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WEIGHTED COMPOSITION OPERATORS BETWEEN H^∞ AND BERGMAN TYPE SPACES

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ABSTRACT. In this paper, we study the boundedness and the compactness of weighted composition operator between H^∞ and Bergman type space on the unit ball of $\mathbb{C}^n$. Also, the norm of corresponding weighted composition operator is computed.

1. Introduction

Let $B = \{z \in \mathbb{C}^n : |z| < 1\}$ be the open unit ball in $\mathbb{C}^n$, $S = \partial B = \{z \in \mathbb{C}^n : |z| = 1\}$ be its boundary. Let dv denote the normalized Lebesgue area measure on the unit ball B such that $v(B) = 1$, and $d\sigma$ be the normalized rotation invariant measure on the boundary S of B such that $\sigma(S) = 1$. Denote by $H(B)$ the space of all holomorphic functions on B and $H^\infty = H^\infty(B)$ the space of all bounded holomorphic functions on B.

Let $\beta(z, w)$ denote the Bergman metric between two points z and w in B. It is well known that(see [6, 8])

$$\beta(z,w) = \frac{1}{2}\log\frac{1+|\varphi_z(w)|}{1-|\varphi_z(w)|},$$

where $\varphi_z(\cdot)$ is an automorphism of B. For any $a \in B$ and $r > 0$ the set

$$D(a,r) = \{z \in B : \beta(a,z) < r\}, \quad a \in B$$

is a Bergman metric ball with center at a and radius r.

To say that a positive continuous function ϕ on $[0, 1)$ is normal means that there exist positive numbers s and t $(0 < s < t)$, such that

$$\frac{\phi(r)}{(1-r)^s} \downarrow 0, \qquad \frac{\phi(r)}{(1-r)^t} \uparrow \infty$$

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as $r \to 1^-$ (see, for example, [2]). For $0 < p < \infty$, $0 < q < \infty$, and a normal function ϕ, let $H(p,q,\phi)$ denote the space of all holomorphic functions f on the unit ball such that

$$\|f\|_{H(p,q,\phi)} = \left(\int_0^1 M_q^p(f,r)\frac{\phi^p(r)}{1-r}r^{2n-1}dr\right)^{1/p} < \infty,$$

where, for $0 < r < 1$, $M_q(f,r)$ is defined by

$$M_q(f,r) = \left(\int_{\partial B} |f(r\zeta)|^q d\sigma(\zeta)\right)^{1/q}.$$

When $1 \le p < \infty$, $H(p,q,\phi)$ becomes a Banach space with the norm $\|f\|_{H(p,q,\phi)}$. When $0 < p < 1$, $H(p,q,\phi)$ is a Fréchet space with the quasi-norm $\|f\|_{H(p,q,\phi)}$. If $0 < p = q < \infty$, $H(p,p,\phi)$, simply denoted by $A^p(\phi)$, is the Bergman-type space, i.e.,

$$A^p(\phi) = \left\{f \in H(B) : \int_B |f(z)|^p \frac{\phi^p(|z|)}{1-|z|}dv(z) < \infty\right\}.$$

Especially, for $\phi(r) = (1-r)^{1/p}$, then $A^p(\phi)$ is the Bergman space A^p. See, for example [8], for some basics about Bergman spaces.

Let φ be a holomorphic self-map of B and $\psi \in H(B)$. Define a linear operator ψC_φ on $H(B)$, called a weighted composition operator, by

$$\psi C_\varphi f = \psi \cdot (f \circ \varphi), \tag{1}$$

where $f \in H(B)$. When $\psi = 1$, we have the composition operator C_φ defined by $C_\varphi f = f \circ \varphi$. When $\varphi(z) = z$, we obtain the multiplication M_ψ defined by $M_\psi f = \psi f$. Therefore the weighted composition operator can be regarded as a generalization of a multiplication operator and a composition operator. It is interesting to explore the connection between the function theoretic properties of φ and ψ and the behavior of ψC_φ on various spaces. See [1] for more information on this topic.

In [3], Ohno has characterized the boundedness and compactness of weighted composition operators between H^∞ and the Bloch space $\mathcal{B}$ in the unit disk. Weighted composition operator between Bloch-type spaces was investigated in [4]. Composition operator from the Hardy space into the Bloch space $\mathcal{B}$ was investigated in [5], to mention only a few related works.

In this paper we consider the weighted composition operators between H^∞ and Bergman type space $A^p(\phi)$ on B. Meanwhile, we will compute the norm of these weighted composition operators.

Throughout this paper, C stand for positive constants not depending on the functions being considered, which may differ from one occurrence to the other. The expression $A \asymp B$ means that $A/C \leq B \leq CA$.

2. Main result and proof

We give the main results and proofs in this section. In order to prove the main results, the following lemmas are needed.

LEMMA 2.1. *Let $0 < p < \infty$ and ϕ be normal on $[0,1)$. If $f \in A^p(\phi)$, then*

$$|f(z)| \leq C\frac{\|f\|_{A^p(\phi)}}{\phi(|z|)(1-|z|^2)^{n/p}}, \qquad z \in B.$$

PROOF. It is well known that (see [8])

$$\frac{1}{|1-\overline{a}z|^{n+1}} \asymp \frac{1}{(1-|z|^2)^{n+1}} \asymp \frac{1}{(1-|a|^2)^{n+1}} \asymp \frac{1}{|D(a,r)|} \tag{2}$$

when $z \in D(a,r)$, where $|D(a,r)|$ denotes the area of the disk $D(a,r)$.

From (2) and since $\phi(r)$ is normal it is not difficult to see that for a fixed $r \in (0,1)$ the following relationship holds (or see [2])

$$\phi(|z|) \asymp \phi(|a|), \quad z \in D(a,r).$$

For $0 < r < 1$ and $z \in B$, from the above statements and the subharmonicity of $|f(z)|^p$, we obtain

$$\begin{aligned} |f(z)|^p &\leq \frac{C}{(1-|z|^2)^{n+1}}\int_{D(z,r)}|f(w)|^p dv(w) \\ &\leq \frac{C}{(1-|z|^2)^n\phi^p(|z|)}\int_{D(z,r)}|f(w)|^p\frac{\phi^p(|w|)}{1-|w|}dv(w) \\ &\leq \frac{C}{(1-|z|^2)^n\phi^p(|z|)}\int_B\frac{\phi^p(|w|)}{1-|w|}|f(w)|^p dv(w) \\ &\leq \frac{C\|f\|^p_{A^p(\phi)}}{(1-|z|^2)^n\phi^p(|z|)}. \end{aligned}$$

The result is achieved. □

LEMMA 2.2. ([7]). *For $\beta > -1$ and $m > 1+\beta$ we have*

$$\int_0^1(1-r)^\beta(1-\rho r)^{-m}dr \leq C(1-\rho)^{1+\beta-m}, \qquad 0 < \rho < 1.$$

To characterize the compactness of weighted composition operator between H^∞ and $A^p(\phi)$, we will need the following lemma, which is an easy modification of the Proposition 3.11 of [1]. We omit the details of the proof.

LEMMA 2.3. *Let X and Y be $A^p(\phi)$ or H^∞. The operator $\psi C_\varphi : X \to Y$ is compact if and only if $\psi C_\varphi : X \to Y$ is bounded and for any bounded sequence $\{f_k\}_{k\in\mathbb{N}}$ in X which converges to zero uniformly on compact subsets of B, $\psi C_\varphi f_k \to 0$ in Y.*

THEOREM 2.1. *Let φ be a holomorphic self-map of B and $\psi \in H(B)$. Assume that $0 < p < \infty$ and ϕ is normal on $[0,1)$. Then $\psi C_\varphi : A^p(\phi) \to H^\infty$ is a bounded operator if and only if*

$$M = \sup_{z\in B} \frac{|\psi(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{\frac{n}{p}}} < \infty. \tag{3}$$

Furthermore, if $\psi C_\varphi : A^p(\phi) \to H^\infty$ is bounded, then

$$\|\psi C_\varphi\| \asymp \sup_{z\in B} \frac{|\psi(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{\frac{n}{p}}}. \tag{4}$$

PROOF. Suppose $\psi C_\varphi : A^p(\phi) \to H^\infty$ is a bounded operator. For $w \in B$, let

$$f_w(z) = \frac{(1-|\varphi(w)|^2)^{t+1}}{\phi(|\varphi(w)|)(1-\langle z, \varphi(w)\rangle)^{n/p+t+1}}.$$

By [2] or [6], we know that

$$M_p(f_w, r) \le C \frac{(1-|\varphi(w)|^2)^{t+1}}{\phi(|\varphi(w)|)(1-r|\varphi(w)|)^{t+1}}.$$

Since ϕ is normal, applying Lemma 2.2, we have

$$\begin{aligned}
\|f_w\|^p_{A^p(\phi)} &= \int_0^1 M_p^p(f_w, r)\frac{\phi^p(r)}{1-r} r^{2n-1} dr \\
&\le C\int_0^1 \frac{(1-|\varphi(w)|^2)^{p(t+1)}}{\phi^p(|\varphi(w)|)(1-r|\varphi(w)|)^{p(t+1)}} \frac{\phi^p(r)}{1-r} dr \\
&\le C\int_0^{|\varphi(w)|} \frac{(1-|\varphi(w)|^2)^{p(t+1)}}{\phi^p(|\varphi(w)|)(1-r|\varphi(w)|)^{p(t+1)}} \frac{\phi^p(r)}{1-r} dr \\
&\quad + \int_{|\varphi(w)|}^1 \frac{(1-|\varphi(w)|^2)^{p(t+1)}}{\phi^p(|\varphi(w)|)(1-r|\varphi(w)|)^{p(t+1)}} \frac{\phi^p(r)}{1-r} dr
\end{aligned}$$

$$\leq C\frac{(1-|\varphi(w)|^2)^{p(t+1)}}{\phi^p(|\varphi(w)|)}\frac{\phi^p(|\varphi(w)|)}{(1-|\varphi(w)|^2)^{pt}}\int_0^{|\varphi(w)|}\frac{(1-r)^{pt-1}}{(1-r|\varphi(w)|)^{p(t+1)}}dr$$
$$+C\frac{(1-|\varphi(w)|^2)^{p(t+1)}}{\phi^p(|\varphi(w)|)}\frac{\phi^p(|\varphi(w)|)}{(1-|\varphi(w)|^2)^{ps}}\int_{|\varphi(w)|}^{1}\frac{(1-r)^{ps-1}}{(1-r|\varphi(w)|)^{p(t+1)}}dr$$
$$\leq C(1-|\varphi(w)|^2)^p\int_0^1\frac{(1-r)^{pt-1}}{(1-r|\varphi(w)|)^{p(t+1)}}dr$$
$$+C(1-|\varphi(w)|^2)^{p(t+1-s)}\int_0^1\frac{(1-r)^{ps-1}}{(1-r|\varphi(w)|)^{p(t+1)}}dr$$
$$\leq C.$$

Hence $f_w \in A^p(\phi)$ (or see [2]) and moreover $\sup_{w\in B}\|f_w\|_{A^p(\phi)} < \infty$. Thus

$$\|\psi C_\varphi f_w\|_\infty \leq \|\psi C_\varphi\|\|f_w\|_{A^p(\phi)} \leq C\|\psi C_\varphi\|,$$

i.e., for any $z \in B$,

$$|\psi(z)||f_w(\varphi(z))| \leq C\|\psi C_\varphi\|.$$

In particular, let $z = w$, we have

$$\frac{|\psi(w)|}{\phi(|\varphi(w)|)(1-|\varphi(w)|^2)^{\frac{n}{p}}} \leq C\|\psi C_\varphi\|. \tag{5}$$

Therefore,

$$\sup_{w\in B}\frac{|\psi(w)|}{\phi(|\varphi(w)|)(1-|\varphi(w)|^2)^{\frac{n}{p}}} < \infty.$$

For the converse, assume that (3) holds. Suppose $f \in A^p(\phi)$, Lemma 2.1 implies that for any $z \in B$,

$$|(\psi C_\varphi f)(z)| = |\psi(z)||f(\varphi(z))|$$
$$\leq \frac{C|\psi(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{\frac{n}{p}}}\|f\|_{A^p(\phi)} \leq MC\|f\|_{A^p(\phi)}. \tag{6}$$

Thus $\psi C_\varphi : A^p(\phi) \to H^\infty$ is bounded.

By (6), we get

$$\|\psi C_\varphi\| = \sup_{\|f\|_{A^p(\phi)}\leq 1}\|\psi C_\varphi f\|_\infty$$
$$\leq \sup_{\|f\|_{A^p(\phi)}\leq 1}\frac{C|\psi(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{\frac{n}{p}}}\|f\|_{A^p(\phi)} \leq CM. \tag{7}$$

(5) together with (7) implies (4). This completes the proof of the theorem. □

THEOREM 2.2. *Let φ be a holomorphic self-map of B and $\psi \in H(B)$. Assume that $0 < p < \infty$ and ϕ is normal on $[0, 1)$. Then $\psi C_\varphi : A^p(\phi) \to H^\infty$ is a compact operator if and only if $\psi \in H^\infty$ and*

$$\lim_{|\varphi(z)|\to 1} \frac{|\psi(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{\frac{n}{p}}} = 0. \tag{8}$$

PROOF. Necessity. Suppose that $\psi C_\varphi : A^p(\phi) \to H^\infty$ is a compact operator. Let $f(z) \equiv 1$. Then

$$\psi = \psi C_\varphi f \in H^\infty.$$

Assume $\{z_k\}$ is a sequence on B such that $\lim_{k\to\infty} |\varphi(z_k)| = 1$. Let

$$f_k(z) = \frac{(1-|\varphi(z_k)|^2)^{t+1}}{\phi(|\varphi(z_k)|)(1-\langle z, \varphi(z_k)\rangle)^{n/p+t+1}}.$$

Then $f_k \in A^p(\phi), \|f_k\|_{A^p(\phi)} \le C$ and $f_k \to 0$ uniformly on compact subsets of B. Since ψC_φ is compact, Lemma 2.3 implies $\|\psi C_\varphi f_k\|_\infty \to 0 (k \to \infty)$. Hence

$$\frac{|\psi(z_k)|}{\phi(|\varphi(z_k)|)(1-|\varphi(z_k)|^2)^{\frac{n}{p}}} \le \sup_{z\in B} |\psi(z)||f_k(\varphi(z))| = \|\psi C_\varphi f_k\|_\infty \to 0$$

as $k \to \infty$. Thus

$$\lim_{k\to\infty} \frac{|\psi(z_k)|}{\phi(|\varphi(z_k)|)(1-|\varphi(z_k)|^2)^{\frac{n}{p}}} = 0,$$

i.e.,

$$\lim_{|\varphi(z)|\to 1} \frac{|\psi(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{\frac{n}{p}}} = 0.$$

Sufficiency. Suppose $\psi \in H^\infty$ and (8) holds, then we get

$$\sup_{z\in B} \frac{|\psi(z)|}{\phi(|\varphi(z)|)(1-|\varphi(z)|^2)^{\frac{n}{p}}} < \infty.$$

By Theorem 2.1, $\psi C_\varphi : A^p(\phi) \to H^\infty$ is a bounded operator.

Assume $\{f_k\}$ is any bounded sequence of $A^p(\phi)$ and $f_k(z) \to 0$ uniformly on compact subsets of B. For any $\varepsilon > 0$, by (8), we know that there exists an $r \in (0, 1)$ such that

$$\frac{|\psi(z_k)|}{\phi(|\varphi(z_k)|)(1-|\varphi(z_k)|^2)^{\frac{n}{p}}} < \varepsilon \tag{9}$$

when $r < |\varphi(z)| < 1$. On the other hand, $f_k(z) \to 0$ uniformly on compact subsets of B. Hence there exists positive integer k_0 such that

$|f_k(\varphi(z))| < \epsilon$ if $|\varphi(z)| \le r$ and $k \ge k_0$. This implies that for $|\varphi(z)| \le r$ and $k \ge k_0$,

$$(10) \qquad |\psi(z)||f_k(\varphi(z))| \le \|\psi\|_\infty |f_k(\varphi(z))| < \|\psi\|_\infty \varepsilon.$$

Combine (9) and (10), we know that $\lim_{k\to\infty} \|\psi C_\varphi f_k\| \to 0$. Thus Lemma 2.3 shows that $\psi C_\varphi : A^p(\phi) \to H^\infty$ is a compact operator. This proves the theorem. □

THEOREM 2.3. *Let φ be a holomorphic self-map of B and $\psi \in H(B)$. Assume that $0 < p < \infty$ and ϕ is normal on $[0,1)$. Then the following conditions are equivalent:*

(1) $\psi C_\varphi : H^\infty \to A^p(\phi)$ is a bounded operator;

(2) $\psi C_\varphi : H^\infty \to A^p(\phi)$ is a compact operator;

(3) $\psi \in A^p(\phi)$.

Furthermore, if $\psi C_\varphi : H^\infty \to A^p(\phi)$ is bounded, then

$$\|\psi C_\varphi\| \asymp \|\psi\|_{A^p(\phi)}.$$

PROOF. (2) ⇒ (1). It is obvious.

(1) ⇒ (3). Suppose that $\psi C_\varphi : H^\infty \to A^p(\phi)$ is bounded, since $\psi = \psi C_\varphi(1)$, it is easy to see that $\psi \in A^p(\phi)$ and

$$(11) \qquad \|\psi\|_{A^p(\phi)} \le C\|\psi C_\varphi\|.$$

(3) ⇒ (1). Assume $\psi \in A^p(\phi)$. For any $f \in H^\infty$, we have

$$\begin{aligned} &\int_B |(\psi C_\varphi f)(z)|^p \frac{\phi^p(|z|)}{1-|z|} dv(z) \\ \le\ & \|f\|_\infty^p \int_B |\psi(z)|^p \frac{\phi^p(|z|)}{1-|z|} dv(z) = \|f\|_\infty^p \|\psi\|_{A^p(\phi)}^p. \end{aligned}$$

Hence

$$(12) \qquad \|\psi C_\varphi f\|_{A^p(\phi)} \le \|\psi\|_{A^p(\phi)} \|f\|_\infty.$$

Therefore $\psi C_\varphi : H^\infty \to A^p(\phi)$ is a bounded operator. By (12), we have

$$(13) \qquad \|\psi C_\varphi\| = \sup_{\|f\|_\infty \le 1} \|\psi C_\varphi f\|_{A^p(\phi)} \le \|\psi\|_{A^p(\phi)}.$$

(11) together with (13) implies

$$\|\psi C_\varphi\| \asymp \|\psi\|_{A^p(\phi)}.$$

(3) ⇒ (2). Suppose $\psi \in A^p(\phi)$. By (3) ⇒ (1), we know that $\psi C_\varphi : H^\infty \to A^p(\phi)$ is bounded. For any $\{f_k\} \subset H^\infty$ such that $\|f_k\|_\infty \le 1$, $\{f_k\}$ is a normal family. By Montel Theorem, there exists a subsequence of $\{f_k\}$, which converges to a holomorphic function $f(z)$ on any compact

subset of B. We may as well denote this subsequence $\{f_k\}$. Since $f_k \in H^\infty$ and $\|f_k\|_\infty \leq 1$, we have $f \in H^\infty$ and $\|f\|_\infty \leq 1$. Therefore $\{f_k - f\}$ converges to 0 on any compact subset of B. By the elementary inequality

$$(a+b)^p \leq \begin{cases} a^p + b^p & , \quad p \in (0,1) \\ 2^p(a^p+b^p) & , \quad p \geq 1 \end{cases} , a > 0, b > 0,$$

it follows that

$$\begin{aligned} & |(\psi C_\varphi f_k)(z) - (\psi C_\varphi f)(z)|^p \\ = \; & |\psi(z)|^p |f_k(\varphi(z)) - f(\varphi(z))|^p \\ \leq \; & 2^p |\psi(z)|^p (|f_k(\varphi(z))|^p + |f(\varphi(z))|^p) \\ \leq \; & 2^p |\psi(z)|^p (\|f_k\|_\infty + \|f\|_\infty) \leq 2^{p+1} |\psi(z)|^p. \end{aligned}$$

Therefore

$$\int_B |(\psi C_\varphi f_k)(z) - (\psi C_\varphi f)(z)|^p \frac{\phi^p(|z|)}{1-|z|} dv(z) < \infty. \tag{14}$$

For any $\varepsilon > 0$, from (14) there exists an $r \in (0,1)$ such that

$$\int_{|z|>r} |(\psi C_\varphi f_k)(z) - (\psi C_\varphi f)(z)|^p \frac{\phi^p(|z|)}{1-|z|} dv(z) < \varepsilon. \tag{15}$$

From the fact that $|\varphi(z)| \to 1$ implies $|z| \to 1$, we obtain

$$\int_{|\varphi(z)|>r} |(\psi C_\varphi f_k)(z) - (\psi C_\varphi f)(z)|^p \frac{\phi^p(|z|)}{1-|z|} dv(z) < \varepsilon. \tag{16}$$

For the above ε, since $\{f_k - f\}$ converges to 0 on any compact subset of B, there exists a k_0 such that $\sup_{|w|\leq r} |f_k(w) - f(w)| < \varepsilon$ as $k > k_0$. Hence we have

$$\begin{aligned} (17) \quad & \|\psi C_\varphi f_k - \psi C_\varphi f\|_{A^p(\phi)} \\ = \; & \int_B |(\psi C_\varphi f_k)(z) - (\psi C_\varphi f)(z)|^p \frac{\phi^p(|z|)}{1-|z|} dv(z) \\ = \; & \Big(\int_{|\varphi(z)|>r} + \int_{|\varphi(z)|\leq r} \Big) |(\psi C_\varphi f_k)(z) - (\psi C_\varphi f)(z)|^p \frac{\phi^p(|z|)}{1-|z|} dv(z) \\ \leq \; & \varepsilon + \varepsilon^p \|\psi\|^p_{A^p(\phi)} \end{aligned}$$

as $k > k_0$. From which we obtain

$$\lim_{k\to\infty} \|\psi C_\varphi f_k - \psi C_\varphi f\|_{A^p(\phi)} = 0.$$

Thus $\psi C_\varphi : H^\infty \to A^p(\phi)$ is a compact operator. □

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