Novel 2-D FDTD Scheme with Isotropic Dispersion and Enhanced Stability

등방성 분산 특성과 개선된 시간 증분을 가지는 2차원 시간 영역 유한 차분법

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Abstract

A two dimensional(2-D) finite-difference time-domain(FDTD) method based on a novel finite difference scheme is developed to eliminate the numerical dispersion errors. In this paper, numerical dispersion and stability analysis of the new scheme are given, which show that the proposed method is nearly dispersionless, and stable for a larger time step than the standard FDTD method.

요 약

본 논문에서는 시간 영역 유한 차분법에 있어 비등방성 분산 특성을 보이는, 기존의 Yee 기법을 개선하기 위해 2차원의 새로운 유한 차분식을 제안하였다. 이 기법은 6개 지점의 샘플링을 통해 공간에 대한 편미분식을 근사화하게 된다. 제안하는 기법의 분산 특성을 보기 위해 분산 관계식을 구하였고 그 관계식에서 수치적 전파 상수를 계산하여 제안하는 기법의 분산 특성이 등방성임을 확인하였다. 또한 기존 기법들에 비해 보다 큰 시간 증분의 모의 실험환경에서 안정함을 수학적으로 확인할 수 있었다.

Key words : FDTD, Isotropic Numerical Dispersion

Ⅰ. Introduction

During the past decades, a finite-difference time-domain(FDTD) method has been intensively used for a wide spectrum of applications since it can provide many advantages such as low computational complexity, great flexibility, easy implementation, etc. However, the standard FDTD method has been suffered from the so-called numerical dispersion, which makes wave propagate at different phase velocity dependant on the propagation direction. Hence, several techniques have been proposed to rectify the numerical dispersion. In [1], in artificial anisotropy is introduced to reduce the dispersion error. Cole replaces the spatial derivatives by a nonstandard finite difference(NSFD), which minimizes the numerical error of the central numerical differentiation scheme[2]. Recently Forgy *et al.* propose a new FDTD scheme based on linearly superposing two different lattices[3]. Advantages and disadvantages of the individual method are discussed in [3].

In this paper, a simple approach to reduce the dispersion error for 2-D(2-Dimensional) FDTD method is proposed. It is based on an observation that the anisotropic dispersion inherently existing in the standard

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FDTD may be caused by the fact of sampling fields at spatially anisotropic points. Therefore the proposed scheme samples fields in an isotropic fashion to approximate the spatial differentiation. A general procedure to formulate a new FDTD scheme is: 1) approximating spatial derivatives with weighted summation of two different numerical differentiation schemes, 2) determine the weighting factor to minimize the fluctuation of the resulting dispersion, 3) removing the artificial ether to achieve the exact phase velocity. In the following sections, a closed-form expression of the weighting factor is formulated, and its properties are investigated. A stability condition of the resulting scheme is also addressed.

Ⅱ. Formulation

2-1 Finite Difference Approximation

The standard FDTD approximates the spatial derivative using the second order central finite difference (FD), as given in (1). As mentioned earlier, the formulation samples fields at a 1-D line as seen in Fig. 1, which may cause the an isotropic dispersion. To sample fields in an isotropic manner in 2-D space, another FD scheme of the same order of accuracy can be used [4] as given by (2). Based on the Yee's grid the fields at each 4 points used in (2) should be estimated by fields at adjacent cells. If a linear interpolation is used to calculate the fields at the 4 points, a new FD scheme is obtained as (3). If a quadrature interpolation scheme is adopted, the coefficients in (3) are changed.

$$\left.\frac{\partial f}{\partial x}\right|_{i,j} \cong \frac{\tilde{\partial}^2_x f_{i,j}}{\triangle x} = \frac{f_{i+1.2,j} - f_{i-1/2,j}}{\triangle x} \tag{1}$$

$$\left.\frac{\partial f}{\partial x}\right|_{i,j} \cong \frac{\tilde{\partial}^4_x f_{i,j}}{2\triangle x} = \frac{\tilde{\partial}^2_x f_{i,j+1/2} + \tilde{\partial}^2_x f_{i,j-1/2}}{2\triangle x} \tag{2}$$

$$\left.\frac{\partial f}{\partial x}\right|_{i,j} \cong \frac{\tilde{\partial}^6_x f_{i,j}}{\triangle x} = \frac{\tilde{\partial}^2_x f_{i,j+1} + 2\tilde{\partial}^2_x f_{i,j} + \tilde{\partial}^2_x f_{i,j-1}}{4\triangle x} \tag{3}$$

Field at a dielectric interface can be discontinuous, but a Lagrange-type interpolation method assumes a continuous function. Hence, any interpolation method fails to predict a correct field at the interface. However, due to Fourier analysis, field at the interface is a mean of fields at two points very close to the interface inside the two media, which is the same field predicted by a linear interpolation. Therefore a linear interpolation method is the best choice for inhomogeneous problems.

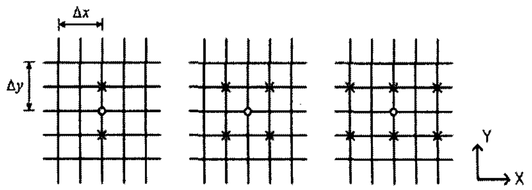


Fig. 1. Graphical depiction of three FD approximations to ∂y in two dimension.

Substituting (3) into the spatial derivatives of 2-D Maxwell's equations, the numerical dispersion of the new FDTD method turns out to be a little bit more anisotropic than that of the Yee scheme, but the angular dependency is opposite to that of the standard FDTD. Hence, by approximating the spatial derivative with a weighted sum of $\tilde{\partial}^2_u$ and $\tilde{\partial}^6_u$, a new scheme can be obtained as

$$\tilde{\partial}_u = (1-\alpha)\,\tilde{\partial}^2_u + \alpha\,\tilde{\partial}^6_u \tag{4}$$

where α is a weighting factor which has to be determined. The resulting dispersion error can be controlled by varying the weighting factor. Finally the new FDTD scheme for a TM wave is simply expressed with the

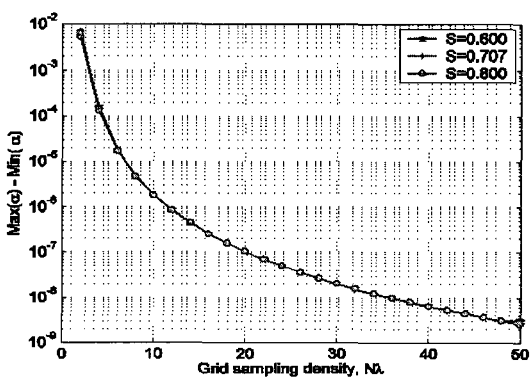


Fig. 2. Plot of variation of as a function of a grid sampling density $N_\lambda = \lambda / \triangle$, and time step.

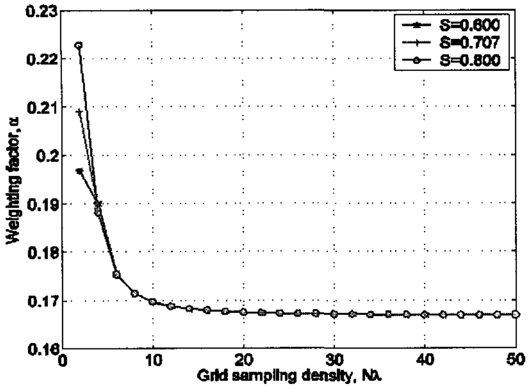


Fig. 3. Optimal α versus the grid sampling density at the stability factor=0.6, 0.707, and 0.8.

new FD scheme (4) as

$$\tilde{\partial}_t^2 H_x^n(i,j+1/2) = -\frac{\Delta t}{\mu\Delta y}\,\tilde{\partial}_y E_z^n(i,j+1/2)$$

$$\tilde{\partial}_t^2 H_y^n(i+1/2,j) = \frac{\Delta t}{\mu\Delta x}\,\tilde{\partial}_x E_z^n(i+1/2,j)$$

$$\tilde{\partial}_t^2 E_z^{n+1}(i,j) = \frac{\Delta t}{\varepsilon\Delta x}\,\tilde{\partial}_x H_y^{n+1/2}(i,j) - \frac{\Delta t}{\varepsilon\Delta y}\,\tilde{\partial}_y H_x^{n+1/2}(i,j) \tag{5}$$

where $\tilde{\partial}_t^2$ is the central time difference operator with respect to time. Unlike the NS-FDTD, the proposed scheme satisfies the duality principle as seen in (5).

2-2 Determination of Optimal Weighting Factor

$$\frac{1}{(c\Delta t)^2}\sin^2\left(\frac{\omega\Delta t}{2}\right) = \frac{1}{\Delta x^2}\left[1-\alpha\sin^2\left(\frac{\tilde{k}_x\Delta x}{2}\right)\right]^2\sin^2\left(\frac{\tilde{k}_y\Delta y}{2}\right) + \frac{1}{\Delta y^2}\left[1-\alpha\sin^2\left(\frac{\tilde{k}_y\Delta y}{2}\right)\right]^2\sin^2\left(\frac{\tilde{k}_x\Delta x}{2}\right) \tag{6}$$

Following [5], the numerical dispersion relation of the proposed method can be derived as (6). After some manipulation, (6) is rewritten as

$$\frac{1}{\Delta^2}C_+C_\times\left(\alpha-\frac{2}{C_+}\right)^2 - \frac{1}{\Delta^2}\left(\frac{4C_\times}{C_+}-C_+\right) - \frac{1}{(c\Delta t)^2}\sin^2\left(\frac{\omega\Delta t}{2}\right) = 0 \tag{7}$$

where $\Delta x = \Delta y = \Delta$, $\tilde{k}_x = \tilde{k}\cos\phi$, $\tilde{k}_y = \tilde{k}\sin\phi$ and

$$C_+ = \sin^2\left(\frac{\tilde{k}_x\Delta}{2}\right) + \sin^2\left(\frac{\tilde{k}_y\Delta}{2}\right)$$

$$C_\times = \sin^2\left(\frac{\tilde{k}_x\Delta}{2}\right) + \sin^2\left(\frac{\tilde{k}_y\Delta}{2}\right)$$

Since (6) is a quadratic equation about α, exact solutions of (7) can be given analytically as

$$\alpha = \frac{2}{C_+}\left[1-\sqrt{1-\frac{\Delta^2 C_+}{4C_\times}\left(\frac{1}{\Delta^2}C_+ - \frac{1}{(c\Delta t)^2}\sin^2\left(\frac{\omega\Delta t}{2}\right)\right)}\right] \tag{8}$$

As seen in (8), α is a function of the azimuth angles (ϕ), cell size, and time step. To determine an optimal α, first, the variation of α as a function of the azimuth angles for a fixed cell size is investigated. Fig. 2 shows the difference between the maximum and minimum of along α the azimuth angles versus the cell size and the time step. As shown in the figure, for the small cell size of $\lambda/10$ to $\lambda/20$, the difference between the maximum and minimum of α ranges from 10^{-6} to 10^{-7}, and thus α can be considered as a constant over the azimuth angles. Therefore, an optimal α can be simply estimated as a mean value of α over the azimuth angles. Fig. 3 shows the calculated optimal α as a function of the cell size, which shows that α remains almost constant for a wide range of the cell size. Therefore the proposed scheme may be suitable for a wide band simulation.

2-3 Removing Numerical Ether

After determining the optimal α, an isotropic nume--rical phase velocity can be archived as shown in Fig. 4 (uncorrected case), but the numerical phase velocity of the proposed method is slower than the exact one. However, by simply scaling ε and/or μ, the numerical phase velocity can easily be adjusted very closed to the exact value over all azimuth angles. The scaling factor is chosen to make the fastest phase velocity be the exact one(ϕ=0°).

$$sc = \frac{k_{exact}}{\frac{2}{\Delta}\sin^{-1}\left(\frac{\Delta}{c\Delta t}\sin\left(\frac{\omega\Delta t}{2}\right)\right)} \tag{9}$$

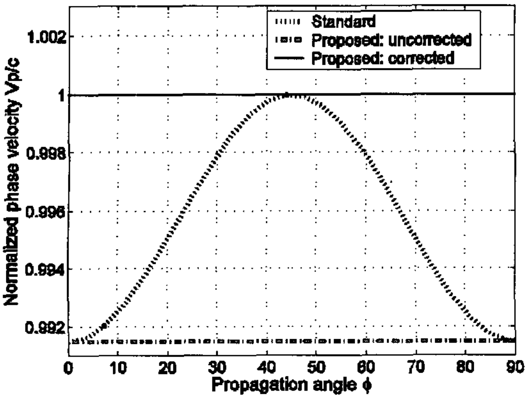


Fig. 4. Plot of the numerical dispersion as a function of the propagation angles.

$$\varepsilon' = sc \times \varepsilon,\ \ \mu' = sc \times \mu \tag{10}$$

Therefore, the final FDTD scheme for 2D TM wave has the same form as (5), but ε and μ are replaced with ε' and μ', respectively. Fig. 4 shows the dispersion after correcting the phase velocity(corrected case).

Ⅲ. Stability Analysis

Based on (6) a maximum $\triangle t$ can be estimated for a stable FDTD scheme. Since the left hand side of (6) is a sinc function, sin c(x)=sin(x)/x, $\triangle t$ can have a maximum value when the right hand side of (6) is minimal. It is easily shown that when $k_{x,y}\triangle/2$ is very small, the right hand side of (6) has a minimum value at ϕ=45°. Since $\tilde{k}_x = \tilde{k}_y$ at ϕ=45°, (6) is rewritten as

$$\sin^2\left(\frac{\omega\triangle t}{2}\right) = \frac{2(c\triangle t)^2}{\triangle^2}[1-\alpha\ \sin^2\beta]\sin^2\beta \tag{11}$$

where $\beta = \tilde{k}\triangle/2\sqrt{2}$. For $\triangle t$ to have a real value, the absolute value of both sides in (11) should be less than unity. Thus $\triangle t$ should satisfy the following inequality:

$$\triangle t \le \frac{\triangle}{\sqrt{2}c}\frac{1}{\max[\sin\beta\sqrt{1-\alpha\ \sin^2\beta}]}$$

The maximum value of the right hand side of (11) occurs at $\sin\beta = 1/\sqrt{3\alpha}$. Therefore the stability condition can be given by

$$\triangle t \le \begin{cases} \dfrac{\triangle}{\sqrt{2}c}\dfrac{1}{1-\alpha} & \text{for } \alpha < 1/3 \\ \dfrac{\triangle}{\sqrt{2}c}\dfrac{3\sqrt{3\alpha}}{2} & \text{for } \alpha > 1/3 \end{cases}$$

The above condition clearly shows the maximum time step of the proposed scheme is larger than that of the Yee scheme.

Ⅳ. Numerical Results

To demonstrate the validity of the proposed FDTD method, two cases were considered in this paper. The first example is a cavity enclosed by perfect conductors whose size is $2\lambda_0 \times 12\lambda_0$. To examine the validation of the scheme for an inhomogeneous problem, the cavity is filled with a medium from the left side to $10\lambda_0$. The dielectric has ε_r=1.0 and μ_r=5.0. A Gaussian pulse with a center frequency of 3 GHz, and a bandwidth of 3 GHz is excited at the center of the cavity. Grid size is chosen to be $\lambda_{min}/10$, where λ_{min} is the wavelength of the highest frequency. Fig. 5 shows the electric field at the source point calculated by the proposed FDTD, standard FDTD, and NS-FDTD. As seen in the figure, the proposed scheme is in excellent agreement with the NS-FDTD, while the standard FDTD dose not. It can be also observed that the proposed scheme is stable beyond

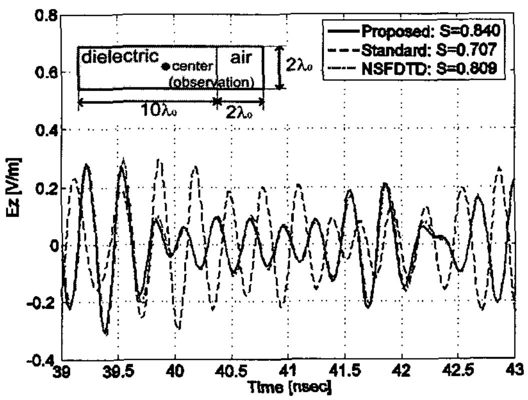
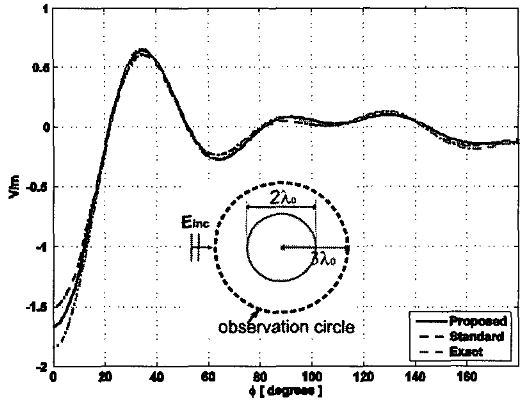
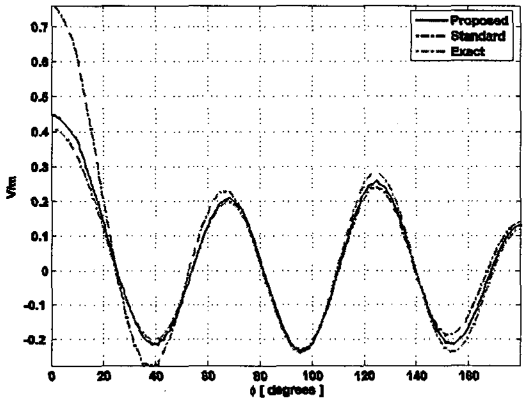


Fig. 5. Electric fields computed by the proposed FDTD scheme, the Yee scheme, and the NSFDTD scheme inside an inhomogeneous 2-D cavity. The size of the cavity is $2\lambda_0 \times 12\lambda_0$, and the dielectric is filled by a dielectric whose relative permeability is 5.



(a) Real part



(b) Imaginary part

Fig. 6. Plots of the real and imaginary parts of scattered electric field by a circular dielectric cylinder with a radius of λ_0 and a dielectric constant of 2. The observation points are moved along the circle with a radius of $3\lambda_0$.

the CFL limit of the standard FDTD.

The second example is a dielectric cylinder in incidence of plane wave. The dielectric constant and radius are 2 and λ_0, respectively. Under this situation, scattered field was observed along the circle with a radius of $3\lambda_0$. For this case, the exact field is known [6] and as Fig. 6, the proposed scheme shows the exact scattered field from the cylinder but the standard scheme does not. This result is the proof that numerical velocity of each scheme is not different from the physical propagation velocity and the proposed scheme can be compensated the exact velocity of material.

V. Conclusion

In this paper, a nearly dispersionless FDTD scheme is proposed. The method replaces the central FD scheme with a new FD scheme which samples fields in a spatially isotropic manner. It is shown that the proposed scheme is stable for a larger time step than the Yee scheme, and has a nearly isotropic dispersion and exact phase velocity. Compared with NS-FDTD, the proposed scheme is numerically verified.

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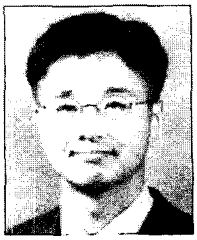

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