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A TOPOLOGICAL MIRROR SYMMETRY ON NONCOMMUTATIVE COMPLEX TWO-TORI

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Abstract. In this paper, we study a topological mirror symmetry on noncommutative complex tori. We show that deformation quantization of an elliptic curve is mirror symmetric to an irrational rotation algebra. From this, we conclude that a mirror reflection of a noncommutative complex torus is an elliptic curve equipped with a Kronecker foliation.

1. Introduction

The noncommutative tori is known to be the most accessible examples of noncommutative geometry developed by A. Connes [3]. It also provides the best examples in applications of noncommutative geometry to the physics of open strings [20, 5]. A noncommutative torus is a universal C^*-algebra generated by two unitary operators subject only to a suitable commutation relation. It arises naturally in a number of different situations. Among others, it can be obtained as a deformation quantization of the algebra of continuous (smooth) functions on an ordinary torus. This approach has been used widely in gauge theories on noncommutative tori (*cf.* [17]) and in applications to string theory such as in [9]. On the other hand, one can consider a noncommutative torus as a foliation C^*-algebra for a Kronecker foliation on a torus via Morita equivalence, [3]. Such an algebra is known to be an irrational rotation algebra, [16]. Here we will choose these two approach to define noncommutative two-tori and we will discuss how they are related with the mirror symmetry [21, 22].

Associated to a topological mirror symmetry, we need to define a complex structure on a noncommutative torus. The complex geometry

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has been developed by A. Schwarz in [19] and some basic calculations have been made in [7] for the two-dimensional case which leads to the study of Kontsevich's homological mirror symmetry ([13]) in [15], [11] and [14] (see also [12]).

In this paper, we will mainly concern on an aspect of a topological mirror symmetry on elliptic curves based on [22] which compares the moduli spaces of stable bundles and supercycles. The stable bundles of a certain topological type are deformed to standard holomorphic bundles on a noncommutative complex torus, along the deformation quantization procedure. We show that the deformation is equivalent to define a linear foliation on a mirror reflection of a given elliptic curve.

In Section 2, we will review some basic facts for noncommutative complex tori. In Section 3, we show how stable bundles on an elliptic curve are deformed to standard holomorphic bundles on a noncommutative complex torus. In Section 4, we find a mirror reflection of a noncommutative complex torus.

2. Some preliminaries on noncommutative complex tori

In this section, we review some basic facts for noncommutative complex tori and bundles on them, following [19] and [15].

2.1. Noncommutative complex two-tori

A noncommutative two-torus T^2_θ is defined by two unitaries U_1, U_2 obeying the relation

$$U_1 U_2 = \exp(2\pi i\theta) U_2 U_1, \tag{1}$$

where $\theta \in \mathbb{R}/\mathbb{Z}$. The commutation relation (1) defines the presentation of the involutive algebra

$$A_\theta = \{ \sum_{n_1, n_2 \in \mathbb{Z}^2} a_{n_1,n_2} U_1^{n_1} U_2^{n_2} \mid a_{n_1,n_2} \in \mathcal{S}(\mathbb{Z}^2)\},$$

where $\mathcal{S}(\mathbb{Z}^2)$ is the Schwartz space of sequences with rapid decay at infinity. According to [4], the algebra A_θ can be understood as the algebra of smooth functions on T^2_θ. As was given in [18], the algebra A_θ can be defined as a deformation quantization of $C^\infty(T^2)$, the algebra of smooth functions on the ordinary torus T^2. The action of T^2 by translation on $C^\infty(T^2)$ gives an action of T^2 on A_θ. The infinitesimal form of the action defines a Lie algebra homomorphism $\delta : L \to \mathrm{Der}(A_\theta)$, where $L = \mathbb{R}^2$ is an abelian Lie algebra and $\mathrm{Der}(A_\theta)$ is the Lie algebra of derivations of A_θ. Generators δ_1, δ_2 of $\mathrm{Der}(A_\theta)$ act in the following way:

$$(2)\qquad \delta_k\left(\sum_{(n_1,n_2)\in\mathbb{Z}^2} a_{n_1,n_2}U^{n_1}U^{n_2}\right) = 2\pi i \sum_{(n_1,n_2)\in\mathbb{Z}^2} n_k a_{n_1,n_2}U^{n_1}U^{n_2}.$$

A complex structure on T^2_θ is defined in terms of a complex structure on the Lie algebra $L=\mathbb{R}^2$ which acts on A_θ. Let us fix a $\tau\in\mathbb{C}$ such that $\mathrm{Im}(\tau)\neq 0$ and then τ defines a one-dimensional subalgebra of $\mathrm{Der}(A_\theta)$ spanned by the derivation δ_τ given by

$$(3)\qquad \begin{aligned}&\delta_\tau\left(\sum_{(n_1,n_2)\in\mathbb{Z}^2} a_{n_1,n_2}U^{n_1}U^{n_2}\right)\\ =\ &2\pi i \sum_{(n_1,n_2)\in\mathbb{Z}^2} (n_1\tau+n_2)a_{n_1,n_2}U^{n_1}U^{n_2}.\end{aligned}$$

The noncommutative torus equipped with such a complex structure is denoted by $T^2_{\theta,\tau}$ and will be called a noncommutative complex torus.

2.2. Holomorphic bundles on $T^2_{\theta,\tau}$

Since the algebra A_θ is considered as the algebra of smooth functions on T^2_θ, the vector bundles on T^2_θ correspond to finitely generated projective right A_θ-modules. Such modules can be constructed by the Heisenberg projective representations and it is known (see [17]) that for each $g=\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in \mathrm{SL}_2(\mathbb{Z})$ such that $d+c\theta\neq 0$, a Heisenberg module $E_g(\theta):=E_{d,c}(\theta)$ over A_θ is given by the Schwarz space $\mathcal{S}(\mathbb{R}\times\mathbb{Z}/c\mathbb{Z})$ equipped with the right action of A_θ:

$$\begin{aligned} fU_1(s,k) &= f(s-\frac{d+c\theta}{c},k-1)\\ fU_2(s,k) &= \exp(s-\frac{kd}{c})f(s,k),\end{aligned}$$

where $s\in\mathbb{R}$ and $k\in\mathbb{Z}/c\mathbb{Z}$. For $g\in\mathrm{SL}_2(\mathbb{Z})$, the modules $E_g(\theta)$ will be referred as *basic modules.*

Connections on a vector bundle on the noncommutative torus T^2_θ are defined in terms of derivations. Let E be a projective right A_θ-module, a connection ∇ on E is a linear map from E to $E\otimes L^*$ such that for all $x\in L$,

$$\nabla_x(\xi u) = (\nabla_x\xi)u + \xi\delta_x(u),\quad \xi\in E, u\in A_\theta.$$

The curvature F_∇ of the connection ∇ is a 2-form on L with values in the algebra of endomorphisms of E. That is, for $x, y \in L$,

$$F_\nabla(x,y) := [\nabla_x, \nabla_y] - \nabla_{[x,y]}.$$

Since L is abelian, we simply have $F_\nabla(x,y) = [\nabla_x, \nabla_y]$.

A *holomorphic structure* on a right A_θ-module E compatible with the complex structure on $T^2_{\theta,\tau}$ is a $\mathbb{C}$-linear map $\overline{\nabla} : E \longrightarrow E$ such that

$$\overline{\nabla}(\xi \cdot u) = \overline{\nabla}(\xi) \cdot u + \xi \cdot \delta_\tau(u), \quad \xi \in E, u \in A_\theta.$$

A projective right A_θ-module equipped with a holomorphic structure is called a *holomorphic bundle* over the noncommutative complex torus $T^2_{\theta,\tau}$. In particular, the basic modules $E_g(\theta)$ equipped with holomorphic structure are called *standard holomorphic bundles* on $T^2_{\theta,\tau}$.

2.3. The Chern character and the slope of basic modules

The K-theory group $K_0(A_\theta)$ classifies the finitely generated projective A_θ-modules and there is a natural map, the Chern character which takes the values in the Grassmann algebra $\wedge^\bullet L^*$, where L^* is the dual vector space of the Lie algebra L. Since there is a lattice Γ in L, there should be elements of $\wedge^\bullet \Gamma^*$ which are integral, where Γ^* is the dual lattice of Γ. The Chern character is the map $\mathrm{Ch} : K_0(A_\theta) \to \wedge^{\mathrm{ev}}(L^*)$ defined by

$$\mathrm{Ch}(\mathcal{E}) = e^{-i(\theta)}\nu(\mathcal{E}), \tag{4}$$

where $i(\theta)$ denotes the contraction with the deformation parameter θ regarded as an element of $\wedge^2 L$ and $\nu(\mathcal{E}) \in \wedge^{\mathrm{even}}\Gamma^*$. See [17] for details for the definition of the Chern character. For a given $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ such that $d + c\theta > 0$, we have

$$\mathrm{Ch}(E_g(\theta)) = (d + c\theta) + c\, dx_1 \wedge dx_2. \tag{5}$$

Let us define

$$\deg(g) = \deg(E_g(\theta)) = c \quad \text{and} \quad \mathrm{rk}(g,\theta) = c\theta + d = \mathrm{rank}(E_g(\theta)).$$

As in the classical case, we may define the *slope* of the basic module $E_g(\theta)$ by the numbers

$$\mu(E_g(\theta)) = \frac{\deg(g)}{\mathrm{rk}(g,\theta)} = \frac{c}{c\theta + d}.$$

3. Stable bundles on an elliptic curve and standard holomorphic bundles

In this section we construct a standard holomorphic bundle over $T^2_{\theta,\tau}$ from a stable bundle over an elliptic curve X_τ. We will show that the moduli space of holomorphic stable bundles is naturally identified with the moduli space of standard holomorphic bundles associated to a matrix in $\mathrm{SL}_2(\mathbb{Z})$ which determines a topological type on both bundles.

3.1. Stable bundles on an elliptic curve

Let $X_\tau = \mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}$ be an elliptic curve whose complex structure is specified by $\tau \in \mathbb{C}$, Im $\tau \neq 0$. For X_τ, the algebraic cohomology ring is

$$A(X_\tau) = H^0(X_\tau, \mathbb{Z}) \oplus H^2(X_\tau, \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}.$$

The Chern character of a holomorphic vector bundle E on X_τ takes the value in $A(X_\tau)$:

$$\mathrm{Ch}(E) = (\text{rank } E, \deg E) \in H^0(X_\tau, \mathbb{Z}) \oplus H^2(X_\tau, \mathbb{Z}),$$

where $\deg E = c_1(E)$. The slope of a vector bundle E is defined by

$$\mu(E) = \frac{\deg E}{\text{rank } E}.$$

A bundle E is said to be stable if, for every proper subbundle E' of E, $0 < \text{rank } E' < \text{rank } E$, we have

$$\mu(E') < \mu(E).$$

Every stable bundles carries a projectively flat Hermitian connection ∇^E. In other words, there is a complex 2-form λ on X_τ such that the curvature of ∇^E is

$$R_{\nabla^E} = \lambda \cdot \mathrm{Id}_E,$$

where Id_E is the identity endomorphism of E. Since

$$c_1(E) = \frac{i}{2\pi}\mathrm{Tr}\ R_{\nabla^E} = \frac{i}{2\pi}\lambda \cdot \text{ rank } E,$$

we have

$$\lambda = \frac{2\pi}{i}\frac{c_1(E)}{\text{rank } E} = \frac{2\pi}{i}\mu(E).$$

Thus

$$R_{\nabla^E} = -2\pi i \mu(E)\mathrm{Id}_E. \tag{6}$$

Note that if E is stable, then the topological type of E is given by the pair (rank E, deg E) which is relatively prime. Thus we may extend the pair to a matrix in $\mathrm{SL}_2(\mathbb{Z})$. For $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, let us denote by $\mathcal{M}_g^s$ the moduli space of holomorphic stable bundles of rank d and degree c on X_τ. Since every stable bundle E on an elliptic curve X_τ is uniquely determined up to translation by its topological type (d, c), the moduli space is

$$\mathcal{M}_g^s \cong X_\tau.$$

3.2. Holomorphic deformations of stable bundles

For a matrix $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, the Chern character of a stable bundle E in $\mathcal{M}_g^s$ is of the form $\mathrm{Ch}(E) = (d, c) \in A(X_\tau)$ and it defines an integral element $d + cdx^{12} \in \wedge^2 L^*$. Let us consider

$$e^{-i(\theta)}(d + cdx^{12}) = (d + c\theta) + c\, dx_1 \wedge dx_2$$

and let

$$\mu = \frac{c}{c\theta + d}.$$

Then we have

PROPOSITION 3.2.1. *For a stable bundle E on X_τ whose topological type is specified by a matrix $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, there is a basic A_θ-module $E_g(\theta)$ equipped with a connection whose curvature is $-2\pi i \mu$.*

Proof. Associated to the curvature condition (6) on the stable bundle E on X_τ, we define a Heisenberg commutation relation by

$$F_\nabla = [\nabla_1, \nabla_2] = -2\pi i \mu. \tag{7}$$

By the Stone-von Neuman theorem, the above relation has a unique representation. As discussed in [6], the representation is just c-copies of the Schrödinger representation of the Heisenberg Lie group $\mathbb{R}^3$ on $L^2(\mathbb{R})$, where the product on $\mathbb{R}^3$ is given by

$$(r, s, t) \cdot (r', s', t') = (r + r', s + s', t + t' + sr').$$

Then the operators ∇_1 and ∇_2 are the infinitesimal form of the representation and is given by

$$(\nabla_1 f)(s, k) = 2\pi i \mu s f(s, k), \tag{8}$$

$$(\nabla_2 f)(s, k) = \frac{df}{ds}(s, k) \tag{9}$$

acting on the Schwartz space $\mathcal{S}(\mathbb{R} \times \mathbb{Z}/c\mathbb{Z}) \cong \mathcal{S}(\mathbb{R}) \otimes \mathbb{C}^c$. Let $E_g(\theta) = \mathcal{S}(\mathbb{R}) \otimes \mathbb{C}^c$. Then one sees that (8) and (9) are in fact desired connection on $E_g(\theta)$. To specify the module $E_g(\theta)$, we need to define a module action which is compatible with the relation (1) for T_θ^2. Let us first consider unitary operators W_1, W_2 on $\mathcal{S}(\mathbb{Z}/c\mathbb{Z}) = \mathbb{C}^c$ defined by

$$W_1 f(k) = f(k-d),$$
$$W_2 f(k) = e^{-2\pi i \frac{k}{c}} f(k).$$

Then

$$W_1 W_2 = e^{2\pi i \frac{d}{c}} W_2 W_1.$$

In other words, W_1 and W_2 provide a representation of the Heisenberg commutation relations for the finite group $\mathbb{Z}/c\mathbb{Z}$. Associated to the connections (8) and (9), we have Heisenberg representations V_1 and V_2 on the space $\mathcal{S}(\mathbb{R})$ as

$$V_1 f(s) = e^{2\pi i (\frac{d}{c} - \theta)s} f(s),$$
$$V_2 f(s) = f(s+1).$$

The operators obey the relation

$$V_1 V_2 = e^{-2\pi i (\frac{d}{c} - \theta)} V_2 V_1.$$

Finally, the operators

(10) $$U_1 = V_1 \otimes W_1 \quad \text{and} \quad U_2 = V_2 \otimes W_2$$

acting on the space $\mathcal{S}(\mathbb{R} \times \mathbb{Z}/c\mathbb{Z})$ satisfy the relation

$$U_1 U_2 = e^{2\pi i \theta} U_2 U_1.$$

This completes the construction of basic module $E_g(\theta)$ equipped with a constant curvature connection ∇_1, ∇_2 such that the curvature is given by (7). □

The basic module $E_g(\theta)$ constructed in Proposition 3.1 admits a constant curvature connection (8, 9) and all other constant curvature connections on $E_g(\theta)$ which satisfies the relation (7) are given as

(11) $$(\nabla_1 f)(s) = 2\pi i \mu(E_g(\theta)) f(s) + 2\pi i \alpha,$$

(12) $$(\nabla_2 f)(s) = \frac{\partial f}{\partial s} + 2\pi i \beta,$$

where α and β are real numbers. Let us fix a complex number τ such that $\mathrm{Im}\,\tau < 0$. The parameter τ defines a complex structure on T_θ^2 via derivation $\delta_\tau = \tau\delta_1 + \delta_2$ spanning $\mathrm{Der}(A_\theta)$ as given in subsection 2.1. Then a holomorphic structure on $E_g(\theta)$ is specified by $\overline{\nabla} = \tau\nabla_1 + \nabla_2$. Along with the connections in (11, 12), and for $z \in \mathbb{C}$, let

$$(\overline{\nabla}_z)(f)(s,k) = \frac{\partial f}{\partial s}(s,k) + 2\pi i (\tau \mu(E_g(\theta)) s + z) f(s,k).$$

Then $\overline{\nabla}_z$ defines a standard holomorphic structure on $E_g(\theta)$. Other holomorphic structures are determined by translations of α and β in (11) and (12). In other words, all the holomorphic structures are determined by the complex number $z = \tau\alpha + \beta$. A basic module $E_g(\theta)$ equipped with a holomorphic structure $\overline{\nabla}_z$ is called a standard holomorphic bundle and is denoted by $E_g^z(\theta)$. Let $\mathcal{M}_g^s(\theta)$ be the moduli space of holomorphic constant curvature connections on $E_g(\theta)$ which satisfy the relation (7). Then our discussion above shows that the following:

PROPOSITION 3.2.2. *With notations above, we have*

$$\mathcal{M}_g^s(\theta) \cong X_\tau \cong \mathcal{M}_g^s.$$

4. Mirror symmetry and noncommutative tori

In this section, we consider a mirror torus $\widehat{X}_\tau$ of the elliptic curve X_τ. We first review the construction of $\widehat{X}_\tau$ following the lines of [22]. Then we will show that a Kronecker foliation on $\widehat{X}_\tau$ is mirror symmetric to a noncommutative complex torus.

4.1. Supercycles

Let $\tau \in \mathbb{C}$ be an element in the lower half-plane as in Subsection 3.2. The complex number defines an elliptic curve $X_\tau = \mathbb{C}^*/q^{\mathbb{Z}}$, $q = \exp(-2\pi i\tau)$. A complex orientation of an elliptic curve X_τ is given by a holomorphic 1-form Ω, which determines a Calabi-Yau manifold structure on X_τ. A special Lagrangian cycle of X_τ is a 1-dimensional Lagrangian submanifold $\mathcal{L}$ such that the restriction of Ω satisfies

$$\operatorname{Im}\Omega|_{\mathcal{L}} = 0 \quad \text{and} \quad \operatorname{Re}\Omega|_{\mathcal{L}} = \operatorname{Vol}(\mathcal{L}),$$

where the volume form is determined by the Euclidean metric on X_τ. A special Lagrangian cycle is just a closed geodesic and hence it is represented by a line with rational slope on the universal covering space of X_τ. Let us fix a smooth decomposition

$$X_\tau = S^1_+ \times S^1_-$$

which induces a decomposition of cohomology group

$$H^1(X_\tau, \mathbb{Z}) \cong \mathbb{Z}_+ \oplus \mathbb{Z}_-.$$

Let $[B] \in \mathbb{Z}_-$ and $[F] \in \mathbb{Z}_+$ be generators of the cohomology group such that the cycles representing $[B]$ and $[F]$ have no self-intersection and two cycles have one intersection point. Then the cohomology class $[F] \in H^1(X_\tau, \mathbb{Z})$ is represented by a smooth cycle in X_τ and is a special

Lagrangian cycle. The family of special Lagrangian cycles representing the class $[F]$ gives a smooth fibration

$$\pi : X_\tau \longrightarrow S^1_- := B \tag{13}$$

and the base space $B = S^1_-$ is just the moduli space of special Lagrangian cycles associated to $[F] \in H^1(X_\tau, \mathbb{Z})$. The unitary flat connections on the trivial line bundle $S^1_+ \times \mathbb{C} \to S^1_+$ are parameterized by $\widehat{S^1} := \mathrm{Hom}(\pi_1(S^1_+), \mathrm{U}(1))$, up to gauge equivalences. Thus we have the dual fibration

$$\widehat{\pi} : \widehat{X}_\tau \longrightarrow B = S^1_- \tag{14}$$

with fibers

$$\widehat{\pi}^{-1}(b) = \mathrm{Hom}(\pi_1(\pi^{-1}(b)), \mathrm{U}(1)) = \widehat{S^1}.$$

The dual fibration admits the section $s_0 \in \widehat{X}_\tau$ with

$$s_0 \cap \widehat{\pi}^{-1}(b) = 1 \in \mathrm{Hom}(\pi_1(\pi^{-1}(b)), \mathrm{U}(1)),$$

so that we have a decomposition

$$\widehat{X}_\tau = \widehat{S^1} \times S^1_-.$$

Hence, associated to the class $[F] \in H^1(X_\tau, \mathbb{Z})$, the space $\widehat{X}_\tau$ is the moduli space of special Lagrangian cycles endowed with unitary flat line bundles. Furthermore, $\widehat{X}_\tau$ admits a Calabi-Yau manifold structure. In other words, $\widehat{X}_\tau$ is the mirror reflection of X_τ in the sense of [21]. Under the Kähler-Hodge mirror map (see [22]), a complexfied Kähler parameter $\rho = b + ik$ defines a complex structure on $\widehat{X}_\tau$, where k is a Kähler form on X_τ and b defines a class in $H^2(X_\tau, \mathbb{R})/H^2(X_\tau, \mathbb{Z})$. Then $\widehat{X}_\tau$ is the elliptic curve $\mathbb{C}^*/e^{2\pi i \rho \mathbb{Z}}$. Similarly, the modular parameter τ of $X_\tau = \mathbb{C}^*/q^{\mathbb{Z}}$, $q = \exp(-2\pi i \tau)$, $\mathrm{Im}\ \tau < 0$, corresponds to a complexfied Kähler parameter $\widehat{\rho}$ on $\widehat{X}_\tau$.

4.2. The moduli space of supercycles

DEFINITION 4.2.1. A *supercycle* or a *brane* on $\widehat{X}_\tau$ is given by a pair $(\mathcal{L}, A)$, where $\mathcal{L}$ is a special Lagrangian submanifold of $\widehat{X}_\tau$ and A a flat connection on the trivial line bundle $\mathcal{L} \times \mathbb{C} \to \mathcal{L}$.

A special Lagrangian cycle in $\widehat{X}_\tau \cong \mathbb{R}^2/\mathbb{Z} \oplus \mathbb{Z}$ is represented by a line of rational slope, so can be given by a pair of relatively prime integers and we extend the pairs to a matrix in $\mathrm{SL}_2(\mathbb{Z})$. The lines of a fixed rational slope are parameterized by the points of interception with the line $y = 0$

on the universal covering space $\mathbb{R}^2$ of $\widehat{X}_\tau$. For $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, let $\mathcal{L}_g$ be a special Lagrangian submanifold of $\widehat{X}_\tau$ given by

$$\mathcal{L}_g = \{(ds + \alpha, cs) \mid s \in \mathbb{R}/\mathbb{Z}\}, \tag{15}$$

so that the line has slope $\frac{c}{d}$ and x-intercept α. The shift of $\mathcal{L}_g$ is represented by the translation of α. Thus the moduli space of special Lagrangian cycles are S^1. Note that a unitary flat line bundle on $\mathcal{L}_g$ is specified by the monodromy around the circle. On the trivial line bundle $\mathcal{L}_g \times \mathbb{C} \to \mathcal{L}_g$, we have a constant real valued connection one-form on $\mathbb{R}^2$, restricted to $\mathcal{L}_g$, given by

$$A = 2\pi i \beta dx, \quad x \in \mathbb{R}^2, \quad \beta \in \mathbb{R}/\mathbb{Z}, \tag{16}$$

so that the monodromy between points (x_1, y_1) and (x_2, y_2) is given by $\exp[2\pi i \beta (x_2 - x_1)]$. Thus, the shift of connections is represented by monodromies. Combining the result in Subsection 3.1, we have the following:

PROPOSITION 4.2.2. *Let $\mathcal{SM}_g$ be the moduli space of supercycles on $\widehat{X}_\tau$, whose special Lagrangian cycle is specified by the matrix $g \in \mathrm{SL}_2(\mathbb{Z})$. Then we have*

$$\mathcal{SM}_g \cong X_\tau \cong \mathcal{M}^s_g.$$

REMARK 4.2.3. In [22], the identification $\mathcal{SM}_g \cong \mathcal{M}^s_g$ was shown in a geometric way. By Proposition 3.2.2, we also have

$$\mathcal{SM}_g \cong \mathcal{M}^s_g \cong \mathcal{M}^s_g(\theta). \tag{17}$$

However in the above identification, one may not see how deformation quantization is related to a mirror symmetry. In below, we shall reprove the identification (17) in a geometric way.

4.3. A mirror reflection of $T^2_{\theta,\tau}$

There are many ways to define a noncommutative torus such as a deformation quantization or as an irrational rotation algebra, etc. In Section 2, we have considered A_θ as a deformation quantization of $C^\infty(T^2)$. An irrational rotation algebra can be obtained from a Kronecker foliation on a torus. We show that those two algebras are related by a mirror symmetry.

THEOREM 4.3.1 *For an irrational number θ, the noncommutative torus A_θ obtained from a deformation quantization of $C^\infty(T^2)$ is mirror symmetric to an irrational rotation algebra which is defined by the*

Kronecker foliation on $\widehat{X}_\tau$ whose leaves are represented by the lines of slope θ^{-1} on the universal covering space of $\widehat{X}_\tau$.

Proof. Let us consider the case when $g = \left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$. It corresponds to trivial line bundles on X_τ. The sections of a trivial line bundle is identified with $C^\infty(X_\tau)$ and it is deformed to a free module over A_θ of rank 1 along the proof of Proposition 3.2.1. Thus we have A_θ as a deformation quantization of $C^\infty(X_\tau)$ with a holomorphic structure specified by the derivations on it.

On the mirror side, let $\widehat{X}_{\tau,\theta^{-1}}$ be a foliated torus defined by the differential equation $dy = \theta^{-1}dx$ with natural coordinate (x, y) on the flat torus determined by the symplectic form on $\widehat{X}_\tau$. Such a foliation is called a Kronecker foliation or a linear foliation (*cf.* [2]). On the covering space of $\widehat{X}_\tau$ the leaves of the foliation are represented by straight lines with fixed slope θ^{-1} and every closed geodesic of $\widehat{X}_\tau$ yields a compact transversal which meets every leaves. For $g = \left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$, a trivial line bundle is mapped to a special Lagrangian cycle represented by the line $y = 0$ under the identification $\mathcal{M}_g^s \cong \mathcal{SM}_g$. The line $y = 0$ is a compact transversal for the θ^{-1}-linear foliation and each leaf meets the line countably many points. Associated to the intersection points, each leaf defines the rotation through the angle θ on the circle S^1, which gives a $\mathbb{Z}$-action on S^1. The action defines the crossed product of $C^\infty(S^1)$ by $\mathbb{Z}$ which is called the irrational rotation algebra (see [16], [1]). Thus a trivial line bundle on X_τ naturally defines the rotation algebra through θ. This completes the proof. □

The rotation algebra considered in the proof of Theorem 4.3.1 is Morita equivalent to the foliation C^*-algebra for $\widehat{X}_{\tau,\theta^{-1}}$ (see [3] for details). Thus we may conclude the following:

COROLLARY 4.3.2. *The foliation C^*-algebra for the θ^{-1}-linear foliation is a mirror reflection of the algebra of functions on $T^2_{\theta,\tau}$. Equivalently, the foliated, complex torus $\widehat{X}_{\tau,\theta^{-1}}$ is a mirror reflection of the noncommutative torus $T^2_{\theta,\tau}$.*

On the other hand, for any $g \in \mathrm{SL}_2(\mathbb{Z})$, the leaves of the θ^{-1}-linear foliation rotate the special Lagrangian cycle $\mathcal{L}_g$ in a different angle θ' from θ. Thus the leaf action on $\mathcal{L}_g$ defines another noncommutative torus $A_{\theta'}$ and two irrational rotation algebras are related by a strongly Morita equivalence (see [3]), in other words, $A_{\theta'} \cong \mathrm{End}_{A_\theta}(E_g(\theta))$. Furthermore, two deformation parameters θ and θ' are in the same orbit of $\mathrm{SL}_2(\mathbb{Z})$-action; $\theta' = g\theta = \frac{a\theta+b}{c\theta+d}$. Thus we can extend Corollary 4.3.2. to a more general foliated manifolds:

COROLLARY 4.3.3. *The foliated, complex torus $\widehat{X}_{\tau,\theta'}$ is a mirror reflection of the noncommutative torus $T^2_{\theta,\tau}$, where $\theta' = g\theta$ for $g \in \mathrm{SL}_2(\mathbb{Z})$.*

REMARK 4.3.4. It was suggested in [8] that the relation of C^*-algebra of foliation and a C^*-algebra obtained from a deformation quantization of a torus can be regarded as a mirror symmetry. From a physical point of view, it was argued in [11] that those two C^*-algebras are related by the T-duality, which is equivalent to the mirror symmetry on tori. Thus, our results, Theorem 4.3.1. and its Corollaries, may be considered as a mathematical interpretation of [11].

Finally, we show that a standard holomorphic bundle $E^z_g(\theta)$ on $T^2_{\theta,\tau}$ is obtained geometrically, using the θ^{-1}-linear foliation structure of $\widehat{X}_{\tau,\theta^{-1}}$. Our argument here is essentially based on [3]. For a matrix $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, the finitely generated projective A_θ-module $E_g(\theta) = \mathcal{S}(\mathbb{R}) \otimes \mathbb{C}^c$ gives a strong Morita equivalence between A_θ and $A_{g\theta}$. As we have shown above, the noncommutative tori A_θ and $A_{g\theta}$ correspond to special Lagrangian cycles $\mathcal{L}_I$, $I = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, and $\mathcal{L}_g$, respectively. Since the cycle $\mathcal{L}_I$ is represented by the line $y = 0$ on $\widehat{X}_\tau = \mathbb{R}^2/\mathbb{Z}^2$, the leaves of θ^{-1}-linear foliation on $\widehat{X}_\tau$ are the lines given by

$$\{(\theta t + x, t) \mid t \in \mathbb{R}\}.$$

On the other hand, the special Lagrangian cycle $\mathcal{L}_g$ is the line

$$\mathcal{L}_g = \{(\frac{d}{c}t + \alpha, t) \mid \alpha \in \mathbb{R}/\mathbb{Z},\ t \in \mathbb{R}\}.$$

Then the space of leaves starting at $\mathcal{L}_I$ and ending at $\mathcal{L}g$ is determined by the equation

$$\frac{d}{c}t + \alpha = \theta t + x, \quad \text{or} \quad (\frac{d}{c} - \theta)t = x - \alpha \mod 1. \tag{18}$$

Let $\mathcal{E}_g = \{((x,0),t) \in \widehat{X}_\tau \times \mathbb{R} \mid (\frac{d}{c} - \theta)t = x - \alpha \mod 1\}$. Since the line $\mathcal{L}_g$ cuts the line $\mathcal{L}_I$ c-times, one finds that the manifold $\mathcal{E}_g$ is the disjoint union of c-copies of $\mathbb{R}$, i.e., $\mathcal{E}_g = \mathbb{R} \times (\mathbb{Z}/c\mathbb{Z})$. Then the algebra of compactly supported smooth functions on $\mathcal{E}_g$ is $C^\infty_c(\mathcal{E}_g) = \mathcal{S}(\mathbb{R} \times \mathbb{Z}/c\mathbb{Z}) = \mathcal{S}(\mathbb{R}) \otimes \mathbb{C}^c$. Let W_1 and W_2 be unitary operators on $\mathbb{C}^c$ such that $W_1^c = W_2^c = 1$ and $W_1W_2 = \exp(2\pi i\frac{d}{c})W_2W_1$. These operators reflect the structure of a stable bundle on X_τ (*cf.* [10]). Associated to the equation (18), we define operators V_1 and V_2 on $\mathcal{S}(\mathbb{R})$ by

$$(V_1 f)(t) = \exp(2\pi\alpha)\exp(2\pi i(\frac{d}{c} - \theta)t)f(t), \tag{19}$$

$$(V_2 f)(t) = f(t+1). \tag{20}$$

Then $U_i = V_i \otimes W_i$, $(i = 1, 2)$, gives a left module action on $C_c^\infty(\mathcal{E}_g)$. The module action (19, 20) also determine a constant curvature connection ∇_1, ∇_2 by the formula given in (11, 12), where the translation of ∇_2 is determined by the monodromy data or a complex phase $\exp(2\pi i\beta)$, $\beta \in \mathbb{R}/\mathbb{Z}$, given in (16). Thus we see that the special Lagrangian cycle $\mathcal{L}_g$ determines a basic module $E_g(\theta) = C_c^\infty(\mathcal{E}_g)$ equipped with a constant curvature connection. The monodromy data specified in (12) determines the holomorphic structure on $E_g(\theta)$ by $\overline{\nabla} = \tau\nabla_1 + \nabla_2$. This completes the geometric construction of $E_g^z(\theta)$, where $z = \tau\alpha + \beta$. From this construction we get the following, which can be interpreted as a noncommutative version of Proposition 4.2.2.

PROPOSITION 4.3.5. *For a matrix $g \in \mathrm{SL}_2(\mathbb{Z})$, let $\mathcal{SM}_g(\theta)$ be the moduli space of compact transversals for θ^{-1}-linear foliation on $\widehat{X}_\tau$ together with the monodromy around the circles (compact transversals) which is given by a complex phase* $\exp(2\pi i\beta)$, $\beta \in \mathbb{R}/\mathbb{Z}$. *Then we have*

$$\mathcal{SM}_g(\theta) \cong \mathcal{M}_g^s(\theta).$$

REMARK 4.3.6. The moduli space $\mathcal{SM}_g(\theta)$ given in Proposition 4.3.5 is essentially the same as the moduli space of supercycles $\mathcal{SM}_g$ given in Proposition 4.2.2. Thus Proposition 4.3.5 can be seen as a geometric construction of the identification of (17) stated in Remark 4.2.3.

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