modules and let $f : M_1 \to M_2$ be R-linear map. Let $g : S_2 \to E$ be an R-linear map and choose $g \circ f : S_1 \to E$ as an R-linear map. And consider the following diagram :



Then, since E is an injective left R-module, there exists a map $h : M_2 \to E$. Now choose $h \circ f : M_1 \to E$ as an R-linear map. Then h and $h \circ f$ complete the above diagram as a commutative diagram. Therefore, $E \xrightarrow{id} E$ is an injective representation. □

2. Direct sum of injective representations

THEOREM 2.1. *A representation $M_1 \xrightarrow{f} M_2$ is an injective representation of a quiver $Q = \bullet \to \bullet$ if and only if M_1, M_2 are injective left R-modules and $M_1 \xrightarrow{f} M_2$ is isomorphic to a direct sum of representations of the types $E_1 \;\to\; 0$ and $E_2 \xrightarrow{id} E_2$ where E_1 and E_2 are injective left R-modules.*

PROOF. Consider the following diagram :

$$\begin{array}{ccccc} (0 \longrightarrow 0) & \longrightarrow & (0 \longrightarrow M_2) & \longrightarrow & (M_2 \xrightarrow{id} M_2) \\ & & \downarrow \qquad \downarrow id & & \\ & & (M_1 \xrightarrow{f} M_2) & & \end{array}$$

Since $M_1 \xrightarrow{f} M_2$ is an injective representation, we can complete the above diagram as a commutative diagram as follows:

$$\begin{array}{ccccc} (0 \longrightarrow 0) & \longrightarrow & (0 \longrightarrow M_2) & \longrightarrow & (M_2 \xrightarrow{id} M_2) \\ & & \downarrow \qquad \downarrow id \;\; \swarrow g & \swarrow id & \\ & & (M_1 \xrightarrow{f} M_2) & & \end{array}$$

Thus, $f \circ g = id_{M_2}$. Therefore, $M_1 \cong M_2 \oplus ker(f)$ and

$$(M_1 \xrightarrow{f} M_2) \;\cong\; (M_2 \xrightarrow{id} M_2) \oplus (ker(f) \longrightarrow 0).$$

□

Now let $Q = \bullet \to \bullet \to \bullet \to \cdots \to \bullet \to \bullet$ be a quiver with n vertices and $n-1$ arrows. Then, we can easily generalize the results of Lemmas 1.3 and 1.5 as follows : the representations

$$\begin{array}{c} E \to 0 \to 0 \to \cdots \to 0 \to 0 \\ E \xrightarrow{id} E \to 0 \to \cdots \to 0 \to 0 \\ \vdots \\ E \xrightarrow{id} E \xrightarrow{id} E \xrightarrow{id} \cdots \to E \xrightarrow{id} E \to 0 \\ E \xrightarrow{id} E \xrightarrow{id} E \xrightarrow{id} \cdots \to E \xrightarrow{id} E \xrightarrow{id} E \end{array}$$

are all injective representations of a quiver $Q = \bullet \to \bullet \to \bullet \to \cdots \to \bullet \to \bullet$, if each E_i is an injective left R-module. We can also generalize Lemma 1.2 so that if $M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} M_n$ of a quiver $Q = \bullet \to \bullet \to \bullet \to \cdots \to \bullet \to \bullet$ is an injective representation, then each M_i is an injective left R-module.

THEOREM 2.2. *A representation $M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3$ is an injective representation of a quiver $Q = \bullet \to \bullet \to \bullet$ if and only if M_1, M_2 and M_3 are injective left R-modules,*

$$\begin{aligned}&(M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3)\\ &\cong (E_1 \xrightarrow{id} E_1 \xrightarrow{id} E_1) \oplus (E_2 \xrightarrow{id} E_2 \to 0) \oplus (E_3 \to 0 \to 0).\end{aligned}$$

PROOF. The diagram

$$\begin{array}{ccccccc}(0 \longrightarrow 0 \longrightarrow 0) \longrightarrow & (0 & \longrightarrow M_2 & \xrightarrow{id} M_2) & \longrightarrow (M_2 \xrightarrow{id} M_2 \xrightarrow{id} M_2) \\ & \downarrow & \downarrow{\scriptstyle id} & \downarrow{\scriptstyle f_2} & \\ & (M_1 & \xrightarrow{f_1} M_2 & \xrightarrow{f_2} M_3) & \end{array}$$

can be completed to a commutative diagram by $g_{21} : M_2 \to M_1$, $id : M_2 \to M_2$ and $g_{23} : M_2 \to M_3$. Then we can get $f_1 \circ g_{21} = id_{M_2}$ so that $M_1 \cong M_2 \oplus ker(f_1)$. Now the diagram

$$\begin{array}{ccccccc}(0 \longrightarrow 0 \longrightarrow 0) \longrightarrow & (0 & \longrightarrow 0 & \longrightarrow M_3) & \longrightarrow (0 \longrightarrow M_3 \xrightarrow{id} M_3) \\ & \downarrow & \downarrow & \downarrow{\scriptstyle id} & \\ & (M_1 & \xrightarrow{f_1} M_2 & \xrightarrow{f_2} M_3) & \end{array}$$

can be completed to a commutative diagram by $i : 0 \to M_1$, $g_{32} : M_3 \to M_2$, $id : M_3 \to M_3$. Then, we can get $f_2 \circ g_{32} = id_{M_3}$ so that $M_2 \cong M_3 \oplus ker(f_2)$. Therefore, $M_1 \cong M_3 \oplus ker(f_2) \oplus ker(f_1)$. Hence,

$$\begin{aligned}&(M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} M_3)\\ &\cong (E_1 \xrightarrow{id} E_1 \xrightarrow{id} E_1) \oplus (E_2 \xrightarrow{id} E_2 \to 0) \oplus (E_3 \to 0 \to 0).\end{aligned}$$

This completes the proof. □

Now, we can easily generalize Theorem 2.2 so that a representation $M_1 \xrightarrow{f_1} M_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} M_n$ of a quiver $Q = \bullet \to \bullet \to \bullet \to \cdots \to \bullet \to \bullet$

is an injective representation if and only if each M_i is an injective left R-module and the representation is the direct sum of the following injective representations:

$$E \to 0 \to 0 \to \cdots \to 0 \to 0$$
$$E \xrightarrow{id} E \to 0 \to \cdots \to 0 \to 0$$
$$\vdots$$
$$E \xrightarrow{id} E \xrightarrow{id} E \xrightarrow{id} \cdots \to E \xrightarrow{id} E \to 0$$
$$E \xrightarrow{id} E \xrightarrow{id} E \xrightarrow{id} \cdots \to E \xrightarrow{id} E \xrightarrow{id} E$$

REMARK 2.3. The representations of a quiver $Q = \bullet \to \bullet \to \bullet \to \cdots \to \bullet \to \bullet$:

$$0 \to 0 \to \cdots \to 0 \to E$$
$$0 \to 0 \to \cdots \to 0 \to E \xrightarrow{id} E$$
$$\vdots$$
$$0 \to 0 \to \cdots \to 0 \to E \xrightarrow{id} E \xrightarrow{id} E$$
$$0 \to E \xrightarrow{id} E \xrightarrow{id} \cdots \to E \xrightarrow{id} E \xrightarrow{id} E$$

are not injective representations if $E \neq 0$.

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