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A COMMON FIXED POINT THEOREM FOR A SEQUENCE OF SELF MAPS IN INTUITIONISTIC FUZZY METRIC SPACES

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Abstract. The purpose of this paper is to obtain a new common fixed point theorem by using a new contractive condition in intuitionistic fuzzy metric spaces. Our result generalizes and extends many known results in fuzzy metric spaces and metric spaces.

Introduction

The concept of fuzzy sets was introduced by Zadeh [9]. Following the concept of fuzzy sets, fuzzy metric spaces have been introduced by Kramosil and Michalek [4], and George and Veeramani [3] modified the notion of fuzzy metric spaces with the help of continuous t-norms. Recently, many authors have proved fixed point theorems involving fuzzy sets [5, 7, 8].

As a generalization of fuzzy sets, Atanassov [1] introduced and studied the concept of intuitionistic fuzzy sets. Recently, using the idea of intuitionistic fuzzy sets, Park [6] introduced the notion of intuitionistic fuzzy metric spaces with the help of continuous t-norms and continuous t-conorms as a generalization of fuzzy metric spaces due to George and Veeramani [3], and showed that every metric induces an intuitionistic fuzzy metric, every fuzzy metric space is an intuitionistic fuzzy metric space and found a necessary and sufficient condition for an intuitionistic fuzzy metric space to be complete.

Choudhury [2] introduced mutually contractive sequence of self maps and proved a fixed point theorem.

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spaces. Our result generalizes and extends many known results in fuzzy metric spaces and metric spaces.

1. Preliminaries

DEFINITION 1. A binary operation $*: [0,1] \times [0,1] \to [0,1]$ is a continuous t-norm if $*$ is satisfying the following conditions:

(a) $*$ is commutative and associative;
(b) $*$ is continuous;
(c) $a * 1 = a$ for all $a \in [0,1]$;
(d) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, and $a, b, c, d \in [0,1]$.

DEFINITION 2. A binary operation $\Diamond : [0,1] \times [0,1] \to [0,1]$ is a continuous t-conorm if $\Diamond$ is satisfying the following conditions:

(a) $\Diamond$ is commutative and associative;
(b) $\Diamond$ is continuous;
(c) $a \Diamond 0 = a$ for all $a \in [0,1]$;
(d) $a \Diamond b \leq c \Diamond d$ whenever $a \leq c$ and $b \leq d$, and $a, b, c, d \in [0,1]$.

DEFINITION 3 ([6]). A 5-tuple $(X, M, N, *, \Diamond)$ is said to be an intuitionistic fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm, $\Diamond$ is a continuous t-conorm and M, N are fuzzy sets on $X^2 \times (0, \infty)$ satisfying the following conditions: for all x, y, $z \in X$, s, $t > 0$,

(a) $M(x,y,t) + N(x,y,t) \leq 1$;
(b) $M(x,y,t) > 0$;
(c) $M(x,y,t) = 1$ if and only if $x = y$;
(d) $M(x,y,t) = M(y,x,t)$;
(e) $M(x,y,t) * M(y,z,s) \leq M(x,z,t+s)$;
(f) $M(x,y,.) : (0,\infty) \to (0,1]$ is continuous;
(g) $N(x,y,t) > 0$;
(h) $N(x,y,t) = 0$ if and only if $x = y$;
(i) $N(x,y,t) = N(y,x,t)$;
(j) $N(x,y,t) \Diamond N(y,z,s) \geq N(x,z,t+s)$;
(k) $N(x,y,.) : (0,\infty) \to (0,1]$ is continuous.

Then (M, N) is called an intuitionistic fuzzy metric on X. The functions $M(x,\ y,t)$ and $N(x,y,t)$ denote the degree of nearness and the degree of non-nearness between x and y with respect to t, respectively.

REMARK 1. Every fuzzy metric space $(X, M, *)$ is an intuitionistic fuzzy metric space of the form $(X, M, 1-M, *, \Diamond)$ such that t-norm $*$ and t-conorm $\Diamond$ are associated, i.e., $x \Diamond y = 1 - ((1-x)*(1-y))$ for any $x, y \in [0,1]$. But the converse is not true.

EXAMPLE 1 (Induced intuitionistic fuzzy metric [6]). Let (X, d) be a metric space. Denote $a * b = ab$ and $a \Diamond b = \min\{1, a+b\}$ for all $a, b \in [0,1]$ and let M_d and N_d be fuzzy sets on $X^2 \times (0, \infty)$ defined as follows:

$$M_d(x,y,t) = \frac{t}{t+d(x,y)}, \quad N_d(x,y,t) = \frac{d(x,y)}{t+d(x,y)} \quad .$$

Then $(X, M_d, N_d, *, \Diamond)$ is an intuitionistic fuzzy metric space. We call this intuitionistic fuzzy metric induced by a metric d the standard intuitionistic fuzzy metric.

REMARK 2. Note that the above example holds even with the t-norm $a * b = \min\{a, b\}$ and the t-conorm $a \Diamond b = \max\{a, b\}$.

EXAMPLE 2. Let $X = \mathbb{N}$. Define $a * b = \max\{0, a+b-1\}$ and $a \Diamond b = a + b - ab$ for all $a, b \in [0,1]$ and let M and N be fuzzy sets on $X^2 \times (0, \infty)$ as follows:

$$M(x,y,t) = \begin{cases} \frac{x}{y} \text{ if } x \le y, \\ \frac{y}{x} \text{ if } y \le x, \end{cases} \quad , \quad N(x,y,t) = \begin{cases} \frac{y-x}{y} \text{ if } x \le y, \\ \frac{x-y}{x} \text{ if } y \le x, \end{cases}$$

for all $x, y \in X$ and $t > 0$. Then $(X, M, N, *, \Diamond)$ is an intuitionistic fuzzy metric space.

REMARK 3. Note that, in the above example, t-norm $*$ and t-conorm $\Diamond$ are not associated. And there exists no metric d on X satisfying

$$M(x,y,t) = \frac{t}{t+d(x,y)}, \qquad N(x,y,t) = \frac{d(x,y)}{t+d(x,y)},$$

where $M(x,y,t)$ and $N(x,y,t)$ are as defined in above example. Also note that the above functions (M, N) is not an intuitionistic fuzzy metric with the t-norm and t-conorm defined as $a * b = \min\{a, b\}$ and $a \Diamond b = \max\{a, b\}$.

DEFINITION 4 ([6]). Let $(X, M, N, *, \Diamond)$ be an intuitionistic fuzzy metric space. Then

(a) a sequence $\{x_n\}$ in X is said to be convergent x in X if for each $\epsilon > 0$ and each $t > 0$, there exists $n_0 \in \mathbf{N}$ such that $M(x_n, x, t) > 1 - \epsilon$ and $N(x_n, x, t) < \epsilon$ for all $n \ge n_0$.

(b) a sequence $\{x_n\}$ in X is said to be Cauchy if for each $\epsilon > 0$ and each $t > 0$, there exists $n_0 \in \mathbf{N}$ such that $M(x_n, x_m, t) > 1 - \epsilon$ and $N(x_n, x_m, t) < \epsilon$ for all $n, m \geq n_0$.

(c) An intuitionistic fuzzy metric space in which every Cauchy sequence is convergent is said to be complete.

REMARK 4. Since $*$ and $\diamondsuit$ are continuous, the limit is uniquely determined from (e) and (j) in Definition 3.

DEFINITION 5. A sequence $\{S_i\}$ of self maps on a complete fuzzy metric space $(X, M, *)$ is said to be mutually contractive if for $t > 0$ and $i \in \mathbb{N}$,

$$M(S_i x, S_j y, t) \geq M(x, y, t/p),$$

where $x, y \in X, p \in (0, 1), i \neq j$ and $x \neq y$.

DEFINITION 6. A sequence $\{S_i\}$ of self maps on a complete intuitionistic fuzzy metric space $(X, M, N, *, \diamondsuit)$ is said to be intuitionistic mutually contractive if for $t > 0$ and $i \in \mathbb{N}$,

$$M(S_i x, S_j y, t) \geq M(x, y, t/p)$$

and

$$N(S_i x, S_j y, t) \leq N(x, y, t/p),$$

where $x, y \in X, p \in (0, 1), i \neq j$ and $x \neq y$.

Throughout this paper, $(X, M, N, *, \diamondsuit)$ will denote the intuitionistic fuzzy metric space with the following conditions:

(l) $\lim_{t \to \infty} M(x, y, t) = 1$;

(m) $\lim_{t \to \infty} N(x, y, t) = 0$ for all $x, y \in X$.

2. Main results

In this section, we prove a common fixed point theorem for a sequence of self maps on intuitionistic fuzzy metric spaces. We also obtain corresponding results in fuzzy metric and metric spaces.

THEOREM 1. *Let $(X, M, N, *, \diamondsuit)$ be a complete intuitionistic fuzzy metric space and $\{S_n\}$ be a sequence of self maps of X satisfying*

(a) $S_i S_j = S_j S_i$ *for all* $i, j = 1, 2, \ldots$,

(b) S_i *is continuous for all* $i = 1, 2, \ldots$,

(c) $\{S_i\}$ *is intuitionistic mutually contractive.*

Then $\{S_i\}$ has a unique common fixed point.

PROOF. Let x_0 be any point in X. We can construct a sequence $\{x_n\}$ in X such that

$$x_1 = S_1x_0, x_2 = S_2x_1, \dots, x_n = S_nx_{n-1}, \dots.$$

Then the following cases may arise:

Case I. If no terms of $\{x_n\}$ are equal. Then, using (c), we get:

$$\begin{aligned} M(x_n, x_{n+1}, t) &= M(S_nx_{n-1}, S_{n+1}x_n, t) \geq M(x_{n-1}, x_n, t/p), \\ N(x_n, x_{n+1}, t) &= N(S_nx_{n-1}, S_{n+1}x_n, t) \leq N(x_{n-1}, x_n, t/p). \end{aligned}$$

By repeated application of above inequalities, we get:

$$\begin{aligned} M(x_n, x_{n+1}, t) &\geq M(x_0, x_1, t/p^n), \\ N(x_n, x_{n+1}, t) &\leq N(x_0, x_1, t/p^n). \end{aligned}$$

Then, using (e) and (j) of Definition 3, we get:

$$\begin{aligned} M(x_n, x_{n+k}, t) &\geq M(x_n, x_{n+1}, t/k) * M(x_{n+1}, x_{n+k}, (k-1)t/k) \\ &\geq M(x_n, x_{n+1}, t/k) * M(x_{n+1}, x_{n+2}, t/k) \\ &\quad * M(x_{n+2}, x_{n+k}, (k-2)t/k) \\ &\geq M(x_n, x_{n+1}, t/k) * M(x_{n+1}, x_{n+2}, t/k) \\ &\quad * \overset{(k-1)}{\cdots} * M(x_{n+k-1}, x_{n+k}, t/k) \\ &\geq M(x_0, x_1, t/kp^n) * M(x_0, x_1, t/kp^{n-1}) \\ &\quad * \overset{(k-1)}{\cdots} * M(x_0, x_1, t/kp^{n+k-1}) \end{aligned}$$

and

$$\begin{aligned} N(x_n, x_{n+k}, t) &\leq N(x_n, x_{n+1}, t/k) \Diamond N(x_{n+1}, x_{n+k}, (k-1)t/k) \\ &\leq N(x_n, x_{n+1}, t/k) \Diamond N(x_{n+1}, x_{n+2}, t/k) \\ &\quad \Diamond N(x_{n+2}, x_{n+k}, (k-2)t/k) \\ &\leq N(x_n, x_{n+1}, t/k) \Diamond N(x_{n+1}, x_{n+2}, t/k) \\ &\quad \Diamond \overset{(k-1)}{\cdots} \Diamond N(x_{n+k-1}, x_{n+k}, t/k) \\ &\leq N(x_0, x_1, t/kp^n) \Diamond N(x_0, x_1, t/kp^{n-1}) \\ &\quad \Diamond \overset{(k-1)}{\cdots} \Diamond N(x_0, x_1, t/kp^{n+k-1}). \end{aligned}$$

According to (l) and (m), we now get:

$$\begin{aligned} \lim M(x_n, x_{n+k}, t) &\geq 1 * 1 * \overset{(k-1)}{\cdots} * 1 = 1, \\ \lim N(x_n, x_{n+k}, t) &\leq 0 \Diamond 0 \Diamond \overset{(k-1)}{\cdots} \Diamond 0 = 0. \end{aligned}$$

That is, $\{x_n\}$ is a Cauchy sequence in X, hence convergent. Call the limit z.

Since two consecutive terms of $\{x_n\}$ are unequal, for an arbitrary integer $i, t > 0$ and $\lambda > 0$, we can find n such that $z \neq x_{n-1}$, $n > i$,

$$M(z, x_n, t/2) > 1 - \lambda, \quad M(z, x_{n-1}, t/2) > 1 - \lambda$$

and

$$N(z, x_n, t/2) < \lambda, \quad N(z, x_{n-1}, t/2) < \lambda.$$

Then, we get:

$$\begin{aligned} M(z, S_i z, t) &\geq M(z, x_n, t/2) * M(x_n, S_i z, t/2) \\ &\geq M(z, x_n, t/2) * M(S_n x_{n-1}, S_i z, t/2) \\ &\geq M(z, x_n, t/2) * M(z, x_{n-1}, t/2) \\ &> 1 - \lambda \end{aligned}$$

and

$$\begin{aligned} N(z, S_i z, t) &\leq N(z, x_n, t/2) \Diamond N(x_n, S_i z, t/2) \\ &\leq N(z, x_n, t/2) \Diamond N(S_n x_{n-1}, S_i z, t/2) \\ &\leq N(z, x_n, t/2) \Diamond N(z, x_{n-1}, t/2) \\ &< \lambda. \end{aligned}$$

Since $t > 0$ and $\lambda > 0$ are arbitrary, $M(z, S_i z, t) = 1$ and $N(z, S_i z, t) = 0$, that is, $z = S_i z$ for all $i = 1, 2, \ldots$.

Case II. $x_i = x_{i-1}$ for some integer i. Then $x_{i-1} = S_i x_{i-1}$. Let $z = x_{i-1}$, that is, $z = S_i z$, $z \neq S_j z$ and further $z \neq S_j^n z$ for all $n = 1, 2, \ldots$. Then, for $t > 0$, we get:

$$\begin{aligned} M(z, S_j^2 z, t) &= M(S_i z, S_j(S_j z), t) \\ &\geq M(z, S_j z, t/p) \end{aligned}$$

and

$$\begin{aligned} N(z, S_j^2 z, t) &= N(S_i z, S_j(S_j z), t) \\ &\leq N(z, S_j z, t/p). \end{aligned}$$

Similarly,

$$M(z, S_j^3 z, t) \geq M(z, S_j z, t/p^2)$$

and

$$N(z, S_j^3 z, t) \leq N(z, S_j z, t/p^2).$$

Consequently,

$$M(z, S_j^n z, t) \geq M(z, S_j z, t/p^{n-1})$$

and

$$N(z, S_j^n z, t) \leq N(z, S_j z, t/p^{n-1})$$

for all $n = 2, 3, \ldots$, where $z \neq S_j^n z$ for all $n = 1, 2, \ldots$. Letting $n \to \infty$, we get

$$S_j^n z \to z \text{ as } n \to \infty.$$

Since S_j is continuous, we get:

$$S_j\left(S_j^n z\right) = S_j^{n+1} \to S_j z \text{ as } n \to \infty.$$

In the view of Remark 4, we get $z = S_j z, j = 1, 2, \ldots$. This is a contradiction, so $z = S_j^k z$ for some k.

Let k be the smallest integer with this property. Then, we get:

$$z \neq S_j^m z \text{ for some } m = 1, 2, \ldots, k-1$$

and for $t > 0$

$$\begin{aligned} M(z, S_j^{k-1}z, t) &= M(S_i z, S_j(S_j^{k-2}z), t) \geq M(z, S_j^{k-2}z, t/2) \\ &= M(S_i z, S_j(S_j^{k-3}z), t/p) \geq M(z, S_j^{k-3}z, t/p^2) \\ &\geq \cdots \geq M(z, S_j z, t/p^{k-2}), \end{aligned}$$

$$\begin{aligned} N(z, S_j^{k-1}z, t) &= N(S_i z, S_j(S_j^{k-2}z), t) \leq N(z, S_j^{k-2}z, t/2) \\ &= N(S_i z, S_j(S_j^{k-3}z), t/p) \leq N(z, S_j^{k-3}z, t/p^2) \\ &\leq \cdots \leq N(z, S_j z, t/p^{k-2}), \end{aligned}$$

hence $z, S_j z, S_j^2 z, \ldots, S_j^{k-1}z$ are all distinct. Then, for $t > 0$,

$$\begin{aligned} M(z, S_j z, t) &= M(S_j^k z, S_j(S_i z), t) \\ &= M(S_j(S_j^{k-1}z), S_i(S_j z), t) \\ &\geq M(S_j^{k-1}z, S_j z, t/p) \\ &\geq M(S_j^{k-2}z, S_j z, t/p^2) \\ &\geq \cdots \geq M(S_j^2 z, S_j z, t/p^{k-2}) \\ &= M(S_j^2(S_i z), S_j z, t/p^{k-2}) \\ &= M(S_i(S_j^2 z), S_j z, t/p^{k-2}) \\ &\geq M(S_j^2 z, z, t/p^{k-1}) \\ &= M(S_j(S_j z), S_i z, t/p^{k-1}) \\ &\geq M(S_j z, z, t/p^k) \end{aligned}$$

and

$$\begin{aligned}
N(z,S_jz,t) &= N(S_j^kz,S_j(S_iz),t)\\
&= N(S_j(S_j^{k-1}z),S_i(S_jz),t)\\
&\le N(S_j^{k-1}z,S_jz,t/p)\\
&\le N(S_j^{k-2}z,S_jz,t/p^2)\\
&\le \cdots \le N(S_j^2z,S_jz,t/p^{k-2})\\
&= N(S_j^2(S_iz),S_jz,t/p^{k-2})\\
&= N(S_i(S_j^2z),S_jz,t/p^{k-2})\\
&\le N(S_j^2z,z,t/p^{k-1})\\
&= N(S_j(S_jz),S_iz,t/p^{k-1})\\
&\le N(S_jz,z,t/p^k).
\end{aligned}$$

But this gives a contradiction, so $z = S_jz$ for all $j = 1, 2, \ldots$.

To show uniqueness, assume z and w be two common fixed points such that $z \neq w$. Then, using (c), we get:

$$\begin{aligned}
M(z,w,t) &= M(S_iz,S_jw,t)\\
&\ge M(z,w,t/p)
\end{aligned}$$

and

$$\begin{aligned}
N(z,w,t) &= N(S_iz,S_jw,t)\\
&\le N(z,w,t/p)
\end{aligned}$$

which is a contradiction. Therefore, $z \neq w$. Hence the common fixed point is unique. □

In the following, we prove the projection of Theorem 1 from complete intuitionistic fuzzy metric space to complete fuzzy metric space.

COROLLARY 1. *Let $(X, M, *)$ be a complete fuzzy metric space and $\{S_n\}$ be a sequence of self maps of X satisfying*

(a) *$S_iS_j = S_jS_i$ for all $i, j = 1, 2, \ldots$,*
(b) *S_i is continuous for all $i = 1, 2, \ldots$,*
(c) *$\{S_i\}$ is mutually contractive.*

Then $\{S_i\}$ has a unique common fixed point.

PROOF. The proof follows from Theorem 1 by considering intuitionistic fuzzy metric space $(X, M, N, *, \Diamond)$, where t-norm $*$ and t-conorm $\Diamond$ are associated, i.e., $x\Diamond y = 1 - ((1-x) * (1-y))$ for any $x, y \in [0, 1]$. □

In the following, we prove the projection of Theorem 1 from complete intuitionistic fuzzy metric space to complete metric space.

COROLLARY 2. *Let (X, d) be a complete fuzzy metric space and $\{S_n\}$ be a sequence of self maps of X satisfying*

(a) $S_iS_j = S_jS_i$,
(b) S_i *is continuous,*
(c) $d(S_ix, S_jy) \leq pd(x, y)$

for all $x, y \in X$ with $x \neq y$, for all $i, j = 1, 2, \ldots$ with $i \neq j$ and $p \in (0, 1)$. Then $\{S_i\}$ has a unique common fixed point.

PROOF. The proof follows from Theorem 1 by considering the induced intuitionistic fuzzy metric space $(X, M_d, N_d, *, \Diamond)$, where M_d and N_d be fuzzy sets on $X^2 \times (0, \infty)$ defined as follows:

$$M_d(x, y, t) = \frac{t}{t + d(x, y)}, \quad N_d(x, y, t) = \frac{d(x, y)}{t + d(x, y)}.$$

□

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