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POSITIVE COEXISTENCE FOR A SIMPLE FOOD CHAIN MODEL WITH RATIO-DEPENDENT FUNCTIONAL RESPONSE AND CROSS-DIFFUSION

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ABSTRACT. The positive coexistence of a simple food chain model with ratio-dependent functional response and cross-diffusion is discussed. Especially, when a cross-diffusion is small enough, the existence of positive solutions of the system concerned can be expected. The extinction conditions for all three interacting species and for one or two of three species are studied. Moreover, when a cross-diffusion is sufficiently large, the extinction of prey species with cross-diffusion interaction to predator occurs. The method employed is the comparison argument for elliptic problem and fixed point theory in a positive cone on a Banach space.

1. Introduction

In this paper, the following 3×3 elliptic system with a ratio-dependent functional response and cross-diffusion is studied:

$$(1)\qquad \begin{cases} -\Delta[(d_1+\beta v)u] = u(a_1 - u - c_1\frac{v}{u+v}) & \\ -d_2\Delta v = v(a_2 - v + m_1\frac{u}{u+v} - c_2\frac{w}{v+w}) & \\ -d_3\Delta w = w(a_3 - w + m_2\frac{v}{v+w}) & \text{in } \Omega, \\ (u,v,w) = (0,0,0) & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subseteq \mathbb{R}^n$ is a bounded region with smooth boundary $\partial\Omega$. While the given coefficients d_i, c_i, m_i, a_1 and β are positive constants, the constants a_i are permitted to be positive or negative for $i = 2, 3$. Here Δ is the Laplacian operator and u, v, w represent the densities of the three interacting species. Model (1) describes predator-prey interactions

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among the three species, more precisely, species v is a predator only on u and w preys only on v. This is called a simple food-chain model. The domain with homogeneous Dirichlet boundary conditions indicates the region with a hostile boundary environment. We say that the system (1) has a positive solution (u, v, w) if $u(x) > 0$, $v(x) > 0$ and $w(x) > 0$ for all $x \in \Omega$. The existence of a positive solution (u, v, w) to the system (1) is called the *positive coexistence.*

Food-chain models have been studied in both spatially homogeneous [8] and spatially inhomogeneous situations [6, 20] for the last two decades. It is well accepted that the dynamics of the three species model is relatively more complicated than that of the two species model. (See [6, 7, 8, 16, 20] and the references therein.) Even with respect to the ODE system, the dynamics for the behavior of positive solutions is much complicated [8]. Additional work relating to the three-species model with predator-prey interacting type with diffusions can be found in [10, 17]. Recently, certain predator-prey models, so called the ratio-dependent predator-prey models (that is, the per capita predator growth rate depends on a function of the ratio of prey to predator abundance), are proposed by R. Arditi and L. R. Ginzburg in [1]. Since then, as well as the actual evidences and justifications [2, 3, 4, 9], the models have been mathematically studied for spatially homogeneous case [11, 12, 13, 14].

On the other hand, there has been a considerable amount of interest to the system with cross-diffusion since the proposal of the model in study of spatial segregation of two interacting species by Shigesada *et al.* [27]. In [21, 22], they investigated the existence of non-constant solutions to the 2×2 competing interaction system with cross-diffusions under homogeneous Neumann boundary conditions. In [24], the authors studied the positive coexistence of the system with self-cross diffusions which describes competing or predator-prey interactions between two species using the method of decomposing operators and the theory of fixed point index. Recently, in [15], for the predator-prey interacting system with only cross diffusions between two species, they showed the multiplicity of positive coexistence under appropriate assumptions. In [25], it was shown that the existence of positive solutions to the 2×2 competing system of a general nonlinear type with self-cross diffusions under Robin boundary conditions where the diffusion rates are strictly positive.

In this paper, we are interested in the situation that predator and prey species have a diffusive interaction among three species with ratio-dependent terms and spatially inhomogeneous distribution with cross-diffusion on a region. We obtain sufficient conditions for the positive coexistence to the system (1) and give the proof for the existence of positive solutions using theory of fixed point index on a positive cone. In addition, the non-existence of positive solutions is studied. In according to our results, as cross-diffusion is sufficiently large, the extinction of prey species with cross-diffusion interaction to predator occurs, and sufficiently small cross-diffusion helps to create the positive coexistence among food-chain species.

One of the mathematical difficulties in the model (1) relates to the method of treating the singular point $(0,0,0)$. To overcome this difficulty, we will adopt the idea of Kuang and Berreta in [14], more precisely, since $\lim_{(u,v)\to(0,0)} \frac{uv}{u+v} = 0$ and $\lim_{(v,w)\to(0,0)} \frac{vw}{v+w} = 0$, the domain of $\frac{uv}{u+v}$ and $\frac{vw}{v+w}$ to $\{(u,v,w) : u \geq 0, v \geq 0, w \geq 0\}$ may be extended so that $(0,0,0)$ becomes a *trivial solution* of (1).

This paper is organized as follows. In Section 2, known results which are useful in later sections are presented. In Section 3, sufficient conditions for the positive coexistence of (1) are given. The nonexistence of positive solutions of the steady state to the model is discussed in Section 4.

2. Preliminaries

In this section, some known lemmas and theorems are stated, which are used in this paper. The following lemmas can be obtained from [24, 25] with a simple modification using homogeneous Dirichlet boundary condition.

For $d(x) > 0$ in $C^2(\overline{\Omega})$ and $b(x) \in L^\infty(\Omega)$, let $\lambda_1(\Delta d(x) + b(x))$ be the principal eigenvalue of the following eigenvalue problem:

$$\begin{cases} \Delta d(x)\phi + b(x)\phi = \lambda\phi & \text{in } \Omega, \\ \phi = 0 & \text{on } \partial\Omega. \end{cases}$$

Then the variational property gives

$$\lambda_1(\Delta d(x) + b(x)) = \sup_{\xi \in W^{1,2}(\Omega)} \frac{\int_\Omega -|\nabla[d(x)\xi]|^2 + d(x)b(x)\xi^2}{||\sqrt{d(x)}\xi||_{L^2}^2}.$$

LEMMA 1. *Assume that* $\frac{b_1(x)}{d_1(x)} > \frac{b_2(x)}{d_2(x)}$, *where* $d_i(x) > 0$ *in* $C^2(\overline{\Omega})$ *and* $b_i(x) \in L^\infty(\Omega)$ *for* $i = 1, 2$.

(i) *If* $\lambda_1(\Delta d_1(x) + b_1(x)) \leq 0$, $\lambda_1(\Delta d_2(x) + b_2(x)) < 0$.

(ii) *If* $\lambda_1(\Delta d_2(x) + b_2(x)) \geq 0$, $\lambda_1(\Delta d_1(x) + b_1(x)) > 0$.

Furthermore, $\lambda_1(\Delta d(x) + b(x))$ *is increasing and decreasing in* $b(x)$ *and* $d(x)$*, respectively.*

LEMMA 2. *Let* $d(x) > 0$ *in* $C^2(\overline{\Omega})$, $b(x) \in L^\infty(\Omega)$ *and* $\phi \geq 0$, $\phi \not\equiv 0$ *in* Ω *with* $\phi = 0$ *on* $\partial\Omega$.

(i) *If* $0 \not\equiv \Delta d(x)\phi + b(x)\phi \geq 0$, *then* $\lambda_1(\Delta d(x) + b(x)) > 0$.

(ii) *If* $0 \not\equiv \Delta d(x)\phi + b(x)\phi \leq 0$, *then* $\lambda_1(\Delta d(x) + b(x)) < 0$.

(iii) *If* $\Delta d(x)\phi + b(x)\phi \equiv 0$, *then* $\lambda_1(\Delta d(x) + b(x)) = 0$.

Let $r(T)$ be the spectral radius of a linear operator $T : E \to E$, where E is a Banach space.

LEMMA 3. *Let* $d(x) > 0$ *in* $C^2(\overline{\Omega})$, $b(x) \in L^\infty(\Omega)$ *and* M *be a positive constant such that* $b(x) + Md(x) > 0$ *for all* $x \in \overline{\Omega}$. *Then we have*

(i) *If* $\lambda_1(\Delta d(x)+b(x)) > 0$, *then* $r\left[\frac{1}{d(x)}(-\Delta+M)^{-1}(b(x)+Md(x))\right] > 1$;

(ii) *If* $\lambda_1(\Delta d(x)+b(x)) < 0$, *then* $r\left[\frac{1}{d(x)}(-\Delta+M)^{-1}(b(x)+Md(x))\right] < 1$;

(iii) *If* $\lambda_1(\Delta d(x)+b(x)) = 0$, *then* $r\left[\frac{1}{d(x)}(-\Delta+M)^{-1}(b(x)+Md(x))\right] = 1$.

Consider the following single equation

$$\begin{cases} -\Delta[d(x)\phi] = \phi f(x,\phi) & \text{in } \Omega, \\ \phi = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

where $f(x,\phi)$ is C^1-function in ϕ and C^α-function in x, and $d(x) \in C^2(\overline{\Omega})$ with $d(x) > 0$ for all $x \in \overline{\Omega}$.

We have the following existence and uniqueness theorem of positive solution for (2).

THEOREM 4. *Assume that* $f_\phi < 0$ *for* $\phi \geq 0$ *and* $f(x,M) < 0$ *for some positive constant* M.

(i) *If* $\lambda_1(\Delta d(x) + f(x,0)) \leq 0$, *then* (2) *has no positive solution.*

(ii) *If* $\lambda_1(\Delta d(x)+f(x,0)) > 0$, *then* (2) *has a unique positive solution.*

Let E be a Banach space and W a total wedge in E, i.e., W is a closed convex subset of E such that $\alpha W \subset E$ for $\alpha \geq 0$ and $\overline{W - W} = E$. A wedge W is said to be a cone if $W \cap (-W) = \{0\}$. Let $y \in W$ and define $W_y = \{x \in E : y + \gamma x \in W \text{ for some } \gamma > 0\}$. Let $S_y = \{x \in \overline{W}_y : -x \in \overline{W}_y\}$. Then $\overline{W}_y$ is a wedge and S_y is a closed subspace of E. We say that $\mathcal{L}$ has property α if there is a $t \in (0,1)$ and a $w \in \overline{W}_y \setminus S_y$ such that $w - t\mathcal{L}w \in S_y$. Assume $\mathcal{A} : W \to W$ is a compact operator with fixed point $y \in W$ and $\mathcal{A}$ is Fréchet differentiable at y. Let $\mathcal{L} = \mathcal{A}'(y)$ be the Fréchet derivative of $\mathcal{A}$ at y. Then $\mathcal{L}$ maps $\overline{W}_y$ to itself.

For an open subset $N \subset W$, index$(\mathcal{A}, N, W)$ is the Leray-Schauder degree $\deg_W(I - \mathcal{A}, N, 0)$ where I is the identity map. If y is an isolated fixed point of $\mathcal{A}$, then the fixed point index of $\mathcal{A}$ at y in W is defined by index$(\mathcal{A}, y, W) =$ index$(\mathcal{A}, N(y), W)$, where $N(y)$ is a small open neighborhood of y in W. We denote index$(\mathcal{A}, y) =$ index$(\mathcal{A}, y, W)$ and index$(\mathcal{A}, N) =$ index$(\mathcal{A}, N, W)$.

In [5], E. N. Dancer introduced the formula to explicitly evaluate the indices of a compact operator at the isolated fixed points on cones in a Banach space. Later, this result was improved by L. Li [18], M. Wang *et al.* [29] and W. H. Ruan and W. Feng [23]. The following can be obtained from the results of [5, 18, 29].

THEOREM 5. *Assume that $I - \mathcal{L}$ is invertible on $\overline{W}_y$.*

(i) *If $\mathcal{L}$ has property α, then* index$(\mathcal{A}, y, W) = 0$.

(ii) *If $\mathcal{L}$ does not have property α, then* index$(\mathcal{A}, y, W) = (-1)^\sigma$, *where σ is the sum of multiplicities of all the eigenvalues of $\mathcal{L}$ which are greater than one.*

3. Existence theorem for food chain model

In this section, the existence of positive solutions of (1) will be discussed. By Theorem 4, (1) has at most three semi-trivial solutions when exactly two of the species are absent. Denote these nonnegative and nonzero solutions by $(u_0, 0, 0)$, $(0, v_0, 0)$ and $(0, 0, w_0)$. Here, u_0 is the positive solution of the following equation:

$$\begin{cases} -d_1 \Delta u = u(a_1 - u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{3}$$

if $\lambda_1(d_1\Delta + a_1) > 0$. Symmetrically, v_0 and w_0 can be obtained by replacing d_1 and a_1 in (3) with d_i and a_i for $i = 2, 3$, respectively.

THEOREM 6. *Any positive solution (u, v, w) of (1) has an a-priori bound:*

$$u \le R := \frac{a_1}{d_1}\Big(d_1 + \frac{\beta}{c_1} a_1(a_1 + a_2 + m_1)\Big), \quad v \le a_2 + m_1 \ \textit{and} \ w \le a_3 + m_2.$$

PROOF. Let (u, v, w) be a positive solution of (1). Since $a_2 - v + m_1 \frac{u}{u+v} - c_2 \frac{w}{v+w} \le a_2 + m_1 - v$, $v \le a_2 + m_1$ follows by the strong maximum principle and Hopf's lemma. Similarly, $w \le a_3 + m_2$ can be obtained.

Consider the following elliptic problem:

$$\begin{cases} -\Delta[(d_1+\beta v)u] = u(a_1 - u - c_1\frac{v}{u+v}) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Assume that $||(d_1+\beta v)u||_\infty = (d_1+\beta v(x_0))u(x_0) > 0$ for some $x_0 \in \Omega$. Then it follows easily that

$$-\Delta[(d_1+\beta v(x_0))u(x_0)] = u(x_0)\Big(a_1 - u(x_0) - c_1\frac{v(x_0)}{u(x_0)+v(x_0)}\Big) \ge 0.$$

Since $a_1 - u(x_0) - c_1\frac{v(x_0)}{u(x_0)+v(x_0)} \ge 0$, $u(x_0) \le a_1$ follows, so that $v(x_0) \le \frac{a_1}{c_1}(u(x_0)+v(x_0)) \le \frac{a_1}{c_1}(a_1+a_2+m_1)$ comes from the fact of $v \le a_2+m_1$. Thus

$$d_1 u \le ||(d_1+\beta v)u||_\infty \le a_1\Big(d_1 + \frac{\beta}{c_1}a_1(a_1+a_2+m_1)\Big)$$

follows, so that one can have the desired result. □

Now we introduce the following notations which are used throughout this paper:

(i) $X := C_D(\overline{\Omega}) \oplus C_D(\overline{\Omega}) \oplus C_D(\overline{\Omega})$, where $C_D(\overline{\Omega}) := \{\phi \in C(\overline{\Omega}) : \phi = 0 \text{ on } \partial\Omega\}$;

(ii) $W := K \oplus K \oplus K$, where $K = \{\phi \in C_D(\overline{\Omega}) : 0 \le \phi(x),\ x \in \overline{\Omega}\}$;

(iii) $Q := \max\{R,\ a_2+m_1,\ a_3+m_2\} + 1$;

(iv) $N(\rho) := \{z \in C_D(\overline{\Omega}) : 0 \le z \le \rho \text{ on } \overline{\Omega}\}$;

(v) $D_Q := N(Q) \oplus N(Q) \oplus N(Q)$.

For $\theta_1,\ \theta_2,\ \theta_3 \in [0,1]$, define the positive and compact operator $\mathcal{G}_{\theta_1,\theta_2,\theta_3} : \overline{D}_Q \to W$ via

$$\begin{aligned} &\mathcal{G}_{\theta_1,\theta_2,\theta_3}(u,v,w) \\ &= (-\Delta+P)^{-1}\begin{pmatrix} \theta_1 u(a_1 - u - c_1\frac{v}{u+v}) + P(d_1+\beta v)u \\ \theta_2 v(a_2 - v + m_1\frac{u}{u+v} - c_2\frac{w}{v+w}) + Pd_2 v \\ \theta_3 w(a_3 - w + m_2\frac{v}{v+w}) + Pd_3 w \end{pmatrix}, \end{aligned}$$

where the constant P is taken so large that $P > \min\{\frac{1}{d_1}|c_1 + 2R - a_1|,\ \frac{1}{d_2}|c_2 + 2(m_1+a_2) - a_2|,\ \frac{1}{d_3}|2(a_3+m_2) - a_3|\}$. Note that for the compactness, define $\frac{uv}{u+v} := 0$ and $\frac{vw}{v+w} := 0$ at $u = v = w = 0$. Denote $\mathcal{G} = \mathcal{G}_{1,1,1}$. Then from (1), observe $\mathcal{D}(u,v,w) = \mathcal{G}(u,v,w)$, where $\mathcal{D}(u,v,w) = ((d_1+\beta v)u, d_2 v, d_3 w)$.

Note that since the Jacobian of $\mathcal{D}$ is positive in D_Q, it has the inverse $\mathcal{D}^{-1}$. Define the positive and compact operator $\mathcal{A}_{\theta_1,\theta_2,\theta_3} : \overline{D}_Q \to W$

via $\mathcal{A}_{\theta_1,\theta_2,\theta_3}(u,v,w) = \mathcal{D}^{-1} \circ \mathcal{G}_{\theta_1,\theta_2,\theta_3}(u,v,w)$. Then (1) has a positive solution if and only if $\mathcal{A} = \mathcal{A}_{1,1,1}$ has a positive fixed point.

LEMMA 7. $\operatorname{index}(\mathcal{A}, D_Q) = 1$.

PROOF. First of all, note that by the definition of D_Q, $\mathcal{A}_{\theta_1,\theta_2,\theta_3}$ has no fixed points on $\partial\Omega$. So using homotopy invariance property of index, $\operatorname{index}(\mathcal{A}, D_Q) = \operatorname{index}(\mathcal{A}_{1,1,1}, D_Q) = \operatorname{index}(\mathcal{A}_{0,0,0}, D_Q)$, where

$$\mathcal{A}_{0,0,0}(u,v,w) = \mathcal{D}^{-1}\mathcal{G}_{0,0,0}(u,v,w) = \mathcal{D}^{-1}(-\Delta + P)^{-1}\begin{pmatrix} P(d_1+\beta v)u \\ Pd_2 v \\ Pd_3 w \end{pmatrix}.$$

One can see that $(0,0,0)$ is the only fixed point of $\mathcal{A}_{0,0,0}$ in D_Q and therefore $\operatorname{index}(\mathcal{A}_{0,0,0}, D_Q) = \operatorname{index}(\mathcal{A}_{0,0,0}, (0,0,0))$ by the excision property of index. Observe that

$$\mathcal{L} := \mathcal{A}'_{0,0,0}(0,0,0) = (\mathcal{D}^{-1})'(\mathcal{G}_{0,0,0}(0,0,0))\mathcal{G}'_{0,0,0}(0,0,0).$$

Since $\mathcal{G}_{0,0,0}(0,0,0) = \mathcal{D}(0,0,0) = (0,0,0)$, the inverse function theorem implies

$$\begin{aligned}(\mathcal{D}^{-1})'(\mathcal{G}_{0,0,0}(0,0,0)) &= (\mathcal{D}^{-1})'(\mathcal{D}(0,0,0)) \\ &= (\mathcal{D}'(0,0,0))^{-1} = \begin{pmatrix} \frac{1}{d_1} & 0 & 0 \\ 0 & \frac{1}{d_2} & 0 \\ 0 & 0 & \frac{1}{d_3} \end{pmatrix}.\end{aligned}$$

Consequently,

$$\begin{aligned}\mathcal{L} &= \begin{pmatrix} \frac{1}{d_1} & 0 & 0 \\ 0 & \frac{1}{d_2} & 0 \\ 0 & 0 & \frac{1}{d_3} \end{pmatrix}\mathcal{G}'_{0,0,0}(0,0,0) \\ &= \begin{pmatrix} \frac{1}{d_1} & 0 & 0 \\ 0 & \frac{1}{d_2} & 0 \\ 0 & 0 & \frac{1}{d_3} \end{pmatrix}(-\Delta + P)^{-1}\begin{pmatrix} Pd_1 & 0 & 0 \\ 0 & Pd_2 & 0 \\ 0 & 0 & Pd_3 \end{pmatrix}\end{aligned}$$

and it is easy to see $\overline{W}_{(0,0,0)} = W$ and $S_{(0,0,0)} = \{(0,0,0)\}$. By Theorem 5, the remained proof can be done throughout the following claims:

Claim 1: $I - \mathcal{L}$ is invertible on $\overline{W}_{(0,0,0)}$, where $\mathcal{L} = \mathcal{A}'_{0,0,0}(0,0,0)$.

Claim 2: The sum of multiplicities of all the eigenvalue of $\mathcal{L}$ which are greater than one is zero.

Claim 3: $\mathcal{L}$ does not have property α.

Proof of Claim 1: Let $\mathcal{L}(\phi_1,\phi_2,\phi_3)^T = (\phi_1,\phi_2,\phi_3)^T \in \overline{W}_{(0,0,0)}$. Then $(-\Delta + P)^{-1}(Pd_i)\phi_i = d_i\phi_i$ for $i = 1,2,3$. This gives $-\Delta d_i\phi_i = 0$, so that $\phi_i \equiv 0$ for $i = 1,2,3$. Hence $I - \mathcal{L}$ is invertible on $\overline{W}_{(0,0,0)}$.

Proof of Claim 2: Since $\lambda_1(\Delta d_i) < 1$ for $i = 1,2,3$, $r[\frac{1}{d_i}(-\Delta + P)^{-1}Pd_i] < 0$ by Lemma 3. Thus $r[\mathcal{L}] < 1$ follows.

Proof of Claim 3: Suppose that $\mathcal{L}$ has property α. Then there exists $t_0 \in (0,1)$ and $(\phi_1,\phi_2,\phi_3) \in \overline{W}_{(0,0,0)}\backslash S_{(0,0,0)}$ such that $(I - t_0\mathcal{L})(\phi_1,\phi_2,\phi_3)^T \in S_{(0,0,0)}$. This is equivalent to $\frac{1}{d_i}(-\Delta+P)^{-1}(Pd_i\phi_i) = \frac{\phi_i}{t_0} > \phi_i$ in Ω and $\phi_i = 0$ on $\partial\Omega$ for $i = 1,2,3$. Hence $r(\mathcal{L}) > 1$ follows. This contradicts to *Claim 2.* □

Now, let's calculate index value of $\mathcal{A}$ at $(0,0,0)$. Since the Fréchet derivative of $\mathcal{A}$ does not defined at $(u,v,w) = (0,0,0)$, we have difficulty in developing our argument. But this adversity can be resolved by using ε-perturbation $\mathcal{A}_\varepsilon$ of $\mathcal{A}$ as in [26]. In the concrete, index$(\mathcal{A},(0,0,0)) = \lim_{\varepsilon\to 0}$ index$(\mathcal{A}_\varepsilon,(0,0,0))$, where

$$\begin{aligned}&\mathcal{A}_\varepsilon(u,v,w)\\ &= \mathcal{D}^{-1}(-\Delta+P)^{-1}\begin{pmatrix} u(a_1 - u - c_1\frac{v}{u+v+\varepsilon}) + P(d_1+\beta v)u \\ v(a_2 - v + m_1\frac{u}{u+v+\varepsilon} - c_2\frac{w}{v+w+\varepsilon}) + Pd_2v \\ w(a_3 - w + m_2\frac{v}{v+w+\varepsilon}) + Pd_3w \end{pmatrix}.\end{aligned}$$

(See Chap. 12 in [28].) In fact, one of the conditions $\lambda_1(d_1\Delta + a_1 - c_1) > 0$, $\lambda_1(d_2\Delta + a_2 - c_2) > 0$, and $\lambda_1(d_3\Delta + a_3) > 0$ is needed to preserve the same index value regardless of a ε-perturbation which gives a Fréchet differentiability at $(u,v,w) = (0,0,0)$. The reason is that $\lambda_1(d_1\Delta + a_1)$ in the below Lemma 8 can be replaced by $\lambda_1(d_1\Delta + a_1 - c_1)$ by applying a ε-perturbation to v as like $c_1\frac{v+\varepsilon}{u+v+\varepsilon} = c_1 - c_1\frac{u}{u+v+\varepsilon}$. Similarly, $\lambda_1(d_2\Delta + a_2)$ in the below Lemma 8 can be replaced by $\lambda_1(d_2\Delta + a_2 - c_2)$. Hence if one of $\lambda_1(d_1\Delta + a_1 - c_1) > 0$, $\lambda_1(d_2\Delta + a_2 - c_2) > 0$ and $\lambda_1(d_3\Delta + a_3) > 0$, then index$(\mathcal{A}_\varepsilon,(0,0,0)) = 0$ is always obtained without respect to the particular choice of ε-perturbation, since $\lambda_1(d_3\Delta + a_3) > 0$ or $\lambda_1(d_i\Delta + a_i) > \lambda_1(d_i\Delta + a_i - c_i) > 0$ hold for $i = 1$ or 2.

LEMMA 8. *Assume that one of* $\lambda_1(d_1\Delta + a_1 - c_1)$, $\lambda_1(d_2\Delta + a_2 - c_2)$ *and* $\lambda_1(d_3\Delta + a_3)$ *is positive. Then* index$(\mathcal{A},(0,0,0)) = 0$.

PROOF. If we show index$(\mathcal{A}_\varepsilon,(0,0,0)) = 0$ which is independent of ε, it's done by the above observation. $\overline{W}_{(0,0,0)} = W$ and $S_{(0,0,0)} = \{(0,0,0)\}$ can be obtained easily. Define

$$\begin{aligned}\mathcal{L}_0 &:= \mathcal{A}'_\varepsilon(0,0,0) = (\mathcal{D}^{-1})'(\mathcal{G}(0,0,0))\mathcal{G}'(0,0,0)\\ &= \begin{pmatrix} \frac{1}{d_1} & 0 & 0 \\ 0 & \frac{1}{d_2} & 0 \\ 0 & 0 & \frac{1}{d_3} \end{pmatrix}(-\Delta+P)^{-1}\begin{pmatrix} a_1+Pd_1 & 0 & 0 \\ 0 & a_2+Pd_2 & 0 \\ 0 & 0 & a_3+Pd_3 \end{pmatrix}.\end{aligned}$$

Without loss of generality, suppose that $\lambda_1(d_1\Delta + a_1 - c_1) > 0$. First, assume that $\lambda_1(d_i\Delta + a_i) \neq 0$ for $i = 2, 3$.

Claim 1 : $I - \mathcal{L}_0$ is invertible on $\overline{W}_{(0,0,0)}$.

Proof of Claim 1 : If $\mathcal{L}_0(\phi_1, \phi_2, \phi_3)^T = (\phi_1, \phi_2, \phi_3) \in W$, then $-d_i\Delta\phi_i = a_i\phi_i$ holds for $i = 1, 2, 3$. Since $a_i \neq d_i\lambda_1(-\Delta)$ for $i = 1, 2, 3$, $\phi_i \equiv 0$ for all i. Hence $I - \mathcal{L}_0$ is invertible on $\overline{W}_{(0,0,0)}$.

Claim 2 : $\mathcal{L}_0$ has property α.

Proof of Claim 2 : Since $\lambda_1(d_1\Delta + a_1) > 0$, $r[\frac{1}{d_1}(-\Delta + P)^{-1}(a_1 + Pd_1)] > 1$ by Lemma 3. So Krein-Rutman theorem implies that $r[\frac{1}{d_1}(-\Delta + P)^{-1}(a_1 + Pd_1)]$ is an eigenvalue of $\frac{1}{d_1}(-\Delta + P)^{-1}(a_1 + Pd_1)I$ and the corresponding eigenfunction ϕ exists in K. If $t_0 = \frac{1}{r[\frac{1}{d_1}(-\Delta+P)^{-1}(a_1+Pd_1)]}$ is taken, then $(\phi, 0, 0) \in W\backslash\{(0, 0, 0)\}$ and $(\phi, 0, 0)^T - t_0\mathcal{L}_0(\phi, 0, 0)^T = (0, 0, 0)$. Hence $\mathcal{L}_0$ has property α. Therefore index$(\mathcal{A}_\varepsilon, (0, 0, 0)) = 0$ by Theorem 5.

Next, consider that $\lambda_1(d_i\Delta + a_i) = 0$ for $i = 2$ or 3. When only one of $\lambda_1(d_i\Delta + a_i)$ for $i = 2, 3$ is zero, the proof is so similar to the case that two of them are zero. Thus we deal with only the latter case: $\lambda_1(d_i\Delta + a_i) = 0$ for $i = 2, 3$. For $\mu \in [0, 1]$, we define a homotopy

$$\begin{aligned} &\mathcal{A}_{\varepsilon,\mu}(u, v, w) \\ &= \mathcal{D}^{-1}(-\Delta + P)^{-1} \begin{pmatrix} u(a_1 - u - c_1\frac{v}{u+v+\varepsilon}) + P(d_1 + \beta v)u \\ v(a_2 - \mu - v + m_1\frac{u}{u+v+\varepsilon} - c_2\frac{w}{v+w+\varepsilon}) + Pd_2v \\ w(a_3 - \mu - w + m_2\frac{v}{v+w+\varepsilon}) + Pd_3w \end{pmatrix}. \end{aligned}$$

It is easy to see that $(0, 0, 0)$ is a fixed point of $\mathcal{A}_{\varepsilon,\mu}$ for all $\mu \in [0, 1]$ and $\mathcal{A}_\varepsilon = \mathcal{A}_{\varepsilon,0}$. Since $\lambda_1(d_i\Delta + a_i - \mu) < 0$ for $\mu \in (0, 1]$, we have index$(\mathcal{A}_{\varepsilon,\mu}, (0, 0, 0)) = 0$ as in the first part of this proof. Finally, the homotopy invariance property of index concludes index$(\mathcal{A}_\varepsilon, (0, 0, 0)) =$ *index*$(\mathcal{A}_{\varepsilon,\mu}, (0, 0, 0)) = 0$. □

LEMMA 9. *Assume that* $\lambda_1(d_1\Delta + a_1) > 0$. *If one of* $\lambda_1(d_2\Delta + a_2 + m_1 - c_2)$ *and* $\lambda_1(d_3\Delta + a_3)$ *is positive, then* index$(\mathcal{A}, (u_0, 0, 0)) = 0$.

PROOF. It is easily seen that $\overline{W}_{(u_0,0,0)} = C_D(\overline{\Omega}) \oplus K \oplus K$ and $S_{(u_0,0,0)} = C_D(\overline{\Omega}) \oplus \{0\} \oplus \{0\}$.

As in Lemma 8, the index value can not be calculated directly. So we introduce the ρ-perturbation $\mathcal{A}_\rho$ of $\mathcal{A}$, where

$$\mathcal{A}_\rho(u,v,w) = \mathcal{D}^{-1}(-\Delta+P)^{-1}\begin{pmatrix} u(a_1-u-c_1\frac{v}{u+v})+P(d_1+\beta v)u \\ v(a_2-v+m_1\frac{u}{u+v}-c_2\frac{w}{v+w+\rho})+Pd_2v \\ w(a_3-w+m_2\frac{v}{v+w+\rho})+Pd_3w \end{pmatrix}.$$

So one can get the Fréchet derivative of $\mathcal{A}_\rho$ at $(u,v,w)=(u_0,0,0)$:

$$\mathcal{L}_{u_0} := \mathcal{A}'_\rho(u_0,0,0) = (\mathcal{D}^{-1})'(\mathcal{G}(u_0,0,0))\mathcal{G}'(u_0,0,0)$$
$$= \begin{pmatrix} d_1 & \beta u_0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{pmatrix}^{-1} (-\Delta+P)^{-1}\begin{pmatrix} \alpha_1 & -c_1 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{pmatrix},$$

where $\alpha_1 := a_1-2u_0+Pd_1$, $\alpha_2 := a_2+m_1+Pd_2$ and $\alpha_3 := a_3+Pd_3$, since $(\mathcal{D}^{-1})'(\mathcal{G}(u_0,0,0)) = (\mathcal{D}^{-1})'(\mathcal{D}(u_0,0,0)) = (\mathcal{D}'(u_0,0,0))^{-1}$ holds from the inverse function theorem.

The above ρ-perturbation is possible under $\lambda_1(d_2\Delta+a_2+m_1-c_2)>0$ or $\lambda_1(d_3\Delta+a_3)>0$. First, assume that $\lambda_1(d_2\Delta+a_2+m_1-c_2)>0$. Note that $\lambda_1(d_1\Delta+a_1-2u_0)<\lambda_1(d_1\Delta+a_1-u_0)=0$ and $\lambda_1(d_2\Delta+a_2+m_1)>\lambda_1(d_2\Delta+a_2+m_1-c_2)>0$ by Lemma 1. Then since one of $\lambda_1(d_1\Delta+a_1-2u_0)$, $\lambda_1(d_2\Delta+a_2+m_1)$ and $\lambda_1(d_3\Delta+a_3)$ is positive, index$(\mathcal{A}_\rho,(u_0,0,0))=0$ can be obtained as in Lemma 8 using a perturbation once again when $\lambda_1(d_3\Delta+a_3)=0$, so we omit it.

If $\lambda_1(d_3\Delta+a_3)>0$ holds, we can have also the desired result, symmetrically. Hence index$(\mathcal{A},(u_0,\ 0,0))=0$. □

LEMMA 10. *Assume that* $\lambda_1(d_2\Delta+a_2)>0$. *If one of* $\lambda_1(\Delta(d_1+\beta v_0)+a_1-c_1)$ *and* $\lambda_1(d_3\Delta+a_3+m_2)$ *is positive, then* index$(\mathcal{A},(0,v_0,0))=0$.

PROOF. Since $\mathcal{A}$ is Fréchet differentiable at $(u,v,w)=(0,v_0,0)$, we do not need any perturbation. Note that

$$\mathcal{A}'(0,v_0,0) = \begin{pmatrix} d_1+\beta v_0 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{pmatrix}^{-1} \mathcal{G}'(0,v_0,0)$$
$$= \begin{pmatrix} d_1+\beta v_0 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{pmatrix}^{-1} (-\Delta+P)^{-1}\begin{pmatrix} \overline{\alpha}_1 & 0 & 0 \\ m_1 & \overline{\alpha}_2 & -c_2 \\ 0 & 0 & \overline{\alpha}_3 \end{pmatrix},$$

where $\overline{\alpha}_1 := a_1-c_1+P(d_1+\beta v_0)$, $\overline{\alpha}_2 := a_2-2v_0+Pd_2$ and $\overline{\alpha}_3 := a_3+m_2+Pd_3$. The remains are so similar to the proof of Lemma 9. □

When w_0 exists (i.e., $\lambda_1(d_3\Delta + a_3) > 0$), another perturbation is needed to calculate index value at $(0, 0, w_0)$:

$$\mathcal{A}_\epsilon(u, v, w)$$
$$= \mathcal{D}^{-1}(-\Delta + P)^{-1} \begin{pmatrix} u(a_1 - u - c_1\frac{v}{u+v+\epsilon}) + P(d_1 + \beta v)u \\ v(a_2 - v + m_1\frac{u}{u+v+\epsilon} - c_2\frac{w}{v+w}) + Pd_2 v \\ w(a_3 - w + m_2\frac{v}{v+w}) + Pd_3 w \end{pmatrix}.$$

Here $\lambda_1(d_1\Delta + a_1 - c_1) > 0$ or $\lambda_1(d_2\Delta + a_2 - c_2) > 0$ is necessary for a well-definedness of index$(\mathcal{A}_\epsilon, (0, 0, w_0))$. The proof will be omitted since it is similar to the one of Lemma 9. So we have the following lemma.

LEMMA 11. *Assume that* $\lambda_1(d_3\Delta+a_3) > 0$. *If one of* $\lambda_1(d_1\Delta+a_1-c_1)$ *and* $\lambda_1(d_2\Delta + a_2 - c_2)$ *is positive, then* index$(\mathcal{A}, (0, 0, w_0)) = 0$.

We use the following notations in the next lemmas. It's for semi-tirivial solutions when exactly one of the species is absent.

For sufficient small δ,

(i) $N_0(\rho) := \{z \in C_D(\overline{\Omega}) : 0 < z \leq \rho \text{ on } \overline{\Omega}\}$;

(ii) $D_1(\delta) := N(\delta) \oplus N_0(Q) \oplus N_0(Q)$;

(iii) $D_3(\delta) := N_0(Q) \oplus N_0(Q) \oplus N(\delta)$.

The slices $D_1(\delta)$ and $D_3(\delta)$ in D_Q contain all fixed points of $\mathcal{A}$ of the form $(0, v, w)$ and $(u, v, 0)$, respectively. Also, $D_2(\delta)$ can be defined in a similar way. However, $D_2(\delta)$ does not have to be considered since there is no positive semi-trivial solution of the form $(u, 0, w)$ for $u, w > 0$. In the case of $v \equiv 0$, it is easy to show that $w \equiv 0$ by a strong maximum principle.

Next lemmas are the calculation for the index value of $\mathcal{A}$ in the slices defined in the above.

LEMMA 12. *Assume that* $\lambda_1(\Delta(d_1+\beta v_*)+a_1-c_1) > 0$ *and* $\lambda_1(d_2\Delta + a_2 - c_2) > 0$, *where* v_* *is an unique positive solution of the equation:*

$$\begin{cases} -d_2\Delta v = v(a_2 - c_2 - v) & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

If $\lambda_1(d_3\Delta + a_3 + m_2) > 0$, *then* index$(\mathcal{A}, D_3(\delta)) = 0$.

PROOF. For $\theta_1, \theta_2 \in [0,1]$, consider the compact operator $\widehat{\mathcal{A}}_{\theta_1,\theta_2}$: $\overline{D}_Q \to W$ via

$$\widehat{\mathcal{A}}_{\theta_1,\theta_2}(u,v,w) = \mathcal{D}^{-1}(-\Delta+P)^{-1}\begin{pmatrix} u(a_1-c_1-u+c_1\theta_1\frac{u}{u+v})+P(d_1+\beta v)u \\ v(a_2-c_2-v+m_1\theta_2\frac{u}{u+v}+c_2\theta_2\frac{v}{v+w})+Pd_2v \\ w(a_3-w+m_2\frac{v}{v+w})+Pd_3w \end{pmatrix}.$$

If the operator $\widehat{\mathcal{A}}_{\theta_1,\theta_2}$ has no fixed points on $\partial D_3(\delta)$, then index($\mathcal{A}$, $D_3(\delta)$) = index($\widehat{\mathcal{A}}_{0,0}$, $D_3(\delta)$) holds by homotopy invariance of index.

Claim 1 : The operator $\widehat{\mathcal{A}}_{\theta_1,\theta_2,1}$ has no fixed points on $\partial D_3(\delta)$.

Proof of Claim 1 : Suppose that this claim is not true. Then for $\{\theta_{1,n}\}$ and $\{\theta_{2,n}\} \in [0,1]$, there are the sequences $\{\delta_n\}$ and fixed points $\{(u_n,v_n,w_n)\}$ of $\widehat{\mathcal{A}}_{\theta_{1,n},\theta_{2,n}}$ such that $(u_n,v_n,w_n) \in \partial D_3(\delta_n)$ and $\delta_n \to 0$. By the choice of Q, $w_n \to 0$ but $w_n \not\equiv 0$ as $\delta_n \to 0$. Note that for $\theta_{2,n} \in [0,1]$, $v_n \geq v_*$. Hence $\lambda_1(d_3\Delta+a_3-w_n+\frac{m_2v_n}{w_n+v_n}) > \lambda_1(d_3\Delta+a_3-w_n+\frac{m_2v_*}{w_n+v_*}) \to \lambda_1(d_3\Delta+a_3+m_2) > 0$ since $w_n \to 0$ as $\delta_n \to 0$. So for a sufficient large n, $\lambda_1(d_3\Delta+a_3-w_n+\frac{m_2v_n}{w_n+v_n}) > 0$. It's a contradiction to $\lambda_1(d_3\Delta+a_3-w_n+\frac{m_2v_n}{w_n+v_n}) = 0$ from Lemma 2.

Claim 2 : index($\widehat{\mathcal{A}}_{0,0}, D_3(\delta)$) = 0

Proof of Claim 2 : For $(u,v,w) \in D_3(\delta)$, $\widehat{\mathcal{A}}_{0,0}(u,v,w) = (u,v,w)$ gives that $u = u_*$, $v = v_*$ and $||w|| < \delta$, where u_* is the unique positive solution of (2) with $d(x) = d_1+\beta v_*$ and $f(x,\phi) = a_1-c_1-u$. Here $w \equiv 0$ follows from the choice of $\delta > 0$. Hence $\widehat{\mathcal{A}}_{0,0,1}$ has only one fixed point in $D_3(\delta)$: $(u_*,v_*,0)$.

Thus index($\widehat{\mathcal{A}}_{0,0}, D_3(\delta)$) = index($\widehat{\mathcal{A}}_{0,0}, (u_*,v_*,0)$). Observe that the Fréchet derivative of $\widehat{\mathcal{A}}_{0,0}$ at $(u_*,v_*,0)$ is

$$\begin{pmatrix} d_1+\beta v_* & \beta u_* & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{pmatrix}^{-1}(-\Delta+P)^{-1}\begin{pmatrix} L_{11} & P\beta u_* & 0 \\ 0 & L_{22} & 0 \\ 0 & 0 & a_3+m_2+Pd_3 \end{pmatrix},$$

where $L_{11} = a_1-c_1-2u_*+P(d_1+\beta v_*)$ and $L_{22} = a_2-c_2-2v_*+Pd_2$. Note that $\lambda_1(d_2\Delta+a_2-c_2-2v_*) < \lambda_1(d_2\Delta+a_2-c_2-v_*) = 0$, $\lambda_1(\Delta(d_1+\beta v_*)+a_1-c_1-2u_*) < \lambda_1(\Delta(d_1+\beta v_*)+a_1-c_1-u_*) = 0$ follow from Lemma 2 and the given assumption. In a result, $\lambda_1(d_2\Delta+a_2-c_2-2v_*) < 0$, $\lambda_1(\Delta(d_1+\beta v_*)+a_1-c_1-2u_*) < 0$ and $\lambda_1(d_3\Delta+a_3+m_2) > 0$ yield that index($\widehat{\mathcal{A}}_{0,0,1}, (u_*,v_*,0)$) = 0 as in Lemma 9. □

LEMMA 13. *Assume that* $\lambda_1(\Delta(d_1+\beta(a_2+m_1))+a_1-c_1)>0$ *and* $\lambda_1(-d_2\Delta-m_1+c_2)<a_2\le\lambda_1(-d_2\Delta)$. *If* $\lambda_1(d_3\Delta+a_3+m_2)>0$, *then* $\mathrm{index}(\mathcal{A},D_3(\delta))=0$.

PROOF. The proof is similar to the one of Lemma 12, so we just sketch the proof. For $\theta_1,\theta_2\in[0,1]$, consider the compact operator $\widetilde{\mathcal{A}}_{\theta_1,\theta_2}:\overline{D}_Q\to W$ via

$$\widetilde{\mathcal{A}}_{\theta_1,\theta_2}(u,v,w)=$$
$$\mathcal{D}^{-1}(-\Delta+P)^{-1}\begin{pmatrix} u(a_1-c_1-u+c_1\theta_1\frac{u}{u+v})+P(d_1+\beta v)u \\ v(a_2-c_2+m_1-v-m_1\frac{\theta_2 v}{u+v}+c_2\frac{\theta_2 v}{v+w})+Pd_2v \\ w(a_3-w+m_2\frac{v}{v+w})+Pd_3w \end{pmatrix}.$$

As in the above lemma, note that for $\theta_{2,n}\in[0,1]$, $v_n\le a_2+m_1$ from Lemma 6. Furthermore, for $\theta_{1,n}\in[0,1]$, it follows that $u_n\ge\frac{\Phi}{d_1+\beta(a_2+m_1)}$, where Φ is the unique positive solution of the equation:

$$\begin{cases} -\Delta U=U(\frac{a_1-c_1}{d_1+\beta(a_2+m_1)}-\frac{U}{d_1^2}) & \text{in } \Omega, \\ U=0 & \text{on } \partial\Omega. \end{cases}$$

The existence of Φ comes from the assumption $\lambda_1(\Delta(d_1+\beta(a_2+m_1))+a_1-c_1)>0$ and Theorem 4. The fact of $u_n\ge\frac{\Phi}{d_1+\beta(a_2+m_1)}$ follows from comparison argument for elliptic problem, since if we denote $U_n=(d_1+\beta v_n)u_n$,

$$-\Delta U_n\ge U_n\Big(\frac{a_1-c_1}{d_1+\beta v_n}-\frac{U_n}{(d_1+\beta v_n)^2}\Big)\ge U_n\Big(\frac{a_1-c_1}{d_1+\beta(a_2+m_1)}-\frac{U_n}{d_1^2}\Big).$$

Using these results, one can obtain that $v_n\ge\Psi$ from $\lambda_1(d_2\Delta+a_2+m_1-c_2)>0$, where Ψ is a unique positive solution of equation:

$$\begin{cases} -d_2\Delta V=V(a_2+m_1-c_2-V-m_1\frac{V}{\frac{\Phi}{d_1+\beta(a_2+m_1)}+V}) & \text{in } \Omega, \\ V=0 & \text{on } \partial\Omega. \end{cases}$$

As in Lemma 12, it can be shown that the operator $\widetilde{\mathcal{A}}_{\theta_1,\theta_2}$ has no fixed points on $\partial D_3(\delta)$ by getting a contradiction as $n\to\infty$. Moreover, $\widetilde{\mathcal{A}}_{0,0}$ has only one fixed point in $D_3(\delta)$: $(u_\sharp,v_\sharp,0)$, where $u_\sharp$ and $v_\sharp$ are unique positive solutions of (2) with $(d(x),\ f(x,\phi))=(d_2,\ a_2+m_1-c_2-v)$ and $(d(x),\ f(x,\phi))=(d_1+\beta v_\sharp,\ a_1-c_1-u)$, respectively. The remained part is routine, since $\lambda_1(d_3\Delta+a_3+m_2)>0$, $\lambda_1(\Delta(d_1+\beta v_\sharp)+a_1-c_1-2u_\sharp)<0$ and $\lambda_1(d_2\Delta+a_2+m_1-c_2-2v_\sharp)<0$. □

LEMMA 14. *Assume that* $\lambda_1(d_2\Delta+a_2-c_2)>0$ *and* $\lambda_1(d_3\Delta+a_3+m_2)>0$. *If* $\lambda_1(\Delta(d_1+\beta v_0)+a_1-c_1)>0$, *then* $\mathrm{index}(\mathcal{A},D_1(\delta))=0$.

PROOF. We sketch the proof. For $\theta_2, \theta_3 \in [0,1]$, introduce the following operator $\overline{\mathcal{A}}_{\theta_2,\theta_3} : \overline{D}_Q \to W$ via

$$\overline{\mathcal{A}}_{\theta_2,\theta_3}(u,v,w) = \mathcal{D}^{-1}(-\Delta+P)^{-1}\begin{pmatrix} u(a_1 - u - c_1\frac{v}{u+v}) + P(d_1+\beta v)u \\ v(a_2 - v + m_1\theta_2\frac{u}{u+v} - c_2\theta_2\frac{w}{v+w}) + Pd_2 v \\ w(a_3 + m_2 - w - m_2\theta_3\frac{w}{v+w}) + Pd_3 w \end{pmatrix}.$$

Note that as in Lemma 12, for $\theta_{2,n} \in [0,1]$, $v_n \geq v_*$, where v_* was defined in Lemma 12. As $n \to \infty$ (i.e., $\delta \to 0$), $u_n \to 0$ gives that for a large n, $v_n \leq v_0$ since $a_2 - v_n + m_1\theta_{2,n}\frac{u_n}{u_n+v_n} - c_2\theta_{2,n}\frac{w_n}{v_n+w_n} \leq a_2 - v_n + m_1\theta_{2,n}\frac{u_n}{u_n+v_*} \to a_2 - v_n$. Thus for a large n, $\lambda_1(\Delta(d_1+\beta v_n) + a_1 - u_n - c_1\frac{v_n}{u_n+v_n}) \geq \lambda_1(\Delta(d_1+\beta v_0)+a_1-c_1-u_n) \to \lambda_1(\Delta(d_1+\beta v_0)+a_1-c_1) > 0$ follows. Hence for a sufficient large n, one can get a contradiction, so that the operator $\overline{\mathcal{A}}_{\theta_2,\theta_3}$ has no fixed points on $\partial D_1(\delta)$. Moreover, $\overline{\mathcal{A}}_{0,0}$ has only one fixed point in $D_1(\delta)$: $(0, v_*, w_*)$, where w_* is the unique positive solution of (2) with $d(x) = d_3$ and $f(x,\phi) = a_3 + m_2 - w$. The remains can be shown easily, since $\lambda_1(\Delta(d_1+\beta v_0) + a_1 - c_1) > 0$. □

Now we can get sufficient conditions which give the positive coexistence for system (1), using Lemma 7-14.

THEOREM 15. *If one of the following principal eigenvalue conditions,* (i) *and* (ii), *is satisfied, then there is a positive solution of* (1).

(i) $\lambda_1(\Delta(d_1+\beta v_0)+a_1-c_1) > 0$, $\lambda_1(d_2\Delta+a_2-c_2) > 0$ *and* $\lambda_1(d_3\Delta + a_3 + m_2) > 0$;

(ii) $\lambda_1(\Delta(d_1 + \beta(a_2+m_2)) + a_1 - c_1) > 0$, $\lambda_1(-d_2\Delta - m_1 + c_2) < a_2 \leq \lambda_1(-d_2\Delta)$ *and* $\lambda_1(d_3\Delta + a_3 + m_2) > 0$.

PROOF. (i) First, note that the existence of positive semi-trivial solutions of the form $(u,v,0)$, $(0,v,w)$ and $(0,0,w_0)$ is not known. However these semi-trivial solutions do not affect our index calculation, regardless of whether these exist or not. The reason is because the assumptions give that $\text{index}(\mathcal{A}, D_3(\delta)) = \text{index}(\mathcal{A}, D_1(\delta)) = \text{index}(\mathcal{A}, (0,0,w_0)) = 0$ even if all these semi-trivial solutions exist. Hence in spite of introducing above semi-trivial solutions in calculating index value, one get the following:

$$\begin{aligned}&\text{index}(\mathcal{A}, D_Q) - \text{index}(\mathcal{A}, (0,0,0)) - \text{index}(\mathcal{A}, (u_0,0,0)) \\ &\quad - \text{index}(\mathcal{A}, (0,v_0,0)) - \text{index}(\mathcal{A}, (0,0,w_0)) - \text{index}(\mathcal{A}, D_1(\delta)) \\ &\quad - \text{index}(\mathcal{A}, D_3(\delta)) \\ &= 1-0-0-0-0-0-0 = 1 \neq 0.\end{aligned}$$

Here we use the comparison property of principal eigenvalue to get the following inequalities:

$$\lambda_1(d_1\Delta+a_1),\ \lambda_1(\Delta(d_1+\beta v_*)+a_1-c_1) > \lambda_1(\Delta(d_1+\beta v_0)+a_1-c_1) > 0;$$
$$\lambda_1(d_2\Delta + a_2),\ \lambda_1(d_2\Delta + a_2 + m_2 - c_2) > \lambda_1(d_2\Delta + a_2 - c_2) > 0.$$

(ii) The assumption $a_2 \leq \lambda_1(-d_2\Delta)$ gives the non-existence of semi-trivial solutions of form $(0, v_0, 0)$ and $(0, v, w)$. Thus v_0 and $D_1(\delta)$ do not need to be considered in this case. Hence using the additivity of index,

$$\begin{aligned}&\mathrm{index}(\mathcal{A}, D_Q) - \mathrm{index}(\mathcal{A}, (0,0,0)) - \mathrm{index}(\mathcal{A}, (u_0, 0, 0))\\ &- \mathrm{index}(\mathcal{A}, (0,0,w_0)) - \mathrm{index}(\mathcal{A}, D_3(\delta)) = 1 - 0 - 0 - 0 - 0 = 1 \neq 0.\end{aligned}$$

□

4. Non-existence theorem for food chain model

In this section, we investigate the non-existence conditions of positive solutions for system (1).

THEOREM 16. (i) *If $a_1 \leq \lambda_1(-d_1\Delta)$, then there is no positive solutions of* (1). *In addition, if $a_2 + m_1 \leq \lambda_1(-d_2\Delta)$ and $a_3 + m_2 \leq \lambda_1(-d_3\Delta)$, then there is no nonnegative and nonzero solutions of* (1).

(ii) *If $\lambda_1(d_2\Delta + a_2 - c_2) > 0$ and $\lambda_1(\Delta(d_1 + \beta v_*) + a_1) \leq 0$, then there is no positive solutions of* (1). *In details, the prey u can not survive.*

PROOF. Using Lemma 2 and the comparison property of eigenvalue, each case can be shown by introducing a contradiction if there is a positive solution (u, v, w) of (1).

(i) This case can be proved with ease. So we omit it.

(ii) One knows that $\lambda_1(d_2\Delta + a_2 - c_2) > 0$ guarantees the existence of v_*. Moreover, $v \geq v_*$ and v_* does not depend on cross-diffusion β. Thus from the following inequality

$$0 = \lambda_1(\Delta(d_1 + \beta v) + a_1 - u - c_1\frac{v}{u+v}) < \lambda_1(\Delta(d_1 + \beta v_*) + a_1) \leq 0,$$

a contradiction is induced. □

REMARK 1. Distinction of $\lambda_1(\Delta(d_1+\beta v_0)+a_1-c_1) > 0$ and $\lambda_1(\Delta(d_1+\beta v_*)+a_1) \leq 0$ in Theorem 15 (i) and Theorem 16 (ii) should be pointed out in view of biological implication : if the middle predator v can survive alone (i.e., $\lambda_1(d_2\Delta + a_2 - c_2) > 0$), a sufficient small cross-diffusion β helps to create a positive solution of our food-chain model (1). On the other hand, sufficient large β makes the prey species u become extinct.

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