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SELECTION PRINCIPLES AND HYPERSPACE TOPOLOGIES IN CLOSURE SPACES

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ABSTRACT. Relations between closure-type properties of hyperspaces over a Čech closure space (X, u) and covering properties of (X, u) are investigated.

Introduction

1. An operator $u : \mathcal{P}(X) \to \mathcal{P}(X)$ defined on the power set $\mathcal{P}(X)$ of a set X satisfying the axioms:
(C1) $u(\emptyset) = \emptyset$,
(C2) $A \subset u(A)$ for every $A \subset X$,
(C3) $u(A \cup B) = u(A) \cup u(B)$ for all $A, B \subset X$,
is called a *Čech closure operator* and the pair (X, u) is a *Čech closure space*. For short, (X, u) will be noted by X as well, and called a *closure space* or a *space*.

A subset A is *closed* in (X, u) if $u(A) = A$ holds. It is *open* if its complement is closed.

The *interior operator* $\mathrm{int}_u : \mathcal{P}(X) \to \mathcal{P}(X)$ is defined by means of the closure operator in the usual way: $\mathrm{int}_u = \mathrm{c} \circ u \circ \mathrm{c}$, where $\mathrm{c} : \mathcal{P}(X) \to \mathcal{P}(X)$ is the complement operator. A subset U is a *neighborhood* of a point x in X if $x \in \mathrm{int}_u U$ holds.

A closure space (X, u) is T_1 if for each two distinct points in X the following holds: $(\{x\} \cap u(\{y\})) \cup (\{y\} \cap u(\{x\})) = \emptyset$ whenever $x \neq y$. It is equivalent to: every one-point subset of X is closed in (X, u).

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A space (X, u) is T_2 (Hausdorff) if each two distinct points in X have disjoint neighborhoods.

All considered spaces are T_1.

2. A collection $\{G_\alpha\}$ is an *interior cover* of a set A in (X, u) if the collection $\{\text{int}_u G_\alpha\}$ covers A. We suppose that the interior of every element of an interior cover is non-empty and that each cover is non-trivial, i.e., that the set X does not belong to the cover.

A subset A in a space (X, u) is *compact* if every interior cover of A has a finite subcover.

Let $\mathcal{C}$ be a collection of subsets of X. An interior cover $\mathcal{U}$ is a $\mathcal{C}$-*cover* of X if for every $C \in \mathcal{C} \setminus \{X\}$ there is a $U \in \mathcal{U}$ such that $C \subset \text{int}_u U$ holds.

The collection of all interior covers $\mathcal{U}$ of (X, u) will be denoted by $\mathcal{I}$ and of all interior $\mathcal{C}$-covers by $\mathcal{IC}$.

The following notations are used:

$\mathcal{H} = \{u(A) \mid A \subset X\}$, $\mathcal{J} = \mathcal{H}^c = \{\text{int}_u(A) \mid A \subset X\}$,

$\Phi_x = \{A \subset X \mid x \in u(A) \setminus A\}$,

$\mathbf{F}(X)$ the family of all finite subsets of X,

$\mathbf{K}(X)$ the family of all compact subsets of X,

$\mathbf{Q}(X)$ the family of all closed subsets of X.

$\mathbf{F}(X)$- , $\mathbf{K}(X)$- and $\mathbf{Q}(X)$-interior covers will be called ω- , κ- and ζ-covers of X, respectively. $\mathcal{I}\Omega$ stands for the collection of all ω-covers of X, $\mathcal{IK}$ for the collection of all k-covers, and $\mathcal{IQ}$ for the collection of all ζ-covers.

For $A \subset X$ the usual notation is $A^+ = \{H \in \mathcal{H} \mid H \subset A\}$.

To the end Δ and Σ are subcollections of $\mathcal{H}$ closed for finite unions and containing all singletons. The *upper* Δ-*topology* for $\mathcal{H}$, denoted by Δ^+, has for a base the collection $\{(D^c)^+ \mid D \in \Delta\} \cup \{\mathcal{H}\}$.

Following [4], the $\mathbf{F}(X)^+$-topology will be denoted by $\mathbf{Z}^+$ and the $\mathbf{K}(X)^+$-topology, the upper Fell topology (or the co-compact topology), by $\mathbf{F}^+$. Also, $\mathbf{V}^+$ will stand for the $\mathbf{Q}(X)^+$-topology, the upper Vietoris topology.

Let $\mathcal{A}$ be a subcollection of $\mathcal{H}$. Each $A \in \mathcal{A}$ is of the form $A = u(B)$. We pick one such $B = B(A)$. The collection $\mathcal{U} = \{B(A)^c\}$ will be denoted by $\mathcal{A}^C$.

The following is our key lemma and the constructions and notations introduced in it are used throughout the paper.

LEMMA. (i) *Let* (X,u) *be a space, a subset* $Y \in \mathcal{J}$, *and* $\mathcal{U}$ *be an interior* Δ*-cover of* Y. *Set* $\mathcal{A} = \{(\mathrm{int}_u U)^{\mathrm{c}} \mid U \in \mathcal{U}\} = \{u(U^{\mathrm{c}}) \mid U \in \mathcal{U}\}$. *Then* $\mathcal{A} \subset \mathcal{H}$ *holds and* $Y^{\mathrm{c}} \in \mathrm{cl}_{\Delta^+}\mathcal{A}$.

(ii) *Conversely, having a collection* $\mathcal{A} \subset \mathcal{H}$ *and a set* $H \in \mathcal{H}$ *such that* $H \in \mathrm{cl}_{\Delta^+}\mathcal{A}$, *the collection* $\mathcal{U} = \mathcal{A}^C$ *is an interior* Δ*-cover of* H^{c}.

Proof. (i) Let $\mathcal{U}$ be an interior Δ-cover of Y and $(D^{\mathrm{c}})^+$ be a basic Δ^+-neighborhood of Y^{c}. Since $D \subset Y$, there is a $U \in \mathcal{U}$ such that $D \subset \mathrm{int}_u U$ holds. Then $A = (\mathrm{int}_u U)^{\mathrm{c}} \in \mathcal{A}$ and $A \subset D^{\mathrm{c}}$ imply $Y^{\mathrm{c}} \in \mathrm{cl}_{\Delta^+}\mathcal{A}$.

(ii) Let $H \in \mathrm{cl}_{\Delta^+}\mathcal{A}$ and $D \in \Delta$ such that $D \subset H^{\mathrm{c}}$. The family $(D^{\mathrm{c}})^+$ is a Δ^+-neighborhood of H. There is an $A \in \mathcal{A}$, such that $A \subset D^{\mathrm{c}}$. For the corresponding $U \in \mathcal{U} = \mathcal{A}^C$, $\mathrm{int}_u U = A^{\mathrm{c}} \supset D$ holds, so $\mathcal{U}$ is an interior Δ-cover of $Y = H^{\mathrm{c}}$. □

In particular, when $\Delta = \mathbf{F}(X)$, (respectively, $\Delta = \mathbf{K}(X)$, $\mathbf{Q}(X)$), then $Y^{\mathrm{c}} \in \mathrm{cl}_{\mathbf{Z}^+}\mathcal{A}$ (resp. $Y^{\mathrm{c}} \in \mathrm{cl}_{\mathbf{F}^+}\mathcal{A}$, $Y^{\mathrm{c}} \in \mathrm{cl}_{\mathbf{V}^+}\mathcal{A}$) for the corresponding interior ω-cover (resp. interior κ-cover, interior ζ-cover) of $Y \in \mathcal{J}$.

Let $\mathcal{A}$ and $\mathcal{B}$ be sets whose members are families of subsets of an infinite set X. Then ([8], [3]):

$\mathbf{S}_1(\mathcal{A},\mathcal{B})$ denotes the selection principle: For each sequence (A_n) of elements of $\mathcal{A}$ there is a sequence (b_n) such that $b_n \in A_n$ for each $n \in \mathbb{N}$ and $\{b_n \mid n \in \mathbb{N}\}$ is an element of $\mathcal{B}$.

$\mathbf{S}_{fin}(\mathcal{A},\mathcal{B})$ denotes the selection principle: For each sequence (A_n) of elements of $\mathcal{A}$ there is a sequence (B_n) of finite sets such that $B_n \subset A_n$ for each $n \in \mathbb{N}$ and $\cup_{n\in\mathbb{N}} B_n \in \mathcal{B}$.

Varying the collections $\mathcal{A}$ and $\mathcal{B}$ in the above defined selection principles, characterizations of the space (X,u) and its hyperspaces are defined. When (X,u) is a topological space, all definitions considered in this paper coincide with the corresponding topological ones. Interior covers are replaced with open covers denoted by $\mathcal{O}$, and $\mathcal{H}$ coincides with $\mathbf{Q}(X)$, the family of all closed subsets of X.

We assume that the space (X,u) is not compact.

All notions not explained here concerning selection principles can be found in [3], while those concerning Čech closure spaces in [1] and [6].

1. The Rothberger-like selection principles

The property $\mathbf{S}_1(\mathcal{O},\mathcal{O})$ for topological spaces was introduced by F. Rothberger in 1938 and it is called nowadays the Rothberger property.

A topological space X has countable strong fan tightness [7] if for each $x \in X$ the selection principle $\mathbf{S}_1(\Phi_x, \Phi_x)$ holds.

We generalize these notions to closure spaces in the following way:

DEFINITION 1. A space (X, u) has the *Rothberger property* if for every sequence $(\mathcal{U}_n)$ of interior covers of X there is a sequence (U_n) such that $U_n \in \mathcal{U}_n$ for each $n \in \mathbb{N}$ and the collection $\{U_n \mid n \in \mathbb{N}\}$ is an interior cover of X.

DEFINITION 2. A space (X, u) has *countable strong fan tightness* if for every $x \in X$ and each sequence (A_n) of subsets of X such that $x \in \bigcap_{n\in\mathbb{N}} u(A_n)$, there is a sequence (x_n), $x_n \in A_n$, such that $x \in u(\{x_n \mid n \in \mathbb{N}\})$.

THEOREM 1. *For a space* (X, u) *and collections* Δ *and* Σ *the following are equivalent:*

(1) $\mathcal{H}$ *satisfies the selection principle* $\mathbf{S}_1(\Phi_H^{\Delta^+}, \Phi_H^{\Sigma^+})$ *for each* $H \in \mathcal{H}$.

(2) *Each* $Y \in \mathcal{J}$ *satisfies* $\mathbf{S}_1(\mathcal{I}\Delta, \mathcal{I}\Sigma)$.

Proof. (1)$\Rightarrow$(2): Let $(\mathcal{U}_n)$ be a sequence of interior Δ-covers of Y. Then for the sequence $(\mathcal{A}_n)$ of elements of $\mathcal{H}$, $\mathcal{A}_n = \{(\mathrm{int}_u U)^c \mid U \in \mathcal{U}_n\} = \{u(U^c) \mid U \in \mathcal{U}_n\}$, $Y^c \in \mathrm{cl}_{\Delta^+}\mathcal{A}_n$ for each $n \in \mathbb{N}$ by Lemma. By assumption, there is a sequence (A_n) such that $A_n \in \mathcal{A}_n$ for every $n \in \mathbb{N}$ and $Y^c \in \mathrm{cl}_{\Sigma^+}\{A_n \mid n \in \mathbb{N}\}$. The corresponding collection $\{U_n \mid n \in \mathbb{N}\}$, $A_n = u(U_n^c)$, is an interior Σ-cover of Y. Indeed, for every $S \in \Sigma$ such that $S \subset Y$, $(S^c)^+$ is a neighborhood of Y^c. There is an A_m such that $A_m \in (S^c)^+$, i.e., $A_m \subset S^c$ implies $S \subset A_m^c = (\mathrm{c} \circ u)(U_m^c) = \mathrm{int}_u U_m$.

(2)$\Rightarrow$(1): Let $(\mathcal{A}_n)$ be a sequence of elements of $\mathcal{H}$ such that $H \in \mathcal{H}$ belongs to $\mathrm{cl}_{\Delta^+}\mathcal{A}_n$ for each $n \in \mathbb{N}$. Then by Lemma, the corresponding sequence $(\mathcal{U}_n)$, $\mathcal{U}_n = \mathcal{A}_n^C$ for $n \in \mathbb{N}$, is a sequence of interior Δ-covers of $H^c = Y$. Since H^c satisfies $\mathbf{S}_1(\mathcal{I}\Delta, \mathcal{I}\Sigma)$, there is a sequence (U_n), $U_n \in \mathcal{U}_n$, such that $\{U_n \mid n \in \mathbb{N}\}$ is an interior Σ-cover of H^c. By applying Lemma again, $H \in \mathrm{cl}_{\Sigma^+}\{A_n \mid n \in \mathbb{N}\}$ which proves that (1) holds. □

In particular, when $\Sigma = \Delta$, the next statement is true.

THEOREM 2. *For a space* (X, u) *and a collection* Δ *the following are equivalent:*

(1) $(\mathcal{H}, \Delta^+)$ *has countable strong fan tightness.*

(2) *Each* $Y \in \mathcal{J}$ *satisfies* $\mathbf{S}_1(\mathcal{I}\Delta, \mathcal{I}\Delta)$.

Setting $\Delta = \mathbf{F}(X)$, resp. $\mathbf{K}(X)$, $\mathbf{Q}(X)$, in Theorem 2, we get generalizations of the results for topological spaces.

COROLLARY 1. (cf. [2]) *For a space* (X,u) *the following are equivalent:*

(1) $(\mathcal{H},\mathbf{Z}^+)$ *has countable strong fan tightness.*

(2) *Each* $Y\in\mathcal{J}$ *satisfies* $\mathbf{S}_1(\mathcal{I}\Omega,\mathcal{I}\Omega)$.

COROLLARY 2. (cf. [2]) *For a Hausdorff space* (X,u) *the following are equivalent:*

(1) $(\mathcal{H},\mathbf{F}^+)$ *has countable strong fan tightness.*

(2) *Each* $Y\in\mathcal{J}$ *satisfies* $\mathbf{S}_1(\mathcal{I}\mathcal{K},\mathcal{I}\mathcal{K})$.

Proof. Note that in a T_2 space (X,u) a compact subset is closed. Thus the family $\mathcal{K}$ of compact subsets is closed for finite unions, contains all singletons and is a subfamily of $\mathcal{H}$. □

COROLLARY 3. *For a space* (X,u) *the following are equivalent:*

(1) $(\mathcal{H},\mathbf{V}^+)$ *has countable strong fan tightness.*

(2) *Each* $Y\in\mathcal{J}$ *satisfies* $\mathbf{S}_1(\mathcal{I}\mathcal{Q},\mathcal{I}\mathcal{Q})$.

Setting in Theorem 1: $\Delta=\mathbf{K}(X)$ and $\Sigma=\mathbf{F}(X)$, (resp. $\Delta=\mathbf{Q}(X)$ and $\Sigma=\mathbf{K}(X)$) we get

COROLLARY 4. (cf. [2]) *For a* T_2 *space* (X,u) *the following are equivalent:*

(1) $\mathcal{H}$ *satisfies* $\mathbf{S}_1(\Phi_H^{\mathbf{F}^+},\Phi_H^{\mathbf{Z}^+})$ *for each* $H\in\mathcal{H}$.

(2) *Each* $Y\in\mathcal{J}$ *satisfies* $\mathbf{S}_1(\mathcal{I}\mathcal{K},\mathcal{I}\Omega)$.

COROLLARY 5. *For a* T_2 *space* (X,u) *the following are equivalent:*

(1) $\mathcal{H}$ *satisfies* $\mathbf{S}_1(\Phi_H^{\mathbf{V}^+},\Phi_H^{\mathbf{F}^+})$ *for each* $H\in\mathcal{H}$.

(2) *Each* $Y\in\mathcal{J}$ *satisfies* $\mathbf{S}_1(\mathcal{I}\mathcal{Q},\mathcal{I}\mathcal{K})$.

We denote by $\mathcal{D}$ the family of dense subsets of a space. When necessary to distinguish between two topologies on the same set we use extra notations; for example: $\mathcal{D}_{\Delta^+}$ (resp. $\mathcal{D}_{\Sigma^+}$) stands for the family of dense collections in the space $(\mathcal{H},\Delta^+)$ (resp. $(\mathcal{H},\Sigma^+)$).

THEOREM 3. *For a space* (X,u) *and collections* Δ *and* Σ *the following are equivalent:*

(1) $\mathcal{H}$ *satisfies* $\mathbf{S}_1(\mathcal{D}_{\Delta^+},\mathcal{D}_{\Sigma^+})$.

(2) (X,u) *satisfies* $\mathbf{S}_1(\mathcal{I}\Delta,\mathcal{I}\Sigma)$.

Proof. (1) ⇒ (2): Let $(\mathcal{U}_n)$ be a sequence of interior Δ-covers of X. For each $n\in\mathbb{N}$ the corresponding $\mathcal{A}_n$ is dense in the space $(\mathcal{H},\Delta^+)$ since for every non-empty basic open set $(D^c)^+$ there is an $U_n\in\mathcal{U}_n$ such that $D\subset \mathrm{int}_u(U_n)$. Hence $A_n=u(U_n^c)\subset D^c$ and $A_n\in\mathcal{A}_n\cap(D^c)^+$ implies $\mathcal{A}_n$ is dense in $(\mathcal{H},\Delta^+)$. By applying (1), there is a sequence (A_n), $A_n\in\mathcal{A}_n$, such that $\mathrm{cl}_{\Sigma^+}\{A_n \mid n\in\mathbb{N}\}=\mathcal{H}$. For each $n \in \mathbb{N}$ choose $U_n \in \mathcal{U}_n$ such that $u(U_n^c) = A_n$. The collection $\{U_n \mid n \in \mathbb{N}\}$ is an interior Σ-covers of X. Indeed, for every $S \in \Sigma$ there is an $A_m \in (S^c)^+$; hence for the corresponding U_m, $S \subset \mathrm{int}_u(U_m)$ holds. Thus (X, u) satisfies $\mathbf{S}_1(\mathcal{I}\Delta, \mathcal{I}\Sigma)$.

(2) ⇒ (1): Let $(\mathcal{D}_n)$ be a sequence of dense subsets in $(\mathcal{H}, \Delta^+)$. For each $n \in \mathbb{N}$ the collection $\mathcal{U}_n = \mathcal{D}_n^C$ is an interior Δ-cover of X. It is true since for every $G \in \Delta$ and $D \in (G^c)^+ \cap \mathcal{D}_n$, $D = u(A) \subset G^c$ implies $G \subset \mathrm{int}_u(A^c) = \mathrm{int}_u U$ for the corresponding $U \in \mathcal{U}_n$. By assumption, there is a sequence (U_n), $U_n \in \mathcal{U}_n$, such that the collection $\{U_n \mid n \in \mathbb{N}\}$ is an interior Σ-cover of X. The collection $\{D_n \mid n \in \mathbb{N}\}$, $D_n = u(U_n^c)$, is dense in $(\mathcal{H}, \Sigma^+)$. For every $S \in \Sigma$ there is an $m \in \mathbb{N}$ such that $S \subset \mathrm{int}_u U_m$ implies $D_m \subset S^c$, i.e., $\{D_n \mid n \in \mathbb{N}\} \cap (S^c)^+ \neq \emptyset$. □

Again, when $\Delta = \mathbf{F}(X)$, (resp. $\mathbf{K}(X)$, $\mathbf{Q}(X)$), or $\Delta = \mathbf{K}(X)$ and $\Sigma = \mathbf{F}(X)$, (resp. $\Delta = \mathbf{Q}(X)$ and $\Sigma = \mathbf{K}(X)$) we get

COROLLARY 6. (cf. [2]) *For a space (X, u) the following are equivalent:*

(1) $(\mathcal{H}, \mathbf{Z}^+)$ *satisfies* $\mathbf{S}_1(\mathcal{D}, \mathcal{D})$.
(2) (X, u) *satisfies* $\mathbf{S}_1(\mathcal{I}\Omega, \mathcal{I}\Omega)$.

COROLLARY 7. (cf. [2]) *For a T_2 space (X, u) the following are equivalent:*

(1) $(\mathcal{H}, \mathbf{F}^+)$ *satisfies* $\mathbf{S}_1(\mathcal{D}, \mathcal{D})$.
(2) (X, u) *satisfies* $\mathbf{S}_1(\mathcal{I}\mathcal{K}, \mathcal{I}\mathcal{K})$.

COROLLARY 8. *For a space (X, u) the following are equivalent:*

(1) $(\mathcal{H}, \mathbf{V}^+)$ *satisfies* $\mathbf{S}_1(\mathcal{D}, \mathcal{D})$.
(2) (X, u) *satisfies* $\mathbf{S}_1(\mathcal{I}\mathcal{Q}, \mathcal{I}\mathcal{Q})$.

COROLLARY 9. (cf. [2]) *For a T_2 space (X, u) the following are equivalent:*

(1) $\mathcal{H}$ *satisfies* $\mathbf{S}_1(\mathcal{D}_{\mathbf{F}^+}, \mathcal{D}_{\mathbf{Z}^+})$.
(2) (X, u) *satisfies* $\mathbf{S}_1(\mathcal{I}\mathcal{K}, \mathcal{I}\Omega)$.

COROLLARY 10. *For a T_2 space (X, u) the following are equivalent:*

(1) $\mathcal{H}$ *satisfies* $\mathbf{S}_1(\mathcal{D}_{\mathbf{V}^+}, \mathcal{D}_{\mathbf{F}^+})$.
(2) (X, u) *satisfies* $\mathbf{S}_1(\mathcal{I}\mathcal{Q}, \mathcal{I}\mathcal{K})$.

A family $\mathcal{N}$ of subsets of X is a π-*network* for a topological space X if every open set in X contains some element of $\mathcal{N}$.

We introduce the following definition.

DEFINITION 3. A family $\mathcal{N} = \{N_\lambda\}_{\lambda \in \Lambda}$ of subsets of a closure space (X, u) is a *π-network* for (X, u) if every non-empty interior of a subset of X contains some N_λ, i.e., for every $A \subset X$ such that $\mathrm{int}_u A \neq \emptyset$, there is $N_\lambda \subset \mathrm{int}_u A$.

By Π_Δ (respectively $\Pi_\omega, \Pi_\kappa, \Pi_\varsigma$) we denote the family of π-networks consisting of elements from $\Delta \subset \mathcal{H}$ (resp. $\mathbf{F}(X), \mathbf{K}(X), \mathbf{Q}(X)$).

THEOREM 4. *For a space (X, u) and collections Δ and Σ the following are equivalent:*

(1) *$\mathcal{H}$ satisfies $\mathbf{S}_1(\mathcal{O}\Delta^+, \mathcal{O}\Sigma^+)$.*

(2) *(X, u) satisfies $\mathbf{S}_1(\Pi_\Delta, \Pi_\Sigma)$.*

Proof. (1)$\Rightarrow$(2): Let (Δ_n) be a sequence from Π_Δ. Then for each $n \in \mathbb{N}$ the collection $\{(D^c)^+ \mid D \in \Delta_n\}$ is a Δ^+-open cover of $\mathcal{H}$. Indeed, for a fixed $n \in \mathbb{N}$ and any $H \in \mathcal{H}$, $H^c = \mathrm{int}_u(A^c)$ for some $A \subset X$. There is a $D \in \Delta_n$ such that $D \subset H^c$. Thus $H \in (D^c)^+$. By (1), there is a sequence (D_n), $D_n \in \Delta_n$, such that the collection $\{(D_n^c)^+ \mid n \in \mathbb{N}\}$ is a Σ^+-open cover of $\mathcal{H}$. Then the collection $\{D_n \mid n \in \mathbb{N}\}$ is a $\Sigma - \pi$-network for (X, u). For, let $A \subset X$ such that $\mathrm{int}_u A \neq \emptyset$. Then for $u(A^c)$ there is a D_n such that $u(A^c) \in (D_n^c)^+$ holds, that is $u(A^c) \subset D_n^c$, i.e., $D_n \subset \mathrm{int}_u A$.

(2)$\Rightarrow$(1): Let $(\mathcal{U}_n)$ be a sequence of Δ^+-open covers of $\mathcal{H}$. We may assume that each cover consists of basic open sets, that is $\mathcal{U}_n = \{(D_{n,\lambda}^c)^+ \mid \lambda \in \Lambda\}$. Then for each n the collection $\Delta_n = \{D_{n,\lambda} \mid \lambda \in \Lambda\}$ is a $\Delta - \pi$-network for (X, u). Indeed, for $A \subset X$ such that $\mathrm{int}_u A \neq \emptyset$, there is a $D_{n,\lambda}$ such that $u(A^c) \in (D_{n,\lambda}^c)^+$, that is $u(A^c) \subset D_{n,\lambda}^c$, i.e., $D_{n,\lambda} \subset \mathrm{int}_u A$. Applying (2), there is a sequence (D_n), $D_n \in \Delta_n$, such that the collection $\{D_n \mid n \in \mathbb{N}\}$ is a $\Sigma - \pi$-network for (X, u). Then the collection $\{(D_n^c)^+ \mid n \in \mathbb{N}\}$ is a Σ^+-open cover of $\mathcal{H}$. □

By taking $\Sigma = \Delta$ we get

THEOREM 5. *For a space (X, u) and a collection Δ the following are equivalent:*

(1) *$(\mathcal{H}, \Delta^+)$ has the Rothberger property.*

(2) *(X, u) satisfies $\mathbf{S}_1(\Pi_\Delta, \Pi_\Delta)$.*

Again, by specifying the family Δ we get special cases.

COROLLARY 11. (cf. [2]) *For a space (X, u) the following are equivalent :*

(1) *$(\mathcal{H}, \mathbf{Z}^+)$ has the Rothberger property.*

(2) *(X, u) satisfies $\mathbf{S}_1(\Pi_\omega, \Pi_\omega)$.*

COROLLARY 12. (cf. [2]) *For a T_2 space (X,u) the following are equivalent:*

(1) $(\mathcal{H},\mathbf{F}^+)$ *has the Rothberger property.*

(2) (X,u) *satisfies* $\mathbf{S}_1(\Pi_\kappa,\Pi_\kappa)$.

COROLLARY 13. *For a space (X,u) the following are equivalent:*

(1) $(\mathcal{H},\mathbf{V}^+)$ *has the Rothberger property.*

(2) (X,u) *satisfies* $\mathbf{S}_1(\Pi_\zeta,\Pi_\zeta)$.

We end this section by proving a result concerning Rothberger-type selection principles.

In [7, Lemma] a relation between the property $\mathbf{S}_1(\mathcal{O},\mathcal{O})$ for each finite product of a topological space X and the property $\mathbf{S}_1(\Omega,\Omega)$ for X was given. For closure spaces we prove the next statement.

THEOREM 6. *If each finite product of (X,u) satisfies $\mathbf{S}_1(\mathcal{I},\mathcal{I})$, then each finite product of (X,u) has the property $\mathbf{S}_1(\mathcal{I}\Omega,\mathcal{I}\Omega)$.*

Proof. (i) First we prove that if each finite product of X satisfies $\mathbf{S}_1(\mathcal{I},\mathcal{I})$, then X has the property $\mathbf{S}_1(\mathcal{I}\Omega,\mathcal{I}\Omega)$. Let $(\mathcal{U}_n)$ be a sequence of interior ω-covers of X. Let $\mathbb{N}=\cup_{m\in\mathbb{N}}N_m$ be a partition of $\mathbb{N}$ into infinite sets. For each $m\in\mathbb{N}$ and each $k\in N_m$ let $\mathcal{V}_k=\{U^m \mid U\in \mathcal{U}_k\}$. Since for the interior operator in the product space (X^m,v) the equality $\mathrm{int}_v(U^m)=(\mathrm{int}_u U)^m$ holds, $(\mathcal{V}_k)$ for $k\in N_m$, is a sequence of interior covers of X^m as for each $x=(x_1,\dots,x_m)\in X^m$ and each $k\in N_m$ there is a $U\in\mathcal{U}_k$ such that $\{x_1,\dots,x_m\}\subset \mathrm{int}_u U$, so that $x\in(\mathrm{int}_u U)^m$. Applying to the sequence $(\mathcal{V}_k)$, $k\in N_m$, the assumption that each finite product of X satisfies $\mathbf{S}_1(\mathcal{I},\mathcal{I})$, there is for each $m\in\mathbb{N}$ a sequence (V_k), $V_k\in\mathcal{V}_k$ for each $k\in N_m$, such that the collection $\{V_k \mid k\in N_m\}$ is an interior cover of X^m. For each $k\in N_m$ let U_k be an element in $\mathcal{U}_k$ with $V_k=U_k^m$.

Then the collection $\{U_k \mid k\in N_m,\, m\in\mathbb{N}\}$ is an interior ω-cover of (X,u) which witnesses for $(\mathcal{U}_n)$ that $\mathbf{S}_1(\mathcal{I}\Omega,\mathcal{I}\Omega)$ holds. Indeed, for every finite set $F=\{x_1,\dots,x_m\}\subset X$ there is a $V_k=U_k^m\in\mathcal{V}_k$ with $(x_1,\dots,x_m)\in \mathrm{int}_v U_k^m$, hence $F\subset \mathrm{int}_u U_k$.

(ii) Now we show that if X has the property $\mathbf{S}_1(\mathcal{I}\Omega,\mathcal{I}\Omega)$, then all finite powers of X satisfy $\mathbf{S}_1(\mathcal{I}\Omega,\mathcal{I}\Omega)$. Fix m. Let $(\mathcal{U}_n)$ be a sequence of interior ω-covers of X^m. For each $n\in\mathbb{N}$ set $\mathcal{V}_n=\{V\subset X \mid V^m\subset U \text{ for some } U\in\mathcal{U}_n\}$. The collection $\mathcal{V}_n$ is an interior ω-cover of X for each n. Indeed, for every finite set $F=\{x_1,\dots,x_k\}\subset X$ there is a $U\in\mathcal{U}_n$ such that $F^m\subset \mathrm{int}_v U$. We choose a $V\subset X$ so that $F^m\subset \mathrm{int}_v V^m=(\mathrm{int}_u V)^m\subset V^m\subset U$. The set V satisfies the required condition, that is $F\subset V\in\mathcal{V}_n$. Applying the assumption to the sequence $\mathcal{V}_n$ there exist $V_n \in \mathcal{V}_n$, $n \in \mathbb{N}$, such that the collection $\{V_n \mid n \in \mathbb{N}\}$ is an interior ω-cover of X. For each n we pick $U_n \in \mathcal{U}_n$ so that $V_n^m \subset U_n$. Since the collection $\{V_n^m \mid n \in \mathbb{N}\}$ is an interior ω-cover of X^m, so is $\{U_n \mid n \in \mathbb{N}\}$. □

2. The Menger-like selection principles

In topological spaces the property $\mathbf{S}_{fin}(\mathcal{O}, \mathcal{O})$ is known as the Menger property. A topological space X has countable fan tightness if for each $x \in X$ the selection principle $\mathbf{S}_{fin}(\Phi_x, \Phi_x)$ holds.

In closure spaces these definitions read as follows.

DEFINITION 4. A space (X, u) has the *Menger property* if for every sequence $(\mathcal{U}_n)$ of interior covers of X there is a sequence $(\mathcal{V}_n)$ of finite collections such that $\mathcal{V}_n \subset \mathcal{U}_n$ for each $n \in \mathbb{N}$ and the collection $\bigcup_{n \in \mathbb{N}} \mathcal{V}_n$ is an interior cover of X.

DEFINITION 5. A space (X, u) has *countable fan tightness* if for every $x \in X$ and for each sequence (A_n) of subsets of X such that $x \in \bigcap_{n \in \mathbb{N}} u(A_n)$, there exist finite sets $B_n \subset A_n$ such that $x \in u(\bigcup_{n \in \mathbb{N}} B_n)$.

The proofs of the next theorems are analogous to those in the previous section. The selection principle $\mathbf{S}_1$ is replaced with $\mathbf{S}_{fin}$.

THEOREM 7. *For a space (X, u) and collections Δ and Σ the following are equivalent:*

(1) *$\mathcal{H}$ satisfies the selection principle $\mathbf{S}_{fin}(\Phi_H^{\Delta^+}, \Phi_H^{\Sigma^+})$ for each $H \in \mathcal{H}$.*

(2) *Each $Y \in \mathcal{J}$ satisfies $\mathbf{S}_{fin}(\mathcal{I}\Delta, \mathcal{I}\Sigma)$.*

THEOREM 8. *For a space (X, u) and a collection Δ the following are equivalent:*

(1) *$(\mathcal{H}, \Delta^+)$ has countable fan tightness.*

(2) *Each $Y \in \mathcal{J}$ satisfies $\mathbf{S}_{fin}(\mathcal{I}\Delta, \mathcal{I}\Delta)$.*

THEOREM 9. *For a space (X, u) and collections Δ and Σ the following are equivalent:*

(1) *$\mathcal{H}$ satisfies $\mathbf{S}_{fin}(\mathcal{D}_{\Delta^+}, \mathcal{D}_{\Sigma^+})$.*

(2) *(X, u) satisfies $\mathbf{S}_{fin}(\mathcal{I}\Delta, \mathcal{I}\Sigma)$.*

THEOREM 10. *For a space (X, u) and collections Δ and Σ the following are equivalent:*

(1) *$\mathcal{H}$ satisfies $\mathbf{S}_{fin}(\mathcal{O}\Delta^+, \mathcal{O}\Sigma^+)$.*

(2) *(X, u) satisfies $\mathbf{S}_{fin}(\Pi_\Delta, \Pi_\Sigma)$.*

In particular, when specifying Δ and Σ the statements corresponding to Corollaries 1-10 are obtained.

By replacing the principle $\mathbf{S}_1$ with $\mathbf{S}_{fin}$ in Theorem 6, we get by a similar proof

THEOREM 11. *If each finite product of* (X, u) *satisfies* $\mathbf{S}_{fin}(\mathcal{I}, \mathcal{I})$, *then each finite product of* (X, u) *has the property* $\mathbf{S}_{fin}(\mathcal{I}\Omega, \mathcal{I}\Omega)$.

3. Set-tightness

Recall that the *set-tightness* t_s of a space X is countable if for each $A \subset X$ and each $x \in \overline{A}$ there is a sequence (A_n) of subsets of A such that $x \in \overline{\cup\{A_n \mid n \in \mathbb{N}\}}$ but $x \notin \overline{A_n}$ for each $n \in \mathbb{N}$.

THEOREM 12. *For a space* (X, u) *and a collection* Δ *the following are equivalent:*

(1) $(\mathcal{H}, \Delta^+)$ *has countable set-tightness.*

(2) *For each* $Y \in \mathcal{J}$ *and each interior* Δ*-cover* $\mathcal{U}$ *of* Y *there is a countable collection* $\{\mathcal{U}_n \mid n \in \mathbb{N}\}$ *of subsets of* $\mathcal{U}$ *such that no* $\mathcal{U}_n$ *is an interior* Δ*-cover of* Y *and* $\cup_{n \in \mathbb{N}} \mathcal{U}_n$ *is an interior* Δ*-cover of* Y.

Proof. Follows from definitions and Lemma. □

In particular, setting $\Delta = \mathbf{F}(X)$ (resp. $\mathbf{K}(X)$, $\mathbf{Q}(X)$), we get

COROLLARY 14. (cf. [2]) *For a space* (X, u) *the following are equivalent:*

(1) $(\mathcal{H}, \mathbf{Z}^+)$ *has countable set-tightness.*

(2) *For each* $Y \in \mathcal{J}$ *and each interior* ω*-cover* $\mathcal{U}$ *of* Y *there is a countable collection* $\{\mathcal{U}_n \mid n \in \mathbb{N}\}$ *of subsets of* $\mathcal{U}$ *such that no* $\mathcal{U}_n$ *is an interior* ω*-cover of* Y *and* $\cup_{n \in \mathbb{N}} \mathcal{U}_n$ *is an interior* ω*-cover of* Y.

COROLLARY 15. (cf. [2]) *For a* T_2 *space* (X, u) *the following are equivalent:*

(1) $(\mathcal{H}, \mathbf{F}^+)$ *has countable set-tightness.*

(2) *For each* $Y \in \mathcal{J}$ *and each interior* κ*-cover* $\mathcal{U}$ *of* Y *there is a countable collection* $\{\mathcal{U}_n \mid n \in \mathbb{N}\}$ *of subsets of* $\mathcal{U}$ *such that no* $\mathcal{U}_n$ *is an interior* κ*-cover of* Y *and* $\cup_{n \in \mathbb{N}} \mathcal{U}_n$ *is an interior* κ*-cover of* Y.

COROLLARY 16. *For a space* (X, u) *the following are equivalent:*

(1) $(\mathcal{H}, \mathbf{V}^+)$ *has countable set-tightness.*

(2) *For each* $Y \in \mathcal{J}$ *and each interior* ζ*-cover* $\mathcal{U}$ *of* Y *there is a countable collection* $\{\mathcal{U}_n \mid n \in \mathbb{N}\}$ *of subsets of* $\mathcal{U}$ *such that no* $\mathcal{U}_n$ *is an interior* ζ*-cover of* Y *and* $\cup_{n \in \mathbb{N}} \mathcal{U}_n$ *is an interior* ζ*-cover of* Y.

4. The Pytkeev property

A topological space X has the *Pytkeev property* if for every $A \subset X$ and each $x \in \overline{A \setminus \{x\}}$ there is a countable collection $\{A_n \mid n \in \mathbb{N}\}$ of countable infinite subsets of A which is a π-network at x, i.e., each neighborhood of x contains some A_n.

Let τ and σ be two topologies on a set X. Then X has the (τ, σ)*−Pytkeev property* if for every $A \subset X$ and each $x \in \mathrm{cl}_\tau(A \setminus \{x\})$ there is a countable collection $\{A_n \mid n \in \mathbb{N}\}$ of countable infinite subsets of A which is a π-network at x with respect to the σ topology. In the next theorem we suppose that for every interior Δ-cover $\mathcal{U}$ of Y, $Y \neq \mathrm{int}_u U$ for each $U \in \mathcal{U}$ holds.

THEOREM 13. *For a space (X, u) and collections Δ and Σ the following are equivalent:*

(1) *$\mathcal{H}$ has the (Δ^+, Σ^+)-Pytkeev property.*

(2) *For each $Y \in \mathcal{J}$ and each interior Δ-cover $\mathcal{U}$ of Y there is a sequence $(\mathcal{U}_n)$ of infinite subsets of $\mathcal{U}$ such that $\{\cap \mathrm{int}_u \mathcal{U}_n \mid n \in \mathbb{N}\}$ is a (not necessarily interior) Σ-cover of Y.* $[\cap \mathrm{int}_u \mathcal{U}_n = \cap\{\mathrm{int}_u U \mid U \in \mathcal{U}_n\}]$

Proof. As in [4, Theorem 8] by using Lemma. □

In particular, setting $\Delta = \Sigma = \mathbf{F}(X)$ (resp. $\Delta = \Sigma = \mathbf{K}(X)$, $\mathbf{Q}(X)$), we get

COROLLARY 17. (cf. [4]) *For a space (X, u) the following are equivalent:*

(1) *$(\mathcal{H}, \mathbf{Z}^+)$ has the Pytkeev property.*

(2) *For each $Y \in \mathcal{J}$ and each interior ω-cover $\mathcal{U}$ of Y there is a sequence $(\mathcal{U}_n)$ of countable infinite subsets of $\mathcal{U}$ such that $\{\cap \mathrm{int}_u \mathcal{U}_n \mid n \in \mathbb{N}\}$ is a (not necessarily interior) ω-cover of Y.*

COROLLARY 18. (cf. [4]) *For a T_2 space (X, u) the following are equivalent:*

(1) *$(\mathcal{H}, \mathbf{F}^+)$ has the Pytkeev property.*

(2) *For each $Y \in \mathcal{J}$ and each interior κ-cover $\mathcal{U}$ of Y there is a sequence $(\mathcal{U}_n)$ of countable infinite subsets of $\mathcal{U}$ such that $\{\cap \mathrm{int}_u \mathcal{U}_n \mid n \in \mathbb{N}\}$ is a (not necessarily interior) κ-cover of Y.*

COROLLARY 19. *For a space (X, u) the following are equivalent:*

(1) *$(\mathcal{H}, \mathbf{V}^+)$ has the Pytkeev property.*

(2) *For each $Y \in \mathcal{J}$ and each interior ζ-cover $\mathcal{U}$ of Y there is a sequence $(\mathcal{U}_n)$ of countable infinite subsets of $\mathcal{U}$ such that $\{\cap \mathrm{int}_u \mathcal{U}_n \mid n \in \mathbb{N}\}$ is a (not necessarily interior) ζ-cover of Y.*

5. The Hurewicz-like selection principles

The notion of groupability was introduced in [5]. A countable interior $\mathcal{C}$-cover $\mathcal{U}$ of a space (X,u) is *groupable* if there is a partition $\mathcal{U} = \cup_{n\in\mathbb{N}}\mathcal{U}_n$ into finite sets such that for each $C \in \mathcal{C}$ and for all but finitely many n there is a $U \in \mathcal{U}_n$ such that $C \subset \mathrm{int}_u U$ holds. Let $\mathcal{IC}^{gp}$ denote the family of groupable interior $\mathcal{C}$-covers of the space.

A countable set $A \in \Phi_x$ is *groupable* if there is a partition $A = \cup_{n\in\mathbb{N}} A_n$ into finite sets such that each neighborhood of x has non-empty intersection with all but finitely many A_n.

Recall that a space X has the *Reznichenko property* [= is weakly Fréchet-Urysohn] if for every $A \subset X$ and each $x \in \overline{A} \setminus A$ there is a countable infinite family $\mathcal{A}$ of finite pairwise disjoint subsets of A such that each neighborhood of x meets all but finitely many elements of $\mathcal{A}$.

THEOREM 14. *For a space (X,u) and a collection Δ the following are equivalent:*

(1) $(\mathcal{H},\Delta^+)$ *has the Reznichenko property.*

(2) *For each $Y \in \mathcal{J}$ and each interior Δ-cover $\mathcal{U}$ of Y, there is a sequence $(\mathcal{U}_n)$ of finite pairwise disjoint subsets of $\mathcal{U}$ such that each $D \in \Delta$ belongs to $\mathrm{int}_u U_n$ for some $U_n \in \mathcal{U}_n$ for all but finitely many n.*

Proof. (1)$\Rightarrow$(2): Let $Y \in \mathcal{J}$ and $\mathcal{U}$ be an interior Δ-cover of Y. By Lemma, $Y^c \in \mathrm{cl}_{\Delta^+}\mathcal{A}$ where $\mathcal{A} = \{u(U^c) \mid U \in \mathcal{U}\}$. By assumption, there is a sequence $(\mathcal{A}_n)$ of finite pairwise disjoint subsets of $\mathcal{A}$ such that each Δ^+-neighborhood of Y^c intersects $\mathcal{A}_n$ for all but finitely many n. The sequence $(\mathcal{U}_n)$ defined by $\mathcal{U}_n = \mathcal{A}_n^C$ satisfies the required conditions. Indeed, $\mathcal{U}_n$ are finite, pairwise disjoint and for $D \in \Delta$ such that $D \subset Y$, the collection $(D^c)^+$ is a Δ^+-neighborhood of Y^c. There is n_0 such that $(D^c)^+ \cap \mathcal{A}_n \neq \emptyset$ for each $n > n_0$. Hence, for every $n > n_0$ there is a set $A_n \in \mathcal{A}_n$ such that $A_n \subset D^c$, that is $D \subset A_n^c = \mathrm{int}_u U_n$ and $U_n \in \mathcal{U}_n$ holds.

(2)$\Rightarrow$(1): Let $\mathcal{A} \subset \mathcal{H}$ and $H \in \mathcal{H}$ so that $H \in \mathrm{cl}_{\Delta^+}\mathcal{A} \setminus \mathcal{A}$. Then $\mathcal{U} = \mathcal{A}^C$ is an interior Δ-cover of $H^c = Y$. Applying (2) to Y and $\mathcal{U}$, there is a sequence $(\mathcal{U}_n)$ of finite pairwise disjoint subsets of $\mathcal{U}$ such that for each $D \subset Y$, $D \in \Delta$, for all but finitely many n there is a $U_n \in \mathcal{U}_n$ such that $D \subset \mathrm{int}_u U_n$. For each n we set $\mathcal{G}_n = \{u(U^c) \mid U \in \mathcal{U}_n\}$. The collection $\{\mathcal{G}_n \mid n \in \mathbb{N}\}$ satisfies (1). □

By combining Theorem 14 with Theorems 2 and 8 we get the following statements.

THEOREM 15. *For a space* (X, u) *and a collection* Δ *the following are equivalent:*

(1) $(\mathcal{H}, \Delta^+)$ *has the Reznichenko property and countable strong fan tightness.*

(2) *Each* $Y \in \mathcal{J}$ *satisfies* $\mathbf{S}_1(\mathcal{I}\Delta, \mathcal{I}\Delta^{gp})$.

THEOREM 16. *For a space* (X, u) *and a collection* Δ *the following are equivalent:*

(1) $(\mathcal{H}, \Delta^+)$ *has the Reznichenko property and countable fan tightness.*

(2) *Each* $Y \in \mathcal{J}$ *satisfies* $\mathbf{S}_{fin}(\mathcal{I}\Delta, \mathcal{I}\Delta^{gp})$.

In particular, setting $\Delta = \mathbf{F}(X)$ (resp. $\Delta = \mathbf{K}(X)$, $\mathbf{Q}(X)$), we get

COROLLARY 20. (cf. [4]) *For a space* (X, u) *the following are equivalent:*

(1) $(\mathcal{H}, \mathbf{Z}^+)$ *has the Reznichenko property and countable strong fan tightness.*

(2) *Each* $Y \in \mathcal{J}$ *satisfies* $\mathbf{S}_1(\mathcal{I}\Omega, \mathcal{I}\Omega^{gp})$.

COROLLARY 21. (cf. [4]) *For a space* (X, u) *the following are equivalent:*

(1) $(\mathcal{H}, \mathbf{Z}^+)$ *has the Reznichenko property and countable fan tightness.*

(2) *Each* $Y \in \mathcal{J}$ *satisfies* $\mathbf{S}_{fin}(\mathcal{I}\Omega, \mathcal{I}\Omega^{gp})$.

COROLLARY 22. (cf. [4]) *For a* T_2 *space* (X, u) *the following are equivalent:*

(1) $(\mathcal{H}, \mathbf{F}^+)$ *has the Reznichenko property and countable strong fan tightness.*

(2) *Each* $Y \in \mathcal{J}$ *satisfies* $\mathbf{S}_1(\mathcal{I}\mathcal{K}, \mathcal{I}\mathcal{K}^{gp})$.

COROLLARY 23. (cf. [4]) *For a* T_2 *space* (X, u) *the following are equivalent:*

(1) $(\mathcal{H}, \mathbf{F}^+)$ *has the Reznichenko property and countable fan tightness.*

(2) *Each* $Y \in \mathcal{J}$ *satisfies* $\mathbf{S}_{fin}(\mathcal{I}\mathcal{K}, \mathcal{I}\mathcal{K}^{gp})$.

COROLLARY 24. *For a space* (X, u) *the following are equivalent:*

(1) $(\mathcal{H}, \mathbf{V}^+)$ *has the Reznichenko property and countable strong fan tightness.*

(2) *Each* $Y \in \mathcal{J}$ *satisfies* $\mathbf{S}_1(\mathcal{I}\mathcal{Q}, \mathcal{I}\mathcal{Q}^{gp})$.

COROLLARY 25. *For a space* (X,u) *the following are equivalent:*

(1) $(\mathcal{H},\mathbf{V}^+)$ *has the Reznichenko property and countable fan tightness.*

(2) *Each* $Y\in\mathcal{J}$ *satisfies* $\mathbf{S}_{fin}(\mathcal{IQ},\mathcal{IQ}^{gp})$.

Using appropriate modifications in the proof of Theorem 6 we get the statement which includes groupability.

THEOREM 17. *If each finite product of* (X,u) *satisfies* $\mathbf{S}_1(\mathcal{I},\mathcal{I}^{gp})$, *then each finite product of* (X,u) *has the property* $\mathbf{S}_1(\mathcal{I}\Omega,\mathcal{I}\Omega^{gp})$.

Proof. (i) First we prove that if each finite product of X satisfies $\mathbf{S}_1(\mathcal{I},\mathcal{I}^{gp})$, then X has the property $\mathbf{S}_1(\mathcal{I}\Omega,\mathcal{I}\Omega^{gp})$. Following the proof of Theorem 6 and applying the assumption to the sequence $(\mathcal{V}_k)$, $k\in N_m$, there is for each $m\in\mathbb{N}$ a sequence (V_k), $V_k\in\mathcal{V}_k$ for each $k\in N_m$, such that the collection $\mathcal{W}_m=\{V_k\mid k\in N_m\}$ is a groupable interior cover of X^m. There is a partition $\mathcal{W}_m=\cup_{\nu\in\mathbb{N}}\mathcal{W}_{m\nu}$ into finite sets such that for each $x\in X^m$ and for all but finitely many ν there is a $V\in\mathcal{W}_{m\nu}$ such that $x\in\mathrm{int}_vV$ in the product space (X^m,v) holds. Each $\mathcal{W}_{m\nu}=\{V_{k_1(\nu)},\dots,V_{k_s(\nu)}\}$ for some $k_1(\nu),\dots,k_s(\nu)\in N_m$. For each $k\in N_m$ let U_k be an element in $\mathcal{U}_k$ with $V_k=U_k^m$ and $\mathcal{Y}_{m\nu}=\{U_{k_1(\nu)},\dots,U_{k_s(\nu)}\}$. The collection $\{\mathcal{Y}_\mu\mid\mu\in\mathbb{N}\}$ where $\mathcal{Y}_\mu=\cup\{\mathcal{Y}_{m\nu}\mid\mu=m+\nu-1;\,m,\nu\in\mathbb{N}\}$ witnesses the groupability of the interior ω-cover $\mathcal{Y}=\cup_{\mu\in\mathbb{N}}\mathcal{Y}_\mu=\{U_k\mid k\in N_m,\,m\in\mathbb{N}\}$ of (X,u).

Indeed, for every finite subset $F=\{x_1,\dots,x_m\}\subset X$ there is a ν_0 such that for all $\nu\in\mathbb{N}$, $\nu\geq\nu_0$ implies there is an $m\in\mathbb{N}$ and a $V\in\mathcal{W}_{m\nu}$ such that $x=(x_1,\dots,x_m)\in\mathrm{int}_vV$ holds. Hence, there is a $\mu_0=m+\nu_0$ such that for all $\mu\in\mathbb{N}$, $\mu\geq\mu_0$ implies there is a $U\in\mathcal{Y}_\mu$ such that $F\subset\mathrm{int}_uU$.

(ii) Now we show that if X has the property $\mathbf{S}_1(\mathcal{I}\Omega,\mathcal{I}\Omega^{gp})$, then all finite powers of X satisfy $\mathbf{S}_1(\mathcal{I}\Omega,\mathcal{I}\Omega^{gp})$. As in Theorem 6, fix m. Given a sequence $(\mathcal{U}_n)$ of interior ω-covers of X^m consider a sequence $(\mathcal{V}_n)$ of interior ω-covers of X where $\mathcal{V}_n=\{V\subset X\mid V^m\subset U$ for some $U\in\mathcal{U}_n\}$. By the assumption there exist $V_n\in\mathcal{V}_n$, $n\in\mathbb{N}$, such that the collection $\mathcal{W}=\{V_n\mid n\in\mathbb{N}\}$ is a groupable interior ω-cover of X. For each n we pick $U_n\in\mathcal{U}_n$ so that $V_n^m\subset U_n$. Since the collection $\{V_n^m\mid n\in\mathbb{N}\}$ is a groupable interior ω-cover of X^m, so is $\{U_n\mid n\in\mathbb{N}\}$. □

In a similar way we prove

THEOREM 18. *If each finite product of* (X,u) *satisfies* $\mathbf{S}_{fin}(\mathcal{I},\mathcal{I}^{gp})$, *then each finite product of* (X,u) *has the property* $\mathbf{S}_{fin}(\mathcal{I}\Omega,\mathcal{I}\Omega^{gp})$.

We conclude this paper with the next statements.

THEOREM 19. (see [4]) *For a space* (X, u) *and collections* Δ *and* Σ *the following are equivalent*:

(1) $\mathcal{H}$ *satisfies* $\mathbf{S}_1(\Phi_H^{\Delta^+}, (\Phi_H^{\Sigma^+})^{gp})$ *for each* $H \in \mathcal{H}$.

(2) *Each* $Y \in \mathcal{J}$ *satisfies* $\mathbf{S}_1(\mathcal{I}\Delta, \mathcal{I}\Sigma^{gp})$.

Proof. The constructions are analogous to those in the previous theorems. □

By replacing the selection principle $\mathbf{S}_1$ with $\mathbf{S}_{fin}$ we get

THEOREM 20. (see [4]) *For a space* (X, u) *and collections* Δ *and* Σ *the following are equivalent*:

(1) $\mathcal{H}$ *satisfies* $\mathbf{S}_{fin}(\Phi_H^{\Delta^+}, (\Phi_H^{\Sigma^+})^{gp})$ *for each* $H \in \mathcal{H}$.

(2) *Each* $Y \in \mathcal{J}$ *satisfies* $\mathbf{S}_{fin}(\mathcal{I}\Delta, \mathcal{I}\Sigma^{gp})$.

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