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FUZZY QUOTIENT STRUCTURES OF BCK-ALGEBRAS INDUCED BY FUZZY BCK-FILTERS

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Abstract. In this paper, we establish a generalization of fundamental homomorphism theorem in BCK-algebras by using fuzzy BCK-filters. We prove that if μ (resp. ν) is a fuzzy BCK-filter of a bounded BCK-algebra X (resp. Y), then $\frac{X\times Y}{\mu\times\nu}\cong X/\mu\times Y/\nu$; and if μ is a fuzzy BCK-filter and F is a BCK-filter in a bounded BCK-algebra X such that F/μ is a BCK-filter of X/μ, then $\frac{X/\mu}{F/\mu}\cong X/F$.

1. Introduction

The study of BCK-algebras was initiated by K. Iséki in 1966 as a generalization of the concept of set-theoretic difference and propositional calculus. Since then a great deal of literature has been produced on the theory of BCK-algebras, In particular, emphasis seems to have been put on the ideal theory of BCK-algebras. For the general development of BCK-algebras the filter theory plays an important role as well as the ideal theory. Meng [4] introduced the notion of BCK-filters in BCK-algebras, and investigated some interesting results. Jun et al. [2] discussed the fuzzification of BCK-filters in BCK-algebras. In [1], Jun et al. constructed the quotient BCK-algebra which is induced by (fuzzy) BCK-filters, and investigated the relation between the quotient BCK-algebra induced by BCK-filters and the quotient BCK-algebra induced by fuzzy BCK-filters. In this paper, we establish a generalization of fundamental homomorphism theorem in BCK-algebras by using fuzzy BCK-filters. We prove that if μ (resp. ν) is a fuzzy BCK-filter of a bounded BCK-algebra X (resp. Y), then $\frac{X\times Y}{\mu\times\nu}\cong X/\mu\times Y/\nu$; and if μ is a fuzzy BCK-filter and F is a BCK-filter in a bounded BCK-algebra X such that F/μ is a BCK-filter of X/μ, then $\frac{X/\mu}{F/\mu}\cong X/F$.

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2. Preliminaries

A nonempty set X with a constant 0 and a binary operation denoted by juxtaposition is called a *BCK-algebra* if for all $x, y, z \in X$ the following conditions hold:

(I) $((xy)(xz))(zy) = 0$,
(II) $(x(xy))y = 0$,
(III) $xx = 0$,
(IV) $0x = 0$,
(V) $xy = 0$ and $yx = 0$ imply $x = y$.

A BCK-algebra can be (partially) ordered by $x \leq y$ if and only if $xy = 0$. The following results are true in any BCK-algebra.

(1) $x0 = x$.
(2) $(xy)z = (xz)y$.
(3) $xy \leq x$.
(4) $(xz)(yz) \leq xy$.
(5) $x \leq y$ implies $xz \leq yz$ and $zy \leq zx$.

If there is a special element e of a BCK-algebra X satisfying $x \leq e$ for all $x \in X$, then e is called a *unit* of X. A BCK-algebra with unit is said to be *bounded*. In a bounded BCK-algebra X, we denote ex by $x^\bullet$ for every $x \in X$. A BCK-algebra X is said to be *commutative* if $x(xy) = y(yx)$ for all $x, y \in X$. A map $f : X \to Y$ of BCK-algebras is called a *homomorphism* if $f(xy) = f(x)f(y)$ for all $x, y \in X$.

In a bounded BCK-algebra X, we have

(6) $e^\bullet = 0$ and $0^\bullet = e$.
(7) $y \leq x$ implies $x^\bullet \leq y^\bullet$.
(8) $x^\bullet y^\bullet \leq yx$.
(9) $x^{\bullet\bullet} = x$ and $x^\bullet y^\bullet = yx$ whenever X is commutative.

DEFINITION 2.1. (Meng [4]) A nonempty subset F of a bounded BCK-algebra X is called a *BCK-filter* of X if

(i) $e \in F$,
(ii) $(x^\bullet y^\bullet)^\bullet \in F$ and $y \in F$ imply $x \in F$ for all $x, y \in X$.

Now we review some fuzzy logic concepts. A *fuzzy set* in a set X is a function $\mu : X \to [0,1]$. Let $f : X \to Y$ be a function from a set X to a set Y and let ν be any fuzzy set in $f(X)$. The fuzzy set μ in X defined by $\mu(x) = \nu(f(x))$ for all $x \in X$ is called the *preimage* of ν under f and is denoted by $f^{-1}(\nu)$. Let μ be a fuzzy set in X. The fuzzy set ν in Y

defined by

$$\nu(y) := \begin{cases} \sup_{x \in f^{-1}(y)} \mu(x) & \text{if } f^{-1}(y) \neq \emptyset, \ \forall y \in Y, \\ 0 & \text{otherwise,} \end{cases}$$

is called the *image* of μ under f and is denoted by $f(\mu)$.

DEFINITION 2.2. (Jun et al. [2]) Let μ be a fuzzy set in a bounded BCK-algebra X. Then μ is called a *fuzzy BCK-filter* of X if, for all $x, y \in X$,

(i) $\mu(e) \geq \mu(x)$,

(ii) $\mu(x) \geq \min\{\mu((x^\bullet y^\bullet)^\bullet), \mu(y)\}$.

PROPOSITION 2.3. (Jun et al. [3, Theorem 3.9]) *Let $f : X \to Y$ be an epimorphism of bounded BCK-algebras. Then*

(a) *If ν is a fuzzy BCK-filter of Y, then $f^{-1}(\nu)$ is a fuzzy BCK-filter of X.*

(b) *If μ is a fuzzy BCK-filter of X, then $f(\mu)$ is a fuzzy BCK-filter of Y.*

3. Quotient structures of BCK-algebras via fuzzy BCK-filters

Let F be a BCK-filter of a bounded BCK-algebra X. We define a relation $\sim_F$ on X as follows:

$$x \sim_F y \quad \text{if and only if} \quad (xy)^\bullet \in F \quad \text{and} \quad (yx)^\bullet \in F.$$

Then $\sim_F$ is an equivalence relation on X. We denote by F_x the equivalence class containing x and by X/F the set of all equivalence classes of X with respect to $\sim_F$, that is, $F_x := \{y \in X \mid x \sim_F y\}$ and $X/F := \{F_x \mid x \in X\}$. Define a binary operation $*_F$ on X/F by $F_x *_F F_y = F_{xy}$ for all $F_x, F_y \in X/F$. Then $(X/F, *_F, F_0)$ is a bounded BCK-algebra with F_0 as unit (see [1]). Let μ be a non-constant fuzzy BCK-filter of a bounded BCK-algebra X and define a binary relation, denoted by $\sim_\mu$, on X as follows:

$$x \sim_\mu y \quad \text{if and only if} \quad \mu((xy)^\bullet) > 0 \quad \text{and} \quad \mu((yx)^\bullet) > 0$$

for every $x, y \in X$. Then $\sim_\mu$ is a congruence relation on X. We denote by $[x]_\mu$ the set $\{y \in X \mid x \sim_\mu y\}$ and by X/μ the set $\{[x]_\mu \mid x \in X\}$. Define a binary operation $*_\mu$ on X/μ by $[x]_\mu *_\mu [y]_\mu = [xy]_\mu$ for all $x, y \in X$. If μ is a fuzzy BCK-filter of a bounded BCK-algebra X, then $(X/\mu, *_\mu, [0]_\mu)$ is a BCK-algebra, which is called the *fuzzy quotient BCK-algebra* induced by the fuzzy BCK-filter μ (see [1]).

We state a generalization of the fundamental homomorphism theorem in BCK-algebras.

THEOREM 3.1. (Fuzzy Fundamental Homomorphism Theorem) *Let $f : X \to Y$ be an epimorphism of bounded BCK-algebras. If ν is a fuzzy BCK-filter of Y, then $X/f^{-1}(\nu) \cong Y/\nu$.*

PROOF. Define a function $g : X/f^{-1}(\nu) \to Y/\nu$ by $g\Big([x]_{f^{-1}(\nu)}\Big) = [f(x)]_\nu$ for all $[x]_{f^{-1}(\nu)} \in X/f^{-1}(\nu)$. Let $[x]_{f^{-1}(\nu)} = [y]_{f^{-1}(\nu)}$ in $X/f^{-1}(\nu)$. Then $x \sim_{f-1(\nu)} y$, and so $f^{-1}(\nu)\Big((xy)^\bullet\Big) > 0$ and $f^{-1}(\nu)\Big((yx)^\bullet\Big) > 0$, i.e.,

$$\nu\Big(f((xy)^\bullet)\Big) > 0 \quad \text{and} \quad \nu\Big(f((yx)^\bullet)\Big) > 0.$$

It follows that

$$\nu\Big((f(x)f(y))^\bullet\Big) = \nu\Big((f(xy))^\bullet\Big) = \nu\Big(f((xy)^\bullet)\Big) > 0$$

and

$$\nu\Big((f(y)f(x))^\bullet\Big) = \nu\Big((f(yx))^\bullet\Big) = \nu\Big(f((yx)^\bullet)\Big) > 0$$

so that $f(x) \sim_\nu f(y)$, i.e., $[f(x)]_\nu = [f(y)]_\nu$. This proves that g is well-defined. We now claim that g is one-one. For any $[x]_{f^{-1}(\nu)}, [y]_{f^{-1}(\nu)} \in X/f^{-1}(\nu)$, if $g\Big([x]_{f^{-1}(\nu)}\Big) = g\Big([y]_{f^{-1}(\nu)}\Big)$ then $[f(x)]_\nu = [f(y)]_\nu$ and so $f(x) \sim_\nu f(y)$. This means that

$$0 < \nu\Big((f(x)f(y))^\bullet\Big) = \nu\Big((f(xy))^\bullet\Big) = \nu\Big(f((xy)^\bullet)\Big) = f^{-1}(\nu)\Big((xy)^\bullet\Big)$$

and

$$0 < \nu\Big((f(y)f(x))^\bullet\Big) = \nu\Big((f(yx))^\bullet\Big) = \nu\Big(f((yx)^\bullet)\Big) = f^{-1}(\nu)\Big((yx)^\bullet\Big).$$

Hence $x \sim_{f^{-1}(\nu)} y$, i.e., $[x]_{f^{-1}(\nu)} = [y]_{f^{-1}(\nu)}$. Therefore g is one-one. Obviously g is onto. We finally show that g is a homomorphism. For any $[x]_{f^{-1}(\nu)}$ and $[y]_{f^{-1}(\nu)}$ in $X/f^{-1}(\nu)$, we have

$$\begin{aligned} g\Big([x]_{f^{-1}(\nu)} *_{f^{-1}(\nu)} [y]_{f^{-1}(\nu)}\Big) &= g\Big([xy]_{f^{-1}(\nu)}\Big) = [f(xy)]_\nu \\ &= [f(x)f(y)]_\nu = [f(x)]_\nu *_\nu [f(y)]_\nu \\ &= g\Big([x]_{f^{-1}(\nu)}\Big) *_\nu g\Big([y]_{f^{-1}(\nu)}\Big). \end{aligned}$$

This completes the proof. □

The homomorphism $\pi : X \to X/\mu$, given by $\pi(x) := [x]_\mu$, is called the *natural* (or *canonical*) *homomorphism* of X onto X/μ. In the above Theorem 3.1, if we consider canonical homomorphisms $p : X \to X/f^{-1}(\nu)$ and $q : Y \to Y/\nu$ then it is easy to show that $g \circ p = q \circ f$, i.e., the following diagram (Figure 1) commutes.

$$\begin{array}{ccc} \mathrm{X} & \xrightarrow{\quad f \quad} & \mathrm{Y} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ X/f^{-1}(\nu) & \xrightarrow{\quad g \quad} & Y/\nu \end{array}$$

Figure 1

As is well known, the characteristic function of a set is a special fuzzy set. Let F be a subset of a bounded BCK-algebra X and denote by χ_F the characteristic function of F, that is,

$$\chi_F(x) := \begin{cases} 1 & \text{if } x \in F, \\ 0 & \text{otherwise.} \end{cases}$$

Note that χ_F is a fuzzy BCK-filter of X if and only if F is a BCK-filter of X (see [2, Theorem 3.4]).

LEMMA 3.2. *Let F be a BCK-filter of a bounded BCK-algebra X and let $x, y \in X$. Then $x \sim_F y$ if and only if $x \sim_{\chi_F} y$.*

PROOF. For every $x, y \in X$, we have

$$\begin{aligned} x \sim_F y \quad & \text{if and only if} \quad (xy)^\bullet \in F \text{ and } (yx)^\bullet \in F \\ & \text{if and only if} \quad \chi_F\Big((xy)^\bullet\Big) = 1 \text{ and } \chi_F\Big((yx)^\bullet\Big) = 1 \\ & \text{if and only if} \quad \chi_F\Big((xy)^\bullet\Big) > 0 \text{ and } \chi_F\Big((yx)^\bullet\Big) > 0 \\ & \text{if and only if} \quad x \sim_{\chi_F} y, \end{aligned}$$

proving the lemma. □

Applying Lemma 3.2, we know that if F is a BCK-filter of a bounded BCK-algebra X then $F_x = [x]_{\chi_F}$ for all $x \in X$ and $X/F = X/\chi_F$. Let $f : X \to Y$ be an epimorphism of bounded BCK-algebras and let G be a BCK-filter of Y. Note that

$$f(e_X) = f(0_X^\bullet) = \Big(f(0_X)\Big)^\bullet = 0_Y^\bullet = e_Y$$

so that $e_Y \in f^{-1}(e_X) \in f^{-1}(G)$. Let $x, y \in X$ be such that $y \in f^{-1}(G)$ and $(x^\bullet y^\bullet)^\bullet \in f^{-1}(G)$. Then $f(y) \in G$ and

$$\Big((f(x))^\bullet (f(y))^\bullet\Big)^\bullet = \Big(f(x^\bullet) f(y^\bullet)\Big)^\bullet = \Big(f(x^\bullet y^\bullet)\Big)^\bullet = f\Big((x^\bullet y^\bullet)^\bullet\Big) \in G.$$

It follows from Definition 2.1(ii) that $f(x) \in G$ so that $x \in f^{-1}(G)$. This shows that $f^{-1}(G)$ is a BCK-filter of X. Using Theorem 3.1, we have

$$X/f^{-1}(G) = X/\chi_{f^{-1}(G)} = X/f^{-1}(\chi_G) \cong Y/\chi_G = Y/G.$$

Therefore we have the following corollary.

COROLLARY 3.3. (Fundamental Homomorphism Theorem) *Let X and Y be bounded BCK-algebras, $f : X \to Y$ an epimorphism and G a BCK-filter of Y. Then $X/f^{-1}(G) \cong Y/G$.*

Given bounded BCK-algebras X and Y, if we define $(x_1, y_1) * (x_2, y_2) := (x_1x_2, y_1y_2)$ in $X \times Y$, then it is easily to check that $(X \times Y; *, (0_X, 0_Y))$ is a BCK-algebra with (e_X, e_Y) as unit. We discuss fuzzy BCK-filters in a bounded BCK-algebra $X \times Y$.

PROPOSITION 3.4. *Let μ and ν be fuzzy BCK-filters of bounded BCK-algebras X and Y respectively. Define a mapping $\mu \times \nu : X \times Y \to [0, 1]$ by $(\mu \times \nu)(x, y) = \min\{\mu(x), \nu(y)\}$ for all $(x, y) \in X \times Y$. Then $\mu \times \nu$ is a fuzzy BCK-filter of $X \times Y$.*

PROOF. For any $(x, y) \in X \times Y$, we have

$$(\mu\times\nu)(x, y) = \min\{\mu(x), \nu(y)\} \leq \min\{\mu(e_X), \nu(e_Y)\} = (\mu\times\nu)(e_X, e_Y).$$

Let $(x_1, y_1), (x_2, y_2) \in X \times Y$. Then

$$\begin{aligned}
&\min\Big\{(\mu \times \nu)\Big((e_X, e_Y) * (((e_X, e_Y) * (x_1, y_1)) * ((e_X, e_Y) \\
&\qquad *(x_2, y_2)))\Big), (\mu \times \nu)(x_2, y_2)\Big\} \\
= \;& \min\Big\{(\mu \times \nu)\Big((e_X, e_Y) * ((x_1^\bullet, y_1^\bullet) * (x_2^\bullet, y_2^\bullet))\Big), (\mu \times \nu)(x_2, y_2)\Big\} \\
= \;& \min\Big\{(\mu \times \nu)\Big((e_X, e_Y) * (x_1^\bullet x_2^\bullet, y_1^\bullet y_2^\bullet)\Big), (\mu \times \nu)(x_2, y_2)\Big\} \\
= \;& \min\Big\{(\mu \times \nu)\Big((x_1^\bullet x_2^\bullet)^\bullet, (y_1^\bullet y_2^\bullet)^\bullet\Big), (\mu \times \nu)(x_2, y_2)\Big\} \\
= \;& \min\Big\{\min\Big\{\mu\Big((x_1^\bullet x_2^\bullet)^\bullet\Big), \nu\Big((y_1^\bullet y_2^\bullet)^\bullet\Big)\Big\}, \min\{\mu(x_2), \nu(y_2)\}\Big\} \\
= \;& \min\Big\{\min\Big\{\mu\Big((x_1^\bullet x_2^\bullet)^\bullet\Big), \mu(x_2)\Big\}, \min\Big\{\nu\Big((y_1^\bullet y_2^\bullet)^\bullet\Big), \nu(y_2)\Big\}\Big\} \\
\leq \;& \min\{\mu(x_1), \nu(y_1)\} = (\mu \times \nu)(x_1, y_1).
\end{aligned}$$

Hence $\mu \times \nu$ is a fuzzy BCK-filter of $X \times Y$. □

We consider the following condition in a bounded BCK-algebra X:

$$\text{(3A)} \qquad (xy)^\bullet = e \Rightarrow xy = 0.$$

The following example shows that the condition (3A) may not be true in a bounded BCK-algebra.

EXAMPLE 3.5. Consider a bounded BCK-algebra $X = \{0, a, b, c, e\}$ with the following Cayley table (Table 1):

	0	a	b	c	e
0	0	0	0	0	0
a	a	0	0	0	0
b	b	a	0	a	0
c	c	c	c	0	0
e	e	e	e	e	0

Table 1

Then $(bc)^\bullet = a^\bullet = e$, but $bc = a \neq 0$.

But we know that there exists a bounded BCK-algebra that satisfies condition (3A).

EXAMPLE 3.6. Consider a bounded BCK-algebra $X = \{0, a, b, e\}$ with Cayley table as follows (Table 2):

	0	a	b	e
0	0	0	0	0
a	a	0	0	0
b	b	b	0	0
e	e	b	a	0

Table 2

It is easy to verify that X satisfies the condition (3A).

THEOREM 3.7. *Let X and Y be bounded BCK-algebras that satisfy condition (3A). If μ and ν are fuzzy BCK-filters of X and Y respectively, then $\frac{X\times Y}{\mu\times\nu} \cong X/\mu \times Y/\nu$.*

PROOF. Let $\Theta : X \times Y \to X/\mu \times Y/\nu$ be the mapping defined by $\Theta(x, y) = ([x]_\mu, [y]_\nu)$ for all $(x, y) \in X \times Y$. It is easy to show that Θ is an epimorphism. Note that $K := \Theta^{-1}([e_X]_\mu, [e_Y]_\nu)$ is a BCK-filter of $X \times Y$. Hence, by Corollary 3.3, we have $\frac{X\times Y}{K} \cong \frac{X/\mu\times Y/\nu}{([e_X]_\mu,[e_Y]_\nu)} \cong$

$X/\mu \times Y/\nu$. We claim that $K_{(x,y)} = [(x,y)]_{\mu\times\nu}$ for every $(x,y) \in X \times Y$. Indeed,

$$\begin{aligned}
& (u,v) \in K_{(x,y)} \\
\Leftrightarrow\ & \Big((xu)^\bullet, (yv)^\bullet\Big) \in K \text{ and } \Big((ux)^\bullet, (vy)^\bullet\Big) \in K \\
\Leftrightarrow\ & \begin{cases} \Big([(xu)^\bullet]_\mu, [(yv)^\bullet]_\nu\Big) = \Big([e_X]_\mu, [e_Y]_\nu\Big), \\ \Big([(ux)^\bullet]_\mu, [(vy)^\bullet]_\nu\Big) = \Big([e_X]_\mu, [e_Y]_\nu\Big) \end{cases} \\
\Leftrightarrow\ & [u]_\mu = [x]_\mu \text{ and } [v]_\nu = [y]_\nu \\
\Leftrightarrow\ & u \sim_\mu x \text{ and } v \sim_\nu y \\
\Leftrightarrow\ & \begin{cases} \min\Big\{\mu\Big((xu)^\bullet\Big), \mu\Big((ux)^\bullet\Big)\Big\} > 0, \\ \min\Big\{\nu\Big((yv)^\bullet\Big), \nu\Big((vy)^\bullet\Big)\Big\} > 0 \end{cases} \\
\Leftrightarrow\ & \begin{cases} \min\Big\{\mu\Big((xu)^\bullet\Big), \nu\Big((yv)^\bullet\Big)\Big\} > 0, \\ \min\Big\{\mu\Big((ux)^\bullet\Big), \nu\Big((vy)^\bullet\Big)\Big\} > 0 \end{cases} \\
\Leftrightarrow\ & \begin{cases} (\mu \times \nu)\Big((xu)^\bullet, (yv)^\bullet\Big) > 0, \\ (\mu \times \nu)\Big((ux)^\bullet, (vy)^\bullet\Big) > 0 \end{cases} \\
\Leftrightarrow\ & \begin{cases} (\mu \times \nu)\Big((e_X, e_Y) * ((x,y) * (u,v))\Big) > 0, \\ (\mu \times \nu)\Big((e_X, e_Y) * ((u,v) * (x,y))\Big) > 0 \end{cases} \\
\Leftrightarrow\ & (u,v) \in [(x,y)]_{\mu\times\nu}.
\end{aligned}$$

Consequently, $\frac{X\times Y}{\mu\times\nu} = \frac{X\times Y}{K} \cong X/\mu \times Y/\nu$, proving the theorem. □

THEOREM 3.8. *Let μ be a fuzzy BCK-filter of a bounded BCK-algebra X. If F^* is a BCK-filter of X/μ, then the set*

$$F := \bigcup\Big\{x : [x]_\mu \in F^*\Big\}$$

is a BCK-filter of X, and $F/\mu = F^*$.

PROOF. Since F^* is a BCK-filter of X/μ, we have $[e]_\mu \in F^*$ and hence $e \in F$. Let $x, y \in X$ be such that $y \in F$ and $(x^\bullet y^\bullet)^\bullet \in F$. Then $y \sim_\mu v$ and $(x^\bullet y^\bullet)^\bullet \sim_\mu u$ for some $[u]_\mu$, $[v]_\mu \in F^*$. This means that $[y]_\mu = [v]_\mu$ and $[(x^\bullet y^\bullet)^\bullet]_\mu = [u]_\mu$. Hence $[x]_\mu \in F^*$, and so $x \in F$.

Therefore F is a BCK-filter of X. Moreover,

$$\begin{aligned} F/\mu &= \Big\{[a]_\mu \mid a \in F\Big\} \\ &= \Big\{[a]_\mu \mid a \in [x]_\mu \text{ for some } [x]_\mu \in F^*\Big\} \\ &= \Big\{[a]_\mu \mid [a]_\mu = [x]_\mu \text{ for some } [x]_\mu \in F^*\Big\} \\ &= \Big\{[a]_\mu \mid [a]_\mu \in F^*\Big\} \\ &= F^*. \end{aligned}$$

This completes the proof. □

THEOREM 3.9. *Let μ be a fuzzy BCK-filter of a bounded BCK-algebra X. If F is a BCK-filter of X such that F/μ is a BCK-filter of X/μ, then $\frac{X/\mu}{F/\mu} \cong X/F$.*

PROOF. Note that $\frac{X/\mu}{F/\mu} = \Big\{(F/\mu)_{[x]_\mu} \mid [x]_\mu \in X/\mu\Big\}$. Define a mapping $\Psi : \frac{X/\mu}{F/\mu} \to X/F$ by $\Psi\Big((F/\mu)_{[x]_\mu}\Big) = F_x$. Suppose that $(F/\mu)_{[x]_\mu} = (F/\mu)_{[y]_\mu}$. Then $[x]_\mu \sim_{F/\mu} [y]_\mu$, and so $[(xy)^\bullet]_\mu \in F/\mu$ and $[(yx)^\bullet]_\mu \in F/\mu$. It follows that $(xy)^\bullet \in F$ and $(yx)^\bullet \in F$, that is, $x \sim_F y$. Thus

$$\Psi\Big((F/\mu)_{[x]_\mu}\Big) = F_x = F_y = \Psi\Big((F/\mu)_{[y]_\mu}\Big),$$

which shows that Ψ is well-defined. For any $(F/\mu)_{[x]_\mu}, (F/\mu)_{[y]_\mu} \in \frac{X/\mu}{F/\mu}$, we have

$$\begin{aligned} &\Psi\Big((F/\mu)_{[x]_\mu} *_{F/\mu} (F/\mu)_{[y]_\mu}\Big) = \Psi\Big((F/\mu)_{[x]_\mu *_\mu [y]_\mu}\Big) \\ =\ &\Psi\Big((F/\mu)_{[xy]_\mu}\Big) = F_{xy} = F_x *_F F_y \\ =\ &\Psi\Big((F/\mu)_{[x]_\mu}\Big) *_F \Psi\Big((F/\mu)_{[y]_\mu}\Big). \end{aligned}$$

Hence Ψ is a homomorphism. Obviously, Ψ is onto. Finally, we claim that Ψ is one-one. Assume that $\Psi\Big((F/\mu)_{[x]_\mu}\Big) = \Psi\Big((F/\mu)_{[y]_\mu}\Big)$. Then $F_x = F_y$ and so $x \sim_F y$. If $[a]_\mu \in (F/\mu)_{[x]_\mu}$, then $[a]_\mu \sim_{F/\mu} [x]_\mu$ which implies that $[(ax)^\bullet]_\mu \in F/\mu$ and $[(xa)^\bullet]_\mu \in F/\mu$. Therefore $(ax)^\bullet \in F$ and $(xa)^\bullet \in F$, i.e., $a \sim_F x$. Since $\sim_F$ is an equivalence relation, we have $a \sim_F y$. Thus $[a]_\mu \in (F/\mu)_{[y]_\mu}$, which shows $(F/\mu)_{[x]_\mu} \subseteq (F/\mu)_{[y]_\mu}$. Similarly we get $(F/\mu)_{[y]_\mu} \subseteq (F/\mu)_{[x]_\mu}$. Consequently, $\frac{X/\mu}{F/\mu} \cong X/F$, proving the theorem. □

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