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MOTION IN PARABOLIC CYLINDRICAL COORDINATES: APPLICATION TO J_2 GRAVITY PERTURBED TRAJECTORIES

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ABSTRACT

In this paper, initial value problem for dynamical astronomy will be established using parabolic cylindrical coordinates. Computation algorithm is developed for the initial value problem of gravity perturbed trajectories. Applications of the algorithm for the problem of final state predication are illustrated by numerical examples of seven test orbits of different eccentricities. The numerical results are extremely accurate and efficient in predicating final state for gravity perturbed trajectories which is of extreme importance for scientific researches as well as for military purposes. Moreover, an additional efficiency of the algorithm is that, for each of the test orbits, the step size used for solving the differential equations of motion is larger than 70% of the step size used for obtaining its reference final state solution.

Key words : Space dynamics, initial value problems, orbit determination

I. INTRODUCTION

The application of the conventional equations of space dynamic for the motion of Earth's artificial satellites gives inaccurate prediction for their positions and velocities. This is because that these equations are unstable in the Liapunow sense (Stiefel & Scheifele 1971) In brief, the deficiency of these equations is due to the choice of the variables, which in turn has led some authors to propose successful methods to change of the dependent and/or independent variables so as to regularize the differential equations of motion. Of these is, the method established by Stiefel and Scheifele, in 1971. This method consists of changing the independent variable from time to a new variable, which is proportional to the eccentric anomaly in the elliptic case or its equivalent in hyperbolic case. The method is then changes the coordinates from three-dimensional Cartesian space to a four -dimensional space by what they called the KS transformation. The resulting equations are four-dimensional harmonic oscillator. Applications of this transformation to problem in space dynamics showed its very great accuracy even for complex systems (e.g., Sharaf et al. 1987). In the present paper, initial value problem for dynamical astronomy will be established using parabolic cylindrical coordinates ,in which the independent variables are only, changed which in turn produce transformations from three dimensional space to another three dimensional space. Computation algorithm is developed for the initial value problem of gravity perturbed trajectories. Applications of the algorithm for the problem of final state predication are illustrated by numerical examples of seven test orbits of different eccentricities. The numerical results are extremely accurate and efficient in predicating final state for gravity perturbed trajectories which is of extreme importance for scientific researches as well as for military purposes. Moreover, an additional efficiency of the algorithm is that, for each of the test orbits, the step size of the independent variable (time) used for solving the differential equations of motion is larger than 70% of the time step size used for obtaining its reference final state solution .

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II. MOTION IN PARABOLIC CYLINDRICAL COORDINATES

(a) Direct transformations

The transformations between Cartesian coordinates (x, y, z) and parabolic cylindrical coordinates (u_1, u_2, u_3) are given as (e.g., Margenau & Murphy 1966):

$$x = \frac{1}{2}(u_1^2 - u_2^2);\ y = u_1 u_2;\ z = u_3, \qquad (1)$$

$$-\infty < u_1 < \infty,\ \ 0 \le u_2 < \infty,\ \ -\infty < u_3 < \infty$$

Differentiating eq. (1) with respect to the time t, we get

$$\dot{x} = u_1\dot{u}_1 - u_2\dot{u}_2;\ \dot{y} = u_2\dot{u}_1 + u_1\dot{u}_2;\ \dot{z} = \dot{u}_3 \qquad (2)$$

where the 'dot' is used to denote the differentiation with respect to the time t.

(b) Inverse transformations

The first two of eq. (1) could be written as:

$$(x + iy)^{1/2} = \frac{1}{2}(u_1 + iu_2);\ \ i = \sqrt{-1},$$

then we get:

$$u_1^2 + u_2^2 = 2(x^2 + y^2)^{1/2}. \tag{3}$$

Since $-\infty < u_1 < \infty$, $u_2 \geq 0$ and $y = u_1 u_2$, it follows from eqs. (1) and (3) that:

$$\begin{aligned} u_1 &= \pm\left\{(x^2+y^2)^{1/2} + x\right\}^{1/2}; \\ u_2 &= \left\{(x^2+y^2)^{1/2} - x\right\}^{1/2}; \\ u_3 &= z, \end{aligned} \tag{4}$$

where the positive (negative), sign is when $y \geq 0 (< 0)$. Differentiating eq. (1) with respect to t and then solving for $\dot{u}_1$ and $\dot{u}_2$ we get:

$$\dot{u}_1 = \frac{u_1\dot{x} + u_2\dot{y}}{u_1^2 + u_2^2}; \quad \dot{u}_2 = \frac{u_1\dot{y} + u_2\dot{x}}{u_1^2 + u_2^2}; \quad \dot{u}_3 = \dot{z}, \tag{5}$$

where u_1 and u_2 are given in terms of (x, y, z) from eq. (4).

(c) Equations of motion

In the present paper we shall suppose that the motion is controlled by a gravitational potential, $V \equiv V(x, y, z)$, so the equations of motion in Cartesian coordinates (x, y, z) are:

$$\ddot{x} = \frac{\partial V}{\partial x}; \quad \ddot{y} = \frac{\partial V}{\partial y}; \quad \ddot{z} = \frac{\partial V}{\partial z} \tag{6}$$

From the above equations we get:

$$\ddot{u}_1 = \frac{1}{U^2}\left(-\dot{u}_1^2 u_1 + u_1\dot{u}_2^2 - 2u_2\dot{u}_1\dot{u}_2 + \frac{\partial V}{\partial u_1}\right), \tag{7a}$$

$$\ddot{u}_1 = \frac{1}{U^2}\left(\dot{u}_1^2 u_2 - u_2\dot{u}_2^2 - 2u_1\dot{u}_1\dot{u}_2 + \frac{\partial V}{\partial u_2}\right), \tag{7b}$$

$$\ddot{u}_3 = \frac{\partial V}{\partial u_3}, \tag{7c}$$

where $U^2 \equiv u_1^2 + u_2^2$ The partial derivatives $\frac{\partial V}{\partial u_j}$; $j = 1, 2, 3$ are given in terms of the known partial derivatives $\frac{\partial V}{\partial x}$, $\frac{\partial V}{\partial y}$ and $\frac{\partial V}{\partial z}$ by:

$$\frac{\partial V}{\partial u_1} = u_1\frac{\partial V}{\partial x} + u_2\frac{\partial V}{\partial y}, \tag{8a}$$

$$\frac{\partial V}{\partial u_2} = -u_2\frac{\partial V}{\partial x} + u_1\frac{\partial V}{\partial y}, \tag{8b}$$

$$\frac{\partial V}{\partial u_3} = \frac{\partial V}{\partial z}. \tag{8c}$$

It should be noted that the equations of the present section are general in the sense that it could be applied for any dynamical system. In what follows we shall consider the applications of these equations for the motion of J_2 gravity perturbed trajectories.

III. APPLICATION TO GRAVITY PERTURBED TRAJECTORIES

(a) The potential V and its partial derivatives

For J_2 gravity perturbed trajectories, the potential V is given as:

$$V \equiv V(x, y, z) = \frac{\mu}{r} + \frac{c}{r^3}\left[3\left(\frac{z}{r}\right)^2 - 1\right], \tag{9}$$

where

$$c = J_2\mu R_\oplus^2/2; \quad r = (x^2 + y^2 + z^2)^{1/2},$$

with the constants: μ the gravitational parameter, which is universal gravitational constant times the Earth's mass; J_2 the Earth's dynamic oblateness (the lowest-degree harmonic component of the gravity field), and $R_\oplus$ is the mean Earth's equatorial radius .The numerical values of these constants are:

$\mu = 398600.8$ km^3 sec^{-2},

$J_2 = 1.0826157 \times 10^{-3}$,

$R_\oplus = 6378.135$ km.

From Equation (9) we have:

$$\frac{\partial V}{\partial x} = -\frac{\mu x}{r^3} + 3c\left(\frac{x}{r^5}\right)\left(1 - \frac{5z^2}{r^2}\right), \tag{10a}$$

$$\frac{\partial V}{\partial y} = -\frac{\mu y}{r^3} + 3c\left(\frac{y}{r^5}\right)\left(1 - \frac{5z^2}{r^2}\right), \tag{10b}$$

$$\frac{\partial V}{\partial z} = -\frac{\mu z}{r^3} + 3c\left(\frac{z}{r^5}\right)\left(3 - \frac{5z^2}{r^2}\right). \tag{10c}$$

(b) Initial value algorithm

In what follows, the initial value algorithm for J_2 gravity perturbed trajectories in parabolic cylindrical coordinates will be considered. The algorithm is described through its basic points: input, output and computational steps.

Input:

- $x_0, y_0, z_0, \dot{x}_0, \dot{y}_0, \dot{z}_0$ at $t = t_0$,
- the flight time, $t = t_f$;
- $\frac{\partial V}{\partial x}$; $\frac{\partial V}{\partial y}$ and $\frac{\partial V}{\partial z}$; (eq. 10)

Output; $x_0, y_0, z_0, \dot{x}_0, \dot{y}_0, \dot{z}_0$ at $t = t_f$

Computational steps:

1- Using eqs. (1) and (10) into eq. (8), to find the

analytical expressions of the partial derivatives $\frac{\partial V}{\partial x}$; $\frac{\partial V}{\partial y}$ and $\frac{\partial V}{\partial z}$ as functions in u_j; $j = 1, 2, 3$ as:

$$\frac{\partial V}{\partial u_1} = \frac{4u_1U^2\left(-U^4A - 8Bu_3^2 - 16\mu u_3^4\right)}{\left(U^4 + 4u_3^2\right)^{7/2}}, \tag{11a}$$

$$\frac{\partial V}{\partial u_2} = \frac{4u_2U^2\left(-U^4A - 8Bu_3^2 - 16\mu u_3^4\right)}{\left(U^4 + 4u_3^2\right)7/2}, \tag{11b}$$

$$\frac{\partial V}{\partial u_3} = \frac{8u_3\left(-240cu_3^2 + 36cD - \mu D^2\right)}{\left[U^4 + 4u_3^2\right]^{7/2}}, \tag{11c}$$

where

$A \equiv -12c + \mu U^4$; $B \equiv 24c + \mu U^4$; and $D \equiv U^4 + 4u_3^2$.

2- The Equations of motion in eq. (7) can be written as first order system in the form:

$$\dot{u}_1 = u_4,$$

$$\dot{u}_2 = u_5,$$

$$\dot{u}_3 = u_6,$$

$$\dot{u}_4 = \frac{1}{U^2}\left(-u_4^2u_1 + u_1u_5^2 - 2u_2u_4u_5 + \frac{\partial V}{\partial u_1}\right),$$

$$\dot{u}_5 = \frac{1}{U^2}\left(u_4^2u_2 - u_2u_5^2 - 2u_1u_4u_5 + \frac{\partial V}{\partial u_2}\right),$$

$$u_6 = \frac{\partial V}{\partial u_3}.$$

3-Compute the initial conditions, u_{0j}; $j = 1, 2, \cdots, 6$ for the above system by applying the transformations: $(x, y, z) \rightarrow (x_0, y_0, z_0)$ and $(\dot{x}, \dot{y}, \dot{z}) \rightarrow (\dot{x}_0, \dot{y}_0, \dot{z}_0)$ in eqs. (4)and (5).

4 - Using these initial conditions to solve numerically the above differential system for u_j; $j = 1, 2, \cdots, 6$ at $t = t_f$, where $u_4 \equiv \dot{u}_1$, $u_5 \equiv \dot{u}_2$ and $u_6 \equiv \dot{u}_3$ at $t = t_f$.

5 - Using u_j;'$\dot{u}_j$; $j = 1, 2, 3$ to compute x, y, z and $\dot{x}, \dot{y}, \dot{z}$ at $t = t_f$ from the direct transformations of Equations (1)

6 - End

(c) Numerical applications

The purpose of this section is to demonstrate the efficiency of the initial value problem using parabolic cylindrical coordinates in producing very accurate final state predictions for J_2 gravity perturbed trajectories.

i) Test orbits

For the applications of the above formulations, we consider seven test orbits given in the Appendix C of Vinti's book, 1998. All these orbits have the initial time $t = t_0$ and each of different flight time t_f, they cover the three basic types of conic motion - elliptic, parabolic and hyperbolic orbits characterized by the initial conditions listed together with t_f, in the first columns of the tables of Appendix A of the present paper. The components of the position vector for each orbit are in km, while the corresponding components of the velocity vector are in km/sec.

ii) Reference orbits

For each orbit, the J_2 gravity perturbed equations of motion in Cartesian coordinate [eqs. (6) and (10)] are solved by the classical Runge-Kutta integrator with variable step size. A final state prediction was determined by reducing the step size until at least five decimal places ($< 10^{-2}$ meter (m)) stabilized in $x(t_f), y(t_f)$ and $z(t_f)$. These values are considered as reference final states solutions to the orbit they refer and are denoted by:

$$\begin{aligned} &\mathbf{r}_R \equiv (x_R(t_f), y_R(t_f), z_R(t_f)) \\ &\text{and } \dot{\mathbf{r}}_R \equiv (\dot{x}_R(t_f), \dot{y}_R(t_f), \dot{z}_R(t_f)) \end{aligned} \tag{12}$$

for the reference position and velocity vectors respectively. The components of these vectors are listed for each orbit in the second columns of the tables of Appendix A.

iii) Efficiency of parabolic cylindrical coordinates

Upon the above reference solutions the efficiency of the initial value problem for J_2 gravity perturbed trajectories using the parabolic cylindrical coordinates (PC- solution) may be checked by testing its ability in predicting final states within certain tolerances as follows. Let $\mathbf{r} \equiv (x(t_f), y(t_f), z(t_f))$ and $\dot{\mathbf{r}} \equiv (\dot{x}(t_f), \dot{y}(t_f), \dot{z}(t_f))$ are the final state of the PC- solution of a given orbit. The efficiency of the PC- solution are then checked by the magnitude of the error criteria ΔR and Δv as:

$$\Delta R = \left[(x - x_R)^2 + (y - y_R)^2 + (z - z_R)^2\right]^{1/2} \times 1000 \text{ in m} \tag{13}$$

$$\Delta v = \left[(\dot{x} - \dot{x}_R)^2 + (\dot{y} - \dot{y}_R)^2 + (\dot{z} - \dot{z}_R)^2\right]^{1/2} \times 1000 \text{ in m/sec} \tag{14}$$

such that, the small the values of ΔR and Δv, the higher the efficiency will be, in this respect, we may define an acceptable solution set (**S.S**) to the problem at hand as:

$$\mathbf{S.S} = ((\mathbf{r}, \dot{\mathbf{r}}) : \Delta R \leq \epsilon_1, \Delta v \leq \epsilon_2) \qquad (15)$$

where $\epsilon_{1,2}$ are given tolerances. For the very accurate predictions required nowadays we may consider the tolerances $\epsilon_{1,2}$ as:

$$\epsilon_1 = 1\ \mathrm{m} \pm 10\mathrm{cm}, \qquad (16a)$$
$$\epsilon_2 = 0.25\mathrm{m/sec}. \qquad (16b)$$

The components of the position and velocity vectors $\mathbf{r}$ and $\dot{\mathbf{r}}$ of the PC solution are listed for each of the test orbits in the third columns of the table of Appendix A., while the values of the errors ΔR and Δv of eqs. (13) and(14) are given at the bottom of each table. These values indicated in accordance of the acceptance solution set that, the PC solution is very accurate and efficient in predicating final state for J_2 gravity perturbed trajectories which is of extreme importance for scientific researches as well as for military purposes. Moreover, an additional efficiency of the algorithm is that, for each of the test orbit, the step size of the independent variable(time) used for solving the differential equations of motion is larger than 70% of time step size used for obtaining its reference final state solution.

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APPENDIX: NUMERICAL RESULTS

LOW EARTH ORBIT

Initial Conditions	Reference Solution	PC-solution
x_0 =2328.96594	x_R=-516.450939	x =-516.450938
y_0 =-5995.21600	y_R =-3026.5115474	y =-3026.5115496
z_0 =1719.97894	z_R =5848.117544	z =5848.117543
$\dot{x}_0$ =2.911101130	$\dot{x}_R$ =3.96659	$\dot{x}$=3.966599
$\dot{y}_0$ =-0.98164053	$\dot{y}_R$ =-6.121618	$\dot{y}$=-6.121618
$\dot{z}_0$ =-7.090499220	$\dot{z}_R$ =3-2.754866	$\dot{z}$=-2.754866
t_f =10000 sec		
	ΔR =0.00302 (m)	Δv = 0.0 (m/sec)