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A CHARACTERIZATION OF HYPERBOLIC TORAL AUTOMORPHISMS

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ABSTRACT. Let $L : C \to C$ be a hyperbolic automorphism. Then the hyperbolic toral automorphism $L_A : T^2 \to T^2$, induced by L, is a chaotic map ([2] pg.192). We characterize hyperbolic toral automorphisms by proving the converse of the above statement.

1. Introduction

Let $(2, 2, 2, 2)$ be ramification indices for the Riemann sphere. It is well known that the regular branched covering map corresponding to this, is the Weierstrass $\mathcal{P}$ function. Lattès [3] (See also [2] pg.291) gives a chaotic rational function $R(z) = \frac{z^4+\frac{1}{2}g_2z^2+\frac{1}{16}g_2^2}{4z^3-g_2z}$ on $\overline{C}$ which is induced by the Weierstrass $\mathcal{P}$ function and the linear map $L(z) = 2z$ on the complex plane C. Recently the author classified chaotic maps of the Riemann sphere $\bar{C}$ which are induced by regular branched coverings from T^2 onto $\bar{C}$ and the linear map $2z$ [4].

Let $L : C \to C$ be a hyperbolic automorphism. Then the hyperbolic toral automorphism $L_A : T^2 \to T^2$, which is induced by L, is a chaotic map ([2], pg.192).

Now let A be a 2×2 integer matrix with $|\det(A)| = 1$. If A is non-hyperbolic, then we have 3 cases for characteristic solutions λ of A : (1) λ's are complex non real numbers, (2) $\lambda = \pm 1$ or (3) $\lambda = 1$ or $\lambda = -1$ with multiple root. We characterize hyperbolic toral automorphisms by proving if A is non-hyperbolic, then L_A is not a chaotic map (Theorem 3.1, 3.2 and 3.4).

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2. Background and definitions

Let $f : M \to M$ be a map of the metric space M. A map $f : M \to M$ is *chaotic* if and only if f has sensitive dependence on initial conditions, f is topologically transitive and the periodic points are dense in M. Remark that f is *topologically transitive* if and only if for any pair of nonempty open sets $U, V \subset M$ there exists $k > 0$ such that $f^k(U) \cap V \neq \emptyset$. We refer to the reader [2] for a detailed definition and examples of chaotic map.

The following simple characterization of chaotic maps which is proved by Touhey [5], is very useful to prove whether a map is chaotic or not. For example, we can easily check that the inverse of a chaotic homeomorphism is also chaotic by this characterization. For the proof of the following proposition, he applied [1] which showed that sensitive dependence on initial conditions is implied by the remaining two conditions.

PROPOSITION 2.1. [5] *A map $f : M \to M$ is chaotic if and only if for all non-empty open sets U and V of M, f has a periodic orbit Γ such that $\Gamma \cap U \neq \emptyset$ and $\Gamma \cap V \neq \emptyset$.*

Let Λ be the lattice induced by $w_1, w_2 \in C$ with $w_1/w_2 \notin R$. Let π be the identification map of C such that $\pi(z) = \pi(z + nw_1 + mw_2)$ for $n, m \in Z$. Then we have the torus T^2 induced by π. In particular if $w_1 = 1$ and $w_2 = i$, then we call Λ the square lattice, which we will use later. Now let $f : C \to C$ be a function such that $f(z) - f(z + nw_1 + mw_2)$ belongs to the lattice points for all points $z \in C$. It follows that $\pi \circ f(z) = \pi \circ f(z + nw_1 + mw_2)$ and therefore f induces a well defined map $\tilde{f} : T^2 \to T^2$ with $\tilde{f}(\pi(z)) = \pi \circ f(z)$ such that the following diagram commutes.

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "\pi"'] & C \arrow[d, "\pi"] \\ T^2 \arrow[r, "\tilde{f}"] & T^2 \end{tikzcd}$$

Let $L : C \to C$ be a linear map whose matrix representation is an integer matrix A. Then $\tilde{L}$ is clearly well-defined on T^2 which is induced by the square lattice. We call $\tilde{L}$ a *toral automorphism*, denoted by L_A.

DEFINITION 2.1. Let $L(x) = Ax$, where A is a 2×2 matrix satisfying
1. All entries of A are integers.

2. $\det(A) = \pm 1$.
3. A is hyperbolic, i.e., A has no eigenvalues on unit circle. Then we call $L_A : T^2 \to T^2$ a *hyperbolic toral automorphism.*

PROPOSITION 2.2. ([2], pg.192) *Let L_A be a hyperbolic toral automorphism of T^2. Then L_A is a chaotic map.*

Let $f : X \to X$ and $g : Y \to Y$ be two maps. f and g are said to be *topologically semi-conjugate* if and only if there exists an onto map $h : X \to Y$ such that $h \circ f = g \circ h$. We call h a topological semi-conjugacy. Then we can easily prove that if $f : X \to X$ is chaotic then $g : Y \to Y$ is also chaotic using Proposition 2.1.

3. A characterization of hyperbolic toral automorphisms

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an integer matrix with $\det(A) = \pm 1$ and let λ be the characteristic solutions of A. Then $\lambda = \frac{(a+d) \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2}$. Here, characteristic solutions of A mean the roots of characteristic polynomial of A.

We now suppose that L_A is not a hyperbolic toral automorphism; i.e., A does not have real eigenvalues which are not in the unit circle. Then we have 3 different cases:
(1) characteristic solutions are complex numbers,
(2) characteristic solutions are ± 1, or
(3) characteristic solutions are 1 or -1 with multiple root.

In this section, we characterize the hyperbolic toral automorphisms by proving that the map L_A can not be chaotic in any of the above 3 cases (Theorem 3.1, 3.2 and 3.4).

3.1. Case (1). Characteristic solutions are complex numbers

LEMMA 3.1. *Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an integer matrix with $\det(A) = \pm 1$. If a characteristic solution of A is a complex number then it has norm 1. Moreover the two solutions are conjugate.*

PROOF. Let λ be a complex characteristic solution of A. Then $ad - bc = 1$ and $(a+d)^2 < 4$ by the formula of λ. Hence $\lambda = \frac{a+d}{2} \pm \frac{\sqrt{4-(a+d)^2}}{2} i$. Therefore $|\lambda| = 1$ and two characteristic solutions are conjugate. $\square$

Remark that if $\det(A) = -1$, then A has real eigenvalues from the formula of λ. But the converse is not true in general. For example

$A = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}$ has a real characteristic solution with multiple root, but $\det(A) = 1$. In general, matrices in Case (3) have a real characteristic solution but $\det(A) = 1$.

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an integer matrix with $\det(A) = \pm 1$. Recall that if characteristic solutions λ are complex numbers then $ad - bc = 1$ and $(a+d)^2 < 4$. Therefore $a + d = 0,\ 1$ or -1. Then $a + d$ is 0, 1 or -1 if and only if λ is $\pm i$, $\frac{1}{2} \pm \frac{\sqrt{3}i}{2}$ or $-\frac{1}{2} \pm \frac{\sqrt{3}i}{2}$ respectively.

PROPOSITION 3.1. *Let* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *be an integer matrix with* $\det(A) = \pm 1$. *If* A *has complex characteristic solutions* $\pm i$, *then* $L_A : T^2 \to T^2$ *is not a chaotic map.*

PROOF. If A has a complex characteristic solutions $\pm i$, then A has the form $\begin{pmatrix} n & b \\ c & -n \end{pmatrix}$. Now we can easily check that A is periodic with period 4 since $\det(A) = 1$. Consequently $L_A : T^2 \to T^2$ can not be chaotic. $\square$

PROPOSITION 3.2. *Let* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *be an integer matrix with* $\det(A) = \pm 1$. *If* A *has complex characteristic solutions* $\frac{1}{2} \pm \frac{\sqrt{3}i}{2}$ *or* $-\frac{1}{2} \pm \frac{\sqrt{3}i}{2}$, *then* $L_A : T^2 \to T^2$ *is not a chaotic map.*

PROOF. We can prove the proposition by matrix multiplication. Note that we have two cases, $a + d = 1$ or $a + d = -1$ by the formula of λ. Consider the case $a + d = 1$. Then the (2,1) component of A^3 is $a(ac + cd) + c(bc + d^2)$. Recall that $\det(A) = 1$ since A has complex characteristic solutions. Then $a(ac+cd)+c(bc+d^2) = 0$ by substituting $d = 1 - a$ and $bc = ad - 1$. We also have the (1,2) component of A^3, $b(a^2 + bc) + d(ab + bd) = 0$ by substituting $d = 1 - a$ and $bc = ad - 1$.

Similarly, we have the (1,2) and (2,1) components of A^3 are 0 in case $a + d = -1$ by direct computation.

Note that $\det(A^3) = 1$, therefore $A^3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. Consequently A is periodic with period 3 or 6 respectively. Consequently $L_A : T^2 \to T^2$ can not be chaotic. $\square$

We now state one result from Proposition 3.1 and Proposition 3.2 in Case (1).

THEOREM 3.1. *Let A be a 2×2 integer matrix with* $\det(A) = \pm 1$. *If A has complex characteristic solutions, then the induced toral automorphism $L_A : T^2 \to T^2$ is not chaotic.*

3.2. Case (2). Characteristic solutions are ± 1

Let A be a 2×2 integer matrix with $\det(A) = \pm 1$ and let $\lambda = \pm 1$ be characteristic solutions of A. If $\det(A) = 1$, then $(a+d)^2 - 4 > 0$ from the formula of λ. Then λ can not be ± 1. We now suppose that $\det(A) = -1$. Then $\lambda = \frac{(a+d) \pm \sqrt{(a+d)^2+4}}{2}$. Consequently $\lambda = \pm 1$ if and only if $a + d = 0$ and $\det(A) = -1$.

THEOREM 3.2. *Let* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *be an integer matrix with* $\det(A) = \pm 1$ *and let $\lambda = \pm 1$ be the characteristic solutions of A. Then $L_A : T^2 \to T^2$ is not chaotic.*

PROOF. Note that A has the form $A = \begin{pmatrix} n & b \\ c & -n \end{pmatrix}$. Since $\det(A) = -1$, $n^2 + bc = 1$. Hence $A^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and therefore A is periodic with period 2. We also can prove the theorem as follows: Since ± 1 are eigenvalues, there exists an invertible matrix P such that $P^{-1}AP = D$, where $D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Then A is periodic with period 2. Consequently the induced toral automorphism $L_A : T^2 \to T^2$ is not chaotic. □

3.3. Case (3). Characteristic solutions are 1 or -1 with multiple root

We will show that we can construct three disjoint simple closed curves which are fixed or invariant such that one of the simple closed curves maps onto itself in Case (3) [Proposition 3.3 and Corollary 3.1]. Then we will show that the induced map L_A in Case (3) does not satisfy topological transitivity and therefore it can not chaotic [Theorem 3.3 and Corollary 3.2].

3.3.1. *Characteristic solution is 1 with multiple root.* Let A be a matrix whose characteristic solution is 1 with multiple root. Then the matrix A satisfies $a + d = 2$ and $\det(A) = 1$ from the characteristic solutions formula. Note that if $\lambda = 1$ with multiple root and its eigen space is R^2, then $Az = z$. Therefore $A = I$.

We now look at the following examples. Each examples have different number of disjoint simple closed curves as the fixed point set under the induced map L_A.

EXAMPLE 1. Let $A = \begin{pmatrix} 4 & -9 \\ 1 & -2 \end{pmatrix}$. Then $\lambda = 1$ and its eigen space is $(3t, t)$. Therefore $L_A : T^2 \to T^2$ has the fixed point set S_1, where S_1 is the simple closed curve in the identification space T^2 induced by eigen vector $(3t, t)$, i.e., induced by $y = \frac{1}{3}x$. Note that S_1 is the only fixed point set.

EXAMPLE 2. Let $A = \begin{pmatrix} 9 & 16 \\ -4 & -7 \end{pmatrix}$. Then L_A has the fixed point sets which is four disjoint union of simple closed curves. Moreover they are induced by $y = -\frac{1}{2}x$, $y = -\frac{1}{2}x + \frac{1}{8}$, $y = -\frac{1}{2}x + \frac{1}{4}$ and $y = -\frac{1}{2}x + \frac{3}{8}$ in the identification space T^2. Remark that the simple closed curve induced by $y = -\frac{1}{2}x + \frac{1}{2}$, $y = -\frac{1}{2}x + \frac{5}{8}$, $y = -\frac{1}{2}x + \frac{3}{4}$ and $y = -\frac{1}{2}x + \frac{7}{8}$ are the same simple closed curves induced by $y = -\frac{1}{2}x$, $y = -\frac{1}{2}x + \frac{1}{8}$, $y = -\frac{1}{2}x + \frac{1}{4}$ and $y = -\frac{1}{2}x + \frac{3}{8}$ respectively. We will explain later why the simple closed curves induced by those lines are the fixed point sets.

EXAMPLE 3. Let $A = \begin{pmatrix} -2 & 3 \\ -3 & 4 \end{pmatrix}$. Then $L_A : T^2 \to T^2$ has three disjoint fixed point sets, each of which is homeomorphic to a simple closed curve, induced by (t, t), $(t, t + \frac{1}{3})$ and $(t, t + \frac{2}{3})$. And those three disjoint simple closed curves are the only fixed point sets.

In general, we can find the fixed point sets by solving $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+m \\ y+n \end{pmatrix}$ (equivalently $\begin{pmatrix} a-1 & b \\ c & d-1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} m \\ n \end{pmatrix}$), where m, n are integers.

Then $(a-1)x+by = k(cx+(d-1)y)$ or $k((a-1)x+by) = cx+(d-1)y$, where k is a rational number.

Case 1: $k = 0$.

Since $k = 0$ we have two cases; $(a-1)x + by = 0$ or $cx + (d-1)y = 0$. Then the matrix is $\begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix}$ or $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ respectively since $a + d = 2$ and $\det(A) = 1$. We denote those matrices A and B respectively.

We consider the matrix A first. Let $\begin{pmatrix} 0 & 0 \\ n & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} m_1 \\ m_2 \end{pmatrix}$, where m_1 and m_2 are integers. Then the solution of the above equations,

$x = \frac{m_2}{n}$, is the fixed point sets. Therefore if $n \geq 3$, then we have more than two disjoint simple closed curves as the fixed point set.

Now we consider the case $n = 1$ and $n = 2$. In case $n = 1$ we can easily check that each simple closed curves S_2 and S_3 induced by $x = \frac{1}{2}$ and $x = \frac{1}{3}$ respectively is invariant under L_A. Therefore we have three simple closed curves such that the simple closed curve S_1, induced by $x = 0$ is fixed and S_2 and S_3 are invariant.

In case $n = 2$ the simple closed curves S_1 and S_2 induced by $x = 0$ and $x = \frac{1}{2}$ respectively are the fixed point sets. Now we also can check that the simple closed curve S_3 induced by $x = \frac{1}{3}$ is invariant under L_A.

We now consider when the matrix is B. Then, by the same argument as the matrix A, we have the fixed point sets induced by $y = \frac{m_1}{n}$. Moreover if $n = 1$, then we have two simple closed curves induced by $y = \frac{1}{2}$ and $y = \frac{1}{3}$ each of which is invariant under L_B (if $n = 2$, then the simple closed curves induced by $y = \frac{1}{3}$ is invariant under L_B). Consequently we have three disjoint simple closed curves each of which is fixed or invariant under the induced map L_B.

Case 2: $k \neq 0$.

It suffice to consider when $(a-1)x + by = k(cx + (d-1)y)$. As shown in the examples, $cx + (d-1)y = n$ is the fixed point set or invariant set in the identification space T^2. In fact, $\begin{pmatrix} a-1 & b \\ c & d-1 \end{pmatrix} \begin{pmatrix} x \\ \frac{c}{1-d}x + \frac{n}{d-1} \end{pmatrix} = \begin{pmatrix} \frac{bn}{d-1} \\ n \end{pmatrix}$ since $a + d = 2$ and $\det(A) = 1$. Therefore if $\frac{bn}{d-1}$ is an integer, then the simple closed curve induced by $y = \frac{c}{1-d}x + \frac{n}{d-1}$ is the fixed point set, otherwise the simple closed curve is invariant for $1 \leq n < |1-d|$.

Now we consider the case when the number of simple closed curves induced by the above one or two.

Note that $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ \frac{c}{1-d}x + \frac{1}{q} \end{pmatrix} = \begin{pmatrix} x + \frac{b}{q} \\ \frac{c}{1-d}(x + \frac{b}{q}) + \frac{1}{q} \end{pmatrix}$ for $q \geq 2$ and $q \in Z$ by $a + d = 2$ and $\det(A) = 1$. Therefore the simple closed curve induced by $\frac{c}{1-d}x + \frac{1}{q}$ for $q \geq 2$ and $q \in Z$ is invariant set under L_A. Moreover the simple closed curve induced by $\frac{c}{1-d}x + \frac{m}{q}$ for $q \geq 2$ and $q \in Z$ and $m < q$ is also invariant set under L_A by the same argument as the above.

Consequently we have more than one disjoint simple closed curves each of them is invariant under L_A when the fixed point sets is one simple closed curve or disjoint union of two simple closed curves.

We now state the above arguments as proposition.

PROPOSITION 3.3. *Let* $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *be an integer matrix with* $\det(A) = \pm 1$ *and let the characteristic solution be 1 with multiple root. Then the fixed point set of the induced toral automorphism* $L_A : T^2 \to T^2$ *is a simple closed curve or a disjoint union of simple closed curves, which are parallel, in the identification space* T^2*. Moreover if the fixed point sets is a simple closed curve or disjoint union of two simple closed curves, then we can find more than one simple closed curves, parallel to the fixed simple closed curves, each of which is invariant under* L_A.

3.3.2. *Characteristic solution is* -1 *with multiple root.* Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an integer matrix with $\det(A) = \pm 1$ and let the characteristic solution be -1 with multiple root. We will show that there exist more than two simple closed curves which are invariant. Moreover one of the invariant simple closed curve S_1 maps onto itself by L_A, i.e., $L_A(S_1) = S_1$.

Note that $-A$ has eigenvalue 1 with multiple root. Therefore the induced toral automorphism $L_{(-A)} : T^2 \to T^2$ has fixed point sets which is a simple closed curve or finite disjoint union of simple closed curves by Proposition 3.3. Then the fixed point sets of $L_{(-A)}$ are invariant set under the map L_A. In fact, let S_i be the fixed point set. Then $L_A(S_i) = S_i$ or $L_A(S_i) = S_j$ and $L_A(S_j) = S_i$, since $L_A(z) = -z$ for elements of the fixed point sets of $L_{(-A)}$.

EXAMPLE 4. Let $A = \begin{pmatrix} 2 & -3 \\ 3 & -4 \end{pmatrix}$. Then $-A$ is just Example 3. So $-A$ has fixed point sets, which is a disjoint union of simple closed curves induced by (t, t), $(t, t + \frac{1}{3})$ and $(t, t + \frac{2}{3})$ denoted by S_1, S_2 and S_3 respectively. Then this fixed points set of L_{-A} is invariant set under L_A . Note that $L_A(S_1) = S_1$, $L_A(S_2) = S_3$ and $L_A(S_3) = S_2$.

We will only consider when $-A$ has the fixed point set which is one simple closed curve or two disjoint union of simple closed curves to get three disjoint union of simple closed curves, which are invariant, since if $-A$ have more than two simple closed curves as the fixed point set, then they are invariant and the simple closed curve induced by the line, passing through the origin, maps onto itself by L_A.

Case 1′: When $-A$ is Case 1.

The matrix has the form $A = \begin{pmatrix} -1 & 0 \\ -n & -1 \end{pmatrix}$ or $\begin{pmatrix} -1 & -n \\ 0 & -1 \end{pmatrix}$ in this case. We will only consider when $n = 1$ or $n = 2$. As we have shown, when $A = \begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix}$, the simple closed curves S_2 and S_3 induced by $x = \frac{1}{2}$ and $x = \frac{1}{3}$ are invariant under L_{-A}. Then we can compute $L_A(S_2) = S_2$ and $L_A(S_3) = S_4$, where S_4 is the simple closed curve induced by $x = \frac{2}{3}$. Recall that S_1, which is the fixed point set of L_{-A}, maps onto itself by L_A since S_1 is induced by y axis. Similarly, we have same result when $n = 2$.

Now let $A = \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}$. Then we can find invariant sets S_1, S_2, S_3 and S_4 which are induced by $y = 0, y = \frac{1}{2}, y = \frac{1}{3}$ and $y = \frac{2}{3}$ respectively such that $L_A(S_1) = S_1$, $L_A(S_2) = S_2$, $L_A(S_3) = S_4$ and $L_A(S_4) = S_3$. We also have same result when $n = 2$.

Case 2′: When $-A$ is Case 2.

Let A be the matrix such that $-A$ is the matrix in Case 2. It suffice to consider when $-A$ have one or two disjoint simple closed curves as the fixed point set. Let S_1 be the simple closed curve induced by the line passing through the origin. Then $L_A(S_1) = S_1$. Now let $y = \frac{1}{k}x + \frac{1}{q}$ be invariant set which induces a simple closed curve S_2 under the induced toral automorphism L_{-A}, where $k = \frac{a-1}{c}$ is rational number and $q > 2$ is an integer. Then $L_A(S_2) = S_3$ and $L_A^2(S_2) = S_2$, where the simple closed curve S_3 is induced by $y = \frac{1}{k}x - \frac{1}{q}$ (equivalently $y = \frac{1}{k}x + \frac{q-1}{q}$). Consequently we have three disjoint simple closed curve such that one of them maps onto itself.

Corollary 3.1. *Let* $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *be an integer matrix with* $\det(A) = \pm 1$ *and let the characteristic solution be* -1 *with multiple root. Then there exist at least three finite disjoint union of simple closed curves which are invariant. In particular the simple closed curve induced by the line passing through the origin maps onto itself by* L_A. *Moreover those simple closed curves are parallel in the identification space* T^2.

Theorem 3.3. *Let* $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *be an integer matrix with* $\det(A) = \pm 1$ *and let the characteristic solution be 1 with multiple root. If the fixed point sets of* $L_A : T^2 \to T^2$ *in Proposition 3.3 is more than two disjoint simple closed curves, then* L_A *is not chaotic.*

Proof. It suffices to consider when the fixed point set of L_A is three disjoint simple closed curves S_1, S_2 and S_3. Since S_1, S_2 and S_3 are parallel, $T^2 - \cup_{i=1}^{i=3} S_i$ are three components, denoted by D_1, D_2 and D_3, where D_1 is the component between S_1 and S_2, D_2 is the component between S_2 and S_3, and D_3 is the component between S_3 and S_1. Let

$D* = D_1 \cup D_3 \cup S_1$. Then $D*$ and D_2 are disjoint components whose common boundary is $S_2 \cup S_3$.

Now let U be a small open set in D_2 and let $V \subset D*$ be a small open neighborhood of $x \in S_1$ such that U and V do not intersects S_2 or S_3. Consider $L_A^n(V)$ for $n \in N$. Note that S_i's are the fixed point sets. Therefore if there exists n such that $L_A^n(V) \cap U \neq \emptyset$, then $L_A^n(V)$ must intersect S_2 or S_3. But this is impossible. In fact, if $y \in L_A^n(V)$ with $y \in S_2 \cup S_3$ then $y = L_A^{-n}(y) \in V$. This contradicts for the choice of V.

Consequently $L_A : T^2 \to T^2$ is not topologically transitive and thereforeL_A is not chaotic. $\square$

We now show that we have the same result as Theorem 3.3 when the invariant set is disjoint union of more than two simple closed curves and one of them maps onto itself. We state and prove this fact as corollary, whose proof is basically same as Theorem 3.3.

COROLLARY 3.2. *Let* $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *be an integer matrix with* $\det(A) = \pm 1$ *and let the characteristic solution be 1 or* -1 *with multiple root. If the invariant set (including fixed point set) of* $L_A : T^2 \to T^2$ *in Proposition 3.3 and Corollary 3.1 is more than two disjoint simple closed curves such that one of them maps onto itself, then* L_A *is not chaotic.*

PROOF. It suffices to consider when the invariant set is three disjoint simple closed curves S_1, S_2 and S_3. Let S_1 be the simple closed curve induced by the line passing through the origin. Recall that S_1 is invariant, i.e., $L_A(S_1) = S_1$. Now let V be a small open set containing $x \in S_1$ and U an open set in the component between S_2 and S_3 such that U and V does not intersect S_2 or S_3.

Suppose that there exists n such that $L_A^n(V) \cap U \neq \emptyset$. Then $L_A^n(V)$ must intersect S_2 or S_3. But this is impossible. In fact, if $y \in L_A^n(V)$ with $y \in S_2 \cup S_3$, then $L_A^{-n}(y) \in V$. But $L_A^{-n}(y) \in S_2 \cup S_3$. This contradicts for the choice of V.

Consequently $L_A : T^2 \to T^2$ is not topologically transitive and thereforeL_A is not chaotic. $\square$

We now state one of main results in Case (3) from Theorem 3.3 and Corollary 3.2.

THEOREM 3.4. *Let* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *be an integer matrix with* $\det(A) = \pm 1$ *and let* $\lambda = 1$ *or* $\lambda = -1$ *be characteristic solutions of* A *with multiple root. Then* $L_A : T^2 \to T^2$ *is not chaotic.*

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