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FUNCTIONAL CENTRAL LIMIT THEOREMS FOR MULTIVARIATE LINEAR PROCESSES GENERATED BY DEPENDENT RANDOM VECTORS

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Abstract. Let $\mathbb{X}_t$ be an m-dimensional linear process defined by $\mathbb{X}_t = \sum_{j=0}^{\infty} A_j \ \mathbb{Z}_{t-j}, \ t = 1, 2, \ldots$, where $\{\mathbb{Z}_t\}$ is a sequence of m-dimensional random vectors with mean $\mathbf{0} : m \times 1$ and positive definite covariance matrix $\Gamma : m \times m$ and $\{A_j\}$ is a sequence of coefficient matrices. In this paper we give sufficient conditions so that $\sum_{t=1}^{[ns]} \mathbb{X}_t$ (properly normalized) converges weakly to Wiener measure if the corresponding result for $\sum_{t=1}^{[ns]} \mathbb{Z}_t$ is true.

1. Introduction

Consider m-dimensional linear process of the form

$$\mathbb{X}_t = \sum_{j=0}^{\infty} A_j \mathbb{Z}_{t-j}, \tag{1.1}$$

where the innovation $\{\mathbb{Z}_t\}$ is a sequence of m-dimensional random vectors with mean $\mathbf{0} : m \times 1$ and positive definite covariance matrix $\Gamma : m \times m$. Throughout we shall assume that

$$\sum_{j=0}^{\infty} \|A_j\| < \infty \text{ and } \sum_{j=0}^{\infty} A_j \neq O_{m \times m}, \tag{1.2}$$

where for any $m \times m, m \geq 1$, matrix $A = (a_{ij}), \ i, j = 1, \ldots, m, \ \|A\| = \sum_{i=1}^{m} \sum_{j=1}^{m} |a_{ij}|$ and $O_{m \times m}$ denotes the $m \times m$ zero matrix. Let W^m denote Wiener measure on $D^m[0, 1]$, the space of all real valued functions

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on [0,1] that are right continuous and have finite left limits, endowed with the sup norm(see, e.g., [3], [10]). Further, let

$$T = \Big(\sum_{j=0}^{\infty} A_j\Big)\Gamma\Big(\sum_{j=0}^{\infty} A_j\Big)', \tag{1.3}$$

$S_n = \sum_{t=1}^{n} \mathbb{X}_t, n \geq 1 (\mathrm{S}_0 = \mathbb{O})$ and define for $n \geq 1$ the stochastic process ξ_n by

$$\xi_n(s) = n^{-\frac{1}{2}} S_{[ns]} \; 0 \leq s \leq 1. \tag{1.4}$$

In this paper we give sufficient conditions so that $\sum_{t=1}^{[ns]} \mathbb{X}_t$ (properly normalized) converges weakly to Wiener measure if the corresponding result for $\sum_{t=1}^{[ns]} \mathbb{Z}_t$ is true. As applications we also discuss functional central limit theorems for linear processes generated by martingale difference and negatively associated random vectors.

2. Main results

THEOREM 2.1. *Let $\mathbb{X}_t$ satisfy model* (1.1) *and $d(n)$ be a positive constant sequence satisfying that $d(n) \longrightarrow \infty$ as $n \longrightarrow \infty$. Assume that $\{A_j\}$ satisfies* (1.2) *and $\{\mathbb{Z}_t\}$ is any random vector sequence satisfying*

$$\sup_j E \max_{1 \leq m \leq n} \|\sum_{k=1}^{m} \mathbb{Z}_{k+j}\|^2 \leq C d^2(n) \text{ for every } n \geq 1 \tag{2.1}$$

and, as $n \longrightarrow \infty$,

$$\frac{1}{d(n)} \max_{-n \leq k \leq n} \|\mathbb{Z}_k\| \longrightarrow^p 0. \tag{2.2}$$

Then,

$$\frac{1}{d(n)} \sum_{t=1}^{k_n(s)} \mathbb{Z}_t \Rightarrow W^m(s) \text{ implies } \frac{1}{d(n)} \sum_{t=1}^{k_n(s)} \mathbb{X}_t \Rightarrow BW^m(s), \tag{2.3}$$

where $k_n(s) = \sup\{m : d^2(m) \leq s d^2(n)\}$ and $B = \sum_{k=0}^{\infty} A_k$.

Theorem 2.1 can be applied to many important cases, such as whether innovation $\{\mathbb{Z}_t\}$ is martingale difference or negatively associated sequence. In the following, we will derive corollaries of Theorem 2.1. We note that Corollary 2.3 below is Theorem 1(i) of [6] and Corollary 2.8 is a new result.

DEFINITION 2.2. Let $\{\mathbb{Z}_k\}$ be a random vector sequence. We say that $\{\mathbb{Z}_k\}$ is a martingale difference sequence if $E(\mathbb{Z}_k|\mathcal{F}_{k-1}) = \mathbf{0}$, a.s. $k = 0, \pm 1, \pm 2, \dots$, where $\mathcal{F}_k = \sigma\{\mathbb{Z}_i, i \leq k\}$.

COROLLARY 2.3. *Define X_t as in (1.1) and ξ_n as in (1.4), respectively. Let $\{\mathbb{Z}_t\}$ be a sequence of m-dimensional martingale difference vectors with $E(\mathbb{Z}_t|\mathcal{F}_{t-1}) = \mathbf{0}$ a.s. and Γ_t denote the conditional covariance matrix of $\mathbb{Z}_t$, $E(\mathbb{Z}_t\mathbb{Z}_t'|\mathcal{F}_{t-1}) = \Gamma_t$ a.s., such that $\frac{1}{n}\sum_{t=1}^{n}\Gamma_t \to^p \Gamma$, where $\mathcal{F}_t$ is sub-σ-algebra generated by $\mathbb{Z}_u, u \leq t$ and the prime denotes transpose and Γ is a positive definite(d.f.) non random matrix. Assume that $\sup_t E\|\mathbb{Z}_t\|^2 < \infty$ and $\frac{1}{n}\sum_{t=1}^{n} E(\mathbb{Z}_t\mathbb{Z}_t' I(\mathbb{Z}_t\mathbb{Z}_t' > n\epsilon)|\mathcal{F}_{t-1}) \to^p 0$ as $n \to \infty$ for every $\epsilon > 0$, where $I(\cdot)$ denotes the indicator function. Then, $\xi_n \Rightarrow W^m$, where W^m is a Wiener measure with covariance matrix $T = (\sum_{j=0}^{\infty} A_j)\Gamma(\sum_{j=0}^{\infty} A_j)'$.*

PROOF. Define for $n \geq 1$ the stochastic process η_n by

$$\eta_n(s) = n^{-\frac{1}{2}} \sum_{i=1}^{[ns]} \mathbb{Z}_i, \ 0 \leq s \leq 1. \tag{2.4}$$

It follows from the multivariate version of Theorem 1 of [2] or Theorem 2 of [1] that $\eta_n(s)$ converges weakly to Wiener measure with covariance matrix Γ (c.f. Theorem 3.1 of [8]). On the other hand, it follows from Doob's maximal inequality and $\sup_t E\|\mathbb{Z}_t\|^2 < \infty$ that for every $n \geq 1$

$$\sup_j E \max_{1 \leq m \leq n} (\sum_{k=1}^{m} \mathbb{Z}_{k+j})^2 \leq C_1 n \sup_j \sup_k E\|\mathbb{Z}_{k+j}\|^2 \leq C_2 n \tag{2.5}$$

and

$$\frac{1}{\sqrt{n}} \max_{-n \leq k \leq n} \|Z_k\| \to^p 0. \tag{2.6}$$

Hence, corollary 2.3 follows immediately from Theorem 2.1 with $d(n) = \sqrt{n}$. □

DEFINITION 2.4. Let $\{\mathbb{Z}_i, 1 \leq i \leq n\}$ be a sequence of m-dimensional random vectors. They are said to be negatively associated(NA) for every pair of disjoint subsets A and B of $\{1, \dots, n\}$ $\mathrm{Cov}(f(\mathbb{Z}_i, i \in A), g(\mathbb{Z}_j, j \in B)) \leq 0$ whenever f and g are coordinatewise increasing and the covariance exists. An infinite family is negatively associated if every finite subfamily is negatively associated.

LEMMA 2.5. *Let $r \geq 2$ and let $\{\mathbb{Z}_i, 1 \leq i \leq n\}$ be a sequence of m-dimensional negatively associated random vectors with $E\mathbb{Z}_i = \mathbf{0}$ and*

$E\|\mathbb{Z}_i\|^r < \infty$, where $\|\mathbb{Z}_i\| = (Z_{i1}^2 + \cdots + Z_{im}^2)^{\frac{1}{2}}$. Then there exists a constant $0 < A_r < \infty$ such that

$$E \max_{1\le k\le n} \|\sum_{i=1}^{k} \mathbb{Z}_i\|^r \le A_r m^r \{(\sum_{i=1}^{n} E\|\mathbb{Z}_i\|^2)^{\frac{r}{2}} + \sum_{i=1}^{n} E\|\mathbb{Z}_i\|^r\}. \tag{2.7}$$

PROOF. Note that

$$\max_{1\le k\le n} \|\sum_{i=1}^{k} \mathbb{Z}_i\| \le \sum_{j=1}^{m} \max_{1\le k\le n} |\sum_{i=1}^{k} Z_{ij}| \tag{2.8}$$

and by the result in [11] we have

$$\begin{aligned} E \max_{1\le k\le n} |\sum_{i=1}^{k} Z_{ij}|^r &\le A_r \{(\sum_{i=1}^{n} E(Z_{ij})^2)^{\frac{r}{2}} + \sum_{i=1}^{n} E|Z_{ij}|^r\} \\ &\le A_r \{(\sum_{i=1}^{n} E\|Z_i\|^2)^{\frac{r}{2}} + \sum_{i=1}^{n} E\|Z_i\|^r\}. \end{aligned} \tag{2.9}$$

Hence, from (2.8) and (2.9) equation (2.7) follows. □

LEMMA 2.6. *Let $\{\mathbb{Z}_i, 1 \le i \le n\}$ be a sequence of m-dimensional negatively associated random vectors with $E(\mathbb{Z}_i) = \mathbf{0}$ and $E\|\mathbb{Z}_i\|^2 < \infty$. Then for all $x > 0$ and $a > 0$,*

$$\begin{aligned} &P(\max_{1\le k\le n} \|\sum_{i=1}^{k} \mathbb{Z}_i\| \ge mx) \\ &\le 2mP(\max_{1\le k\le n} \|\mathbb{Z}_k\| > a) + 4m \exp(-\frac{x^2}{8\sum_{i=1}^{n} E\|\mathbb{Z}_i\|^2}) \\ &\quad + 4m(\frac{\sum_{i=1}^{n} E\|\mathbb{Z}_i\|^2}{4(xa + \sum_{i=1}^{n} E\|\mathbb{Z}_i\|^2)})^{x/(12a)}. \end{aligned} \tag{2.10}$$

PROOF. From (2.8) and the result of [11], (2.10) follows easily. □

THEOREM 2.7. *Let $\{\mathbb{Z}_i, i \ge 1\}$ be a strictly stationary sequence of m-dimensional negatively associated random vectors with $E(\mathbb{Z}_1) = \mathbf{0}$ and $E\|\mathbb{Z}_1\|^2 < \infty$. Define, for $t \in [0,1]$, $n \ge 1$ $\xi_n(t) = n^{-\frac{1}{2}} \sum_{i=1}^{[nt]} \mathbb{Z}_i$.*

If $E\|\mathbb{Z}_1\|^2 + 2\sum_{i=2}^{\infty}\sum_{j=1}^{m} E(Z_{1j}Z_{ij}) = \sigma^2 < \infty$, then, as $n \to \infty$, $\xi_n \Rightarrow B^m$, where B^m is an m-dimensional Wiener measure with covariance matrix $\Gamma = (\sigma_{kj})$ and $\sigma_{kj} = E(Z_{1k}Z_{1j}) + \sum_{i=2}^{\infty}[E(Z_{1k}Z_{ij}) + E(Z_{1j}Z_{ik})]$.

PROOF. By means of the simple device due to Cramer Wold (see [3], [4]), from the Newman's central limit theorem for negatively associated random variables(see [9]) we obtain $n^{-\frac{1}{2}}\sum_{i=1}^{n}\mathbb{Z}_i \to^D N(\mathbf{0},\Gamma)$, where $N(\mathbf{0},\Gamma)$ denotes an m-dimensional normal random vector and the symbol $\to^D$ indicates convergence in distribution. Hence, as in the proof of Theorem 2 of [5] on weakly associated random vectors, the limit point of $\xi_n(\cdot)$ is an m-dimensional Wiener measure with covariance matrix $\Gamma = (\sigma_{kj})$. It remains to verify the tightness of $\xi_n(\cdot)$ (see Theorem 15.1 of [3]). By Theorem 8.4 of [3] we only need to show that for any $\epsilon > 0$, there exist a positive number λ and an integer n such that for every $n \geq n_0$

$$P(\max_{1\leq k\leq n}\|\sum_{i=1}^{k}\mathbb{Z}_i\| > \lambda n^{\frac{1}{2}}) \leq m^3\epsilon\lambda^{-2}. \tag{2.11}$$

Applying Lemma 2.6 with $\lambda = m\lambda^{'}$, $x = \lambda^{'}n^{\frac{1}{2}}$ and $a = \lambda^{'}n^{\frac{1}{2}}/48$

$$\begin{aligned}
& P(\max_{1\leq k\leq n}\|\sum_{i=1}^{k}\mathbb{Z}_i\| > \lambda n^{\frac{1}{2}}) \\
= \ & P(\max_{1\leq k\leq n}\|\sum_{i=1}^{k}\mathbb{Z}_i\| > m\lambda^{'} n^{\frac{1}{2}}) \\
\leq \ & 2mP(\max_{1\leq k\leq n}\|\mathbb{Z}_k\| > \lambda^{'}n^{\frac{1}{2}}/48) \\
& +4m\exp(-\frac{\lambda^{'2}n}{8nE\|\mathbb{Z}_1\|^2}) + 4m(\frac{nE\|\mathbb{Z}_1\|^2}{4(nE\|\mathbb{Z}_1\|^2+\lambda^{'2}n/48)})^4 \\
\leq \ & 2m(48)^2\lambda^{'-2}E\|\mathbb{Z}_1\|^2 I\{\|\mathbb{Z}_1\| > \lambda^{'}n^{\frac{1}{2}}/48\} \\
& +4m\exp(-\frac{\lambda^{'2}}{8E\|\mathbb{Z}_1\|^2}) + 4m(\frac{12E\|\mathbb{Z}_1\|^2}{\lambda^{'2}})^4 \\
\leq \ & m\epsilon\lambda^{'-2} = m^3\epsilon\lambda^{-2}
\end{aligned}$$

provided that λ is sufficiently large. This proves (2.11), and hence the proof of Theorem 2.7 is complete. □

COROLLARY 2.8. *Let $\{\mathbb{Z}_i, i \geq 1\}$ be a strictly stationary negatively associated sequence of m-dimensional random vectors centered at expectations and $E\|\mathbb{Z}_1\|^2 < \infty$ and $\mathbb{X}_t$ be defined as in* (1.1). *Let the stochastic process ξ_n be defined as in* (1.4). *Assume* (1.2) *and $E\|\mathbb{Z}_1\|^2 + 2\sum_{i=2}^{\infty}\sum_{j=1}^{m}E(Z_{1j}Z_{ij}) = \sigma^2 < \infty$ hold. Then $\xi_n \Rightarrow W^m$.*

PROOF. First note that $\xi_n(s) = n^{-\frac{1}{2}}\sum_{i=1}^{[ns]}\mathbb{Z}_i$ converses weakly to Wiener measure B^m with covariance matrix Γ by Theorem 2.7. On the other hand, it follows from Lemma 2.5 and the condition $E\|Z_1\|^2 < \infty$ that (2.5) and (2.6) hold. Hence, Corollary 2.8 follows immediately from Theorem 2.1 with $d(n) = \sqrt{n}$. □

3. Proof of Theorem 2.1

For every fixed $l \geq 1$, put

$$\mathbb{X}_{1j}^{(l)} = \sum_{k=0}^{l} A_k \mathbb{Z}_{j-k} \text{ and } \mathbb{X}_{2j}^{(l)} = \sum_{k=l+1}^{\infty} A_k \mathbb{Z}_{j-k}. \tag{3.1}$$

From the idea in [7] (p.320) we obtain that for any $m \geq 1$,

$$\begin{aligned}\sum_{j=1}^{m} \mathbb{X}_{1j}^{(l)} &= \sum_{j=1}^{m}\sum_{k=0}^{l} A_k \mathbb{Z}_{j-k} \\ &= \sum_{k=0}^{l} A_k \sum_{j=1}^{m} \mathbb{Z}_j + \sum_{s=1}^{l} \mathbb{Z}_{1-s} \sum_{j=s}^{l} A_j + \sum_{s=0}^{l-1} \mathbb{Z}_{m-s} \sum_{j=s+1}^{l} A_j \qquad (3.2)\\ &= \sum_{k=0}^{l} A_k \sum_{j=1}^{m} \mathbb{Z}_j + R(m,l), \text{ (say)}.\end{aligned}$$

Therefore, it follows that for every fixed $l \geq 1$,

$$\begin{aligned}\frac{1}{d(n)} \sum_{t=1}^{k_n(s)} \mathbb{X}_t &= (\sum_{k=0}^{l} A_k) \frac{1}{d(n)} \sum_{j=1}^{k_n(s)} \mathbb{Z}_j + \frac{1}{d(n)} R(k_n(s), l) \\ &\quad + \frac{1}{d(n)} \sum_{j=1}^{k_n(s)} \mathbb{X}_{2j}^{(l)}.\end{aligned} \tag{3.3}$$

By (3.3), Theorem 4.1 given in [3] (p.25) and noting that $\sum_{k=0}^{l} \|A_k\| \to B$ as $l \to \infty$, to prove (2.3), it suffices to show that for any $\delta > 0$,

$$\limsup_{n\to\infty} P\{ \sup_{0\leq t\leq 1} \|R(k_n(t), l)\| \geq \delta d(n)\} = 0, \tag{3.4}$$

for every fixed $l \geq 1$ and

$$\lim_{l\to\infty} \limsup_{n\to\infty} P\{ \sup_{0\leq t\leq 1} \| \sum_{j=1}^{k_n(t)} \mathbb{X}_{2j}^{(l)} \| \geq \delta d(n)\} = 0. \tag{3.5}$$

By condition (2.2) since $\sum_{k=0}^{\infty} \|A_k\| < \infty$, as $n \to \infty$,

$$\frac{1}{d(n)} \sup_{0\le t\le 1} \|R(k_n(s), l)\|$$

$$\le \frac{1}{d(n)} \max_{-l\le j\le n} \|\mathbb{Z}_j\| \sum_{s=0}^{l} (\sum_{j=s}^{l} \|A_j\| + \sum_{j=s+1}^{\infty} \|A_u\|) \to^P 0$$

and hence (3.4) holds.

Noting that $\sum_{j=1}^{m} \mathbb{X}_{2j}^{(l)} = \sum_{k=l+1}^{\infty} A_k \sum_{j=1}^{m} \mathbb{Z}_{j-k}$ for any $m \ge 1$, by applying Hölder inequality and (2.1), we have

$$E \sup_{1\le t\le 1} \| \sum_{j=1}^{k_n(t)} \mathbb{X}_{2j}^{(l)} \|^2 \le (\sum_{k=l+1}^{\infty} \|A_k\|)^2 E \max_{1\le m\le n} \| \sum_{j=1}^{m} \mathbb{Z}_{j-k}\|^2$$

$$\le Cd^2(n)(\sum_{k=l+1}^{\infty} \|A_k\|)^2.$$

Hence, (3.5) follows immediately from the Markov inequality and $\sum_{k=l+1}^{\infty} \|A_k\| \to 0$ as $l \to \infty$. The proof of Theorem 2.1 is complete. □

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