

IDENTIFICATION OF CONSTANT PARAMETERS IN PERTURBED SINE-GORDON EQUATIONS

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ABSTRACT. We study the identification problems of constant parameters appearing in the perturbed sine-Gordon equation with the Neumann boundary condition. The existence of optimal parameters is proved, and necessary conditions are established for several types of observations by utilizing quadratic optimal control theory due to Lions [13].

1. Introduction

In previous paper [7] we studied the problem of identification of the parameters α, β, γ and δ for the system governed by a damped sine-Gordon equation

$$(1.1) \quad \frac{\partial^2 y}{\partial t^2} + \alpha \frac{\partial y}{\partial t} - \beta \Delta y + \gamma \sin y = \delta f.$$

In [7] the existence and necessary conditions of optimal parameters $q^* = (\alpha, \beta^*, \gamma^*, \delta^*)$ are established with a quadratic cost function which does not include the parameter $q^* = (\alpha, \beta^*, \gamma^*, \delta^*)$ explicitly.

Several types of perturbed sine-Gordon equations different from (1.1) are proposed to describe dynamics of the phase difference in various Josephson junctions. In Kivshar and Malomed [10] the perturbed equation

$$(1.2) \quad \frac{\partial^2 y}{\partial t^2} - \frac{\partial^2 y}{\partial x^2} + \sin y = \epsilon \frac{\partial^2}{\partial x^2} \left(\frac{\partial y}{\partial t} \right)$$

is proposed by taking into account of losses or dissipation due to the current along a dielectric barrier in Josephson junctions. The nonlinear

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perturbed equation

$$(1.3) \quad \frac{\partial^2 y}{\partial t^2} - \frac{\partial^2 y}{\partial x^2} + \sin y = \epsilon \sin 2y$$

is also proposed by Kivshar and Malomed [9] to determine the inelastic interaction of a fast kink and weakly bounded breather. A additional nonlinear perturbation $\sum_{i=1}^L \epsilon_i \sin \kappa_i y$ is possible in (1.3). Further, in Ghidaglia and Marzocchi [4] the following equation having damping and amplification perturbed terms

$$(1.4) \quad \frac{\partial^2 y}{\partial t^2} - \Delta y + \sin y = \epsilon_1 \frac{\partial y}{\partial t} + \epsilon_2 y$$

is considered and they prove that (1.4) has a finite dimensional global attractor.

Recently in Ramos [15] numerical solutions of a more generalized perturbed sine-Gordon equation

$$(1.5) \quad \frac{\partial^2 y}{\partial t^2} - \frac{\partial^2 y}{\partial x^2} + \sin y = \epsilon_1 \frac{\partial y}{\partial t} + \epsilon_2 y + \epsilon_3 \sin 2y + \epsilon_4 \frac{\partial^2}{\partial x^2} \left(\frac{\partial y}{\partial t} \right)$$

with homogeneous Neumann boundary conditions are given by utilizing implicit finite difference method. In [15], he observes some interesting solutions in according to choosing the perturbation parameters ϵ_i . However, it is not given the proof of existence and uniqueness of solutions in [15] and there is no information on amplitude of constant parameters ϵ_i . It is an important physical problem to identify such parameters.

In this paper we will study identifying physical parameters $\alpha, \beta, \gamma_i, \delta, \kappa_i, \nu$ and a source function f in a general perturbed sine-Gordon equation

$$(1.6) \quad \frac{\partial^2 y}{\partial t^2} + \alpha \frac{\partial \Delta y}{\partial t} - \beta \Delta y + \sum_{i=1}^L \gamma_i \sin \kappa_i y + \delta y = \nu f.$$

That is, our aim is to extend the results in [7] to the ones for the equation (1.6) under the homogeneous Neumann boundary condition. We take as in [7] the framework of variational method due to Dautray and Lions [3] and Park [8] and set a solution space for the equation (1.6). We will prove the existence and uniqueness of solutions for (1.6) and solve the identification problems of (1.6) by utilizing optimal control method by Lions [13]. We note that the restriction on the space dimensions in [7] is removed and more general costs including parameter terms explicitly than those in [7] are considered.

This paper is composed of three sections. In studying the problems of identification for (1.6) we need some fundamental results on solutions for (1.6). Hence in Section 2, we explain the existence, uniqueness and regularity of solutions for (1.6) with Neumann boundary condition. In Section 3 we solve the problems of identification for (1.6). Finally we deduce the bang-bang property of optimal parameters via necessary conditions on optimality.

2. Perturbed sine-Gordon equation

Let Ω be an open bounded set of the n dimensional Euclidean space \mathbf{R}^n with a piecewise smooth boundary $\Gamma = \partial\Omega$. Let $Q = (0, T) \times \Omega$ and $\Sigma = (0, T) \times \Gamma$. Let us consider a perturbed sine-Gordon equation

$$(2.1) \quad \frac{\partial^2 y}{\partial t^2} - \alpha \frac{\partial \Delta y}{\partial t} - \beta \Delta y + \sum_{i=1}^L \gamma_i \sin \kappa_i y + \delta y = f \text{ in } Q,$$

where $\alpha, \beta > 0$, $\delta, \gamma_i, \kappa_i \in \mathbf{R} = (-\infty, \infty), i = 1, \dots, L$, Δ is a Laplacian in \mathbf{R}^n and f is a given function. The boundary condition is the homogeneous Neumann condition

$$(2.2) \quad \frac{\partial y}{\partial n} = 0 \text{ on } \Sigma.$$

The initial value is given by

$$(2.3) \quad y(0, x) = y_0(x) \text{ in } \Omega \text{ and } \frac{\partial y}{\partial t}(0, x) = y_1(x) \text{ in } \Omega.$$

First we introduce two Hilbert spaces H and V by $H = L^2(\Omega)$ and $V = H^1(\Omega)$, respectively. We endow $H = L^2(\Omega)$ with the inner product and norm $(\psi, \phi) = \int_{\Omega} \psi(x)\phi(x)dx, \|\psi\| = (\psi, \psi)^{1/2}, \forall \phi, \psi \in L^2(\Omega)$. For $\phi, \psi \in V = H^1(\Omega)$ we define $((\psi, \phi)) = \sum_{i=1}^n \int_{\Omega} \frac{\partial}{\partial x_i} \psi(x) \frac{\partial}{\partial x_i} \phi(x)dx$. The inner product and norm of $V = H^1(\Omega)$ are defined by

$$((\psi, \phi))_1 = ((\psi, \phi)) + (\psi, \phi), \quad \|\psi\| = ((\psi, \psi))_1^{1/2}, \quad \forall \phi, \psi \in H^1(\Omega).$$

The duality pairing between V and V' is denoted by $\langle \cdot, \cdot \rangle$. Then the pair (V, H) is a Gelfand triple space with a notation, $V \hookrightarrow H \equiv H' \hookrightarrow V'$, which means that embeddings $V \subset H$ and $H \subset V'$ are continuous, dense and compact. The norm of the dual space V' is denoted by $\|\cdot\|_*$.

Now we introduce a bilinear form

$$(2.4) \quad a(\phi, \varphi) = \int_{\Omega} \nabla \phi \cdot \nabla \varphi dx = ((\phi, \varphi)), \quad \forall \phi, \varphi \in H^1(\Omega).$$

The form (2.4) is symmetric, bounded on $H^1(\Omega) \times H^1(\Omega)$ and coercive $a(\phi, \phi) \geq \|\phi\|^2 - |\phi|^2, \forall \phi \in H^1(\Omega)$. Then we can define the bounded operator $A \in \mathcal{L}(V, V')$ through (2.4). The operator A is an isomorphism from V onto V' and it is also considered as a self-adjoint operator in $H = L^2(\Omega)$ with dense domain $\mathcal{D}(A)$ in V and in $H, \mathcal{D}(A) = \{\phi \in V : A\phi \in H\} = \{\phi \in H^2(\Omega) : \frac{\partial \phi}{\partial n} = 0 \text{ on } \Gamma\}$. Also we define the sine function for $z \in H = L^2(\Omega)$ by $(\sin z)(x) = \sin z(x)$ for a.e. $x \in \Omega$. Using the operator A and sine function $\sin y$, the system (2.1)-(2.3) is converted to the following Cauchy problem in H .

$$(2.5) \quad \begin{cases} \frac{d^2 y(t)}{dt^2} + \alpha A \frac{dy(t)}{dt} + \beta Ay(t) + \sum_{i=1}^L \gamma_i \sin \kappa_i y + \delta y = f(t), & t \in (0, T), \\ y(0) = y_0, \quad \frac{dy}{dt}(0) = y_1. \end{cases}$$

We define a solution space available to (2.5) and its inner product as follows:

$$W_V(0, T) = \{g | g \in L^2(0, T; V), g' \in L^2(0, T; V), g'' \in L^2(0, T; V')\},$$

$$(f, g)_{W_V(0, T)} = \int_0^T (\langle f(t), g(t) \rangle + \langle f'(t), g'(t) \rangle + \langle f''(t), g''(t) \rangle_{V'}) dt,$$

where $(\cdot, \cdot)_{V'}$ is the inner product of V' .

We give the definition of a weak solution for the Cauchy problem (2.5).

DEFINITION 2.1. A function y is said to be a weak solution of (2.5) if $y \in W_V(0, T)$ and y satisfies

$$\begin{aligned} & \langle y''(\cdot), \phi \rangle + \langle \alpha y'(\cdot), \phi \rangle + \langle \beta y(\cdot), \phi \rangle \\ & + \sum_{i=1}^L \langle \gamma_i \sin \kappa_i y(\cdot), \phi \rangle + \langle \delta y(\cdot), \phi \rangle \\ & = \langle f(\cdot), \phi \rangle \quad \text{for all } \phi \in V \text{ in the sense of } \mathcal{D}'(0, T), \\ & y(0) = y_0, \quad y'(0) = y_1, \end{aligned}$$

where $\mathcal{D}'(0, T)$ denotes the space of distributions on $(0, T)$.

For the existence, uniqueness and regularity of the weak solutions for (2.5), we can prove Theorem 2.2. For the proof of Theorem 2.2, see Ha and Nakagiri [6].

THEOREM 2.2. *Let $\alpha, \beta > 0$, $\delta, \gamma_i, \kappa_i \in \mathbf{R}$, $i = 1, \dots, L$ and f, y_0, y_1 be given satisfying*

$$(2.6) \quad f \in L^2(0, T; V'), \quad y_0 \in H^1(\Omega), \quad y_1 \in L^2(\Omega).$$

Then the problem (2.5) has a unique weak solution y in $W_V(0, T)$. The solution y has the regularity

$$(2.7) \quad y \in C([0, T]; H^1(\Omega)), \quad y' \in C([0, T]; L^2(\Omega))$$

and it is estimated by

$$(2.8) \quad |y'(t)|^2 + \|y(t)\|^2 \leq C_1(\|y_0\|^2 + |y_1|^2 + \|f\|_{L^2(0, T; V')}^2), \quad t \in [0, T],$$

$$(2.9) \quad \int_0^T \|y'(t)\|^2 dt \leq C_2(\|y_0\|^2 + |y_1|^2 + \|f\|_{L^2(0, T; V')}^2),$$

where C_1, C_2 are constants depending only on $\alpha, \beta, \delta, \gamma_i$ and κ_i .

Next we consider the linearized Cauchy problem of (2.5).

$$(2.10) \quad \begin{cases} \frac{d^2 y(t)}{dt^2} + \alpha A \frac{dy(t)}{dt} + \beta A y(t) + \delta y(t) + B(t)y(t) = f(t), & t \in (0, T), \\ y(0) = y_0 \in V, \quad \frac{dy}{dt}(0) = y_1 \in H, \end{cases}$$

where $B(\cdot) \in L^\infty(0, T; \mathcal{L}(H))$. The definition of a weak solution for (2.10) is same as given in Definition 2.1. Then we can prove Corollary 2.3.

COROLLARY 2.3. *Under the same conditions in Theorem 2.2 without the conditions on γ_i and κ_i , the problem (2.10) has a unique weak solution y in $W_V(0, T)$. The solution y satisfies (2.7), (2.8) and (2.9), in which C_1, C_2 are constants depending only on α, β, δ and the $L^\infty(0, T; \mathcal{L}(H))$ norm of $B(\cdot)$.*

REMARK 2.4. The constant C_1 in Theorem 2.2 can be chosen uniformly bounded on each bounded set of $\alpha, \beta, \delta, \gamma_i, \kappa_i$, and δ .

3. Identification of constant parameters

From now on we will omit to attach the interval notation $(0, T)$ to all equations. In this section we study the problems of identification for

the following perturbed sine-Gordon system.

$$(3.1) \quad \begin{cases} y'' + (\alpha_0 + \alpha^2)Ay' + (\beta_0 + \beta^2)Ay + \sum_{i=1}^L \gamma_i \sin \kappa_i y + \delta y = \nu f, \\ y(0) = y_0, \quad y'(0) = y_1, \end{cases}$$

where $\alpha_0 > 0$ and $\beta_0 > 0$ are fixed. In (3.1) we multiply a constant ν to the forcing term f and replace the diffusion parameters α to $\alpha_0 + \alpha^2$ and β to $\beta_0 + \beta^2$ to obtain the linear space of parameters $\alpha, \beta, \gamma_i, \delta, \kappa_i, \nu$. Hence the diffusion terms in (3.1) never disappear and are uniformly coercive for all $\alpha, \beta \in \mathbf{R}$. For setting the identification problems of the parameters $\alpha, \beta, \gamma_i, \delta, \kappa_i$ and ν in (3.1), we take $\mathcal{P} = \mathbf{R}^{2L+4}$ as a set of parameters $q = (\alpha, \beta, \gamma_1, \dots, \gamma_L, \delta, \kappa_1, \dots, \kappa_L, \nu)$. The Euclidean norm and inner product of \mathcal{P} are denoted simply by $|\cdot|$ and (\cdot, \cdot) , respectively. For simplicity of notations we write $q = (\alpha, \beta, \gamma_i, \delta, \kappa_i, \nu) \in \mathcal{P}$. By Theorem 2.2, for each $q \in \mathcal{P}$ there exists a unique weak solution $y = y(q) \in W_V(0, T)$ of (3.1). Hence we can find a solution map from \mathcal{P} to $W_V(0, T)$ such that we give a relation between q and $y(q)$.

Let K be a Hilbert space of observations and let $\|\cdot\|_K$ be its norm. The observation of $y(q)$ is assumed to be given by

$$(3.2) \quad z(q) = \mathcal{C}y(q) \in K,$$

where \mathcal{C} is a bounded linear observation operator of $W_V(0, T)$ into K . We introduce a cost functional $J(q)$ subject to (3.1) and (3.2) as follows:

$$(3.3) \quad J(q) = \|\mathcal{C}y(q) - z_d\|_K^2 + (Mq, q) \quad \text{for } q \in \mathcal{P},$$

where $z_d \in K$ is a desired value of $y(q)$ and M is a symmetric and non-negative $(2L + 4) \times (2L + 4)$ matrix on $\mathcal{P} = \mathbf{R}^{2L+4}$.

Assume that an admissible subset \mathcal{P}_{ad} of \mathcal{P} is convex and closed. As in [7] we shall solve the following two problems.

(i) Find $q^* \in \mathcal{P}_{ad}$ satisfying

$$(3.4) \quad \inf_{q \in \mathcal{P}_{ad}} J(q) = J(q^*);$$

(ii) Deduce necessary conditions on q^* .

As usual we call q^* an optimal parameter and $y(q^*)$ the optimal state corresponding to q^* . For solving the problem (i) we give some sufficient conditions such that \mathcal{P}_{ad} is a compact subset of \mathcal{P} or M is a positive matrix. For solving (ii) we use an inequality given by

$$(3.5) \quad DJ(q^*)(q - q^*) \geq 0 \quad \text{for all } q \in \mathcal{P}_{ad},$$

where $DJ(q^*)$ denotes the Gâteaux derivative of $J(q)$ at $q = q^*$ in the direction $q - q^*$. That is, we analyze the inequality (3.5) by introducing an adjoint state equation for (3.1) and deduce necessary conditions on q^* .

3.1. Existence of optimal parameters

The continuity of $q \rightarrow y(q)$ is crucial to solve the problems (i) and (ii).

THEOREM 3.1. *The map $q \rightarrow y(q) : \mathcal{P} \rightarrow W_V(0, T)$ is weakly continuous in the sense that $y(q_n) \rightarrow y(q)$ weakly in $W_V(0, T)$ as $q_n \rightarrow q$ in \mathbf{R}^{2L+4} .*

Proof. Suppose $q_n = (\alpha_n, \beta_n, \gamma_{in}, \delta_n, \kappa_{in}, \nu_n) \rightarrow q = (\alpha, \beta, \gamma_i, \delta, \kappa_i, \nu)$ in \mathbf{R}^{2L+4} , i.e., $\alpha_n \rightarrow \alpha, \beta_n \rightarrow \beta, \gamma_{in} \rightarrow \gamma_i, \delta_n \rightarrow \delta, \kappa_{in} \rightarrow \kappa_i, \nu_n \rightarrow \nu$ in \mathbf{R} . Let $y_n = y(q_n)$ be the weak solution of (3.6)

$$\begin{cases} y'' + (\alpha_n^2 + \alpha_0)Ay' + (\beta_n^2 + \beta_0)Ay + \sum_{i=1}^L \gamma_{in} \sin \kappa_{in}y + \delta_n y = \nu_n f, \\ y(q_n; 0) = y_0, \quad y'(q_n; 0) = y_1. \end{cases}$$

It follows from (2.8) and (2.9) that

$$|y'_n(t)|^2 + \|y_n(t)\|^2 \leq C_1(q_n)(\|y_0\|^2 + |y_1|^2 + \|f\|_{L^2(0,T;H)}^2), \quad \forall t \in [0, T],$$

$$\int_0^T \|y'_n(t)\|^2 dt \leq C_2(q_n)(\|y_0\|^2 + |y_1|^2 + \|f\|_{L^2(0,T;V')}^2),$$

where $C_1(q_n)$ and $C_2(q_n)$ depend on $\alpha_n, \beta_n, \gamma_{in}, \kappa_{in}, \delta_n$ and ν_n . Since $\alpha_n^2 + \alpha_0 \geq \alpha_0$ and $\beta_n^2 + \beta_0 \geq \beta_0$ for all n , the sequences $\{C_1(q_n)\}$ and $\{C_2(q_n)\}$ are bounded in $\mathbf{R}^+ = [0, \infty)$. Hence $\{y_n\}$ is bounded in $L^\infty(0, T; V)$ and $\{y'_n\}$ is bounded in $L^2(0, T; V)$. Also we can easily verify that $\{y''_n\}$ is bounded in $L^2(0, T; V')$ by applying the boundednesses of $\{y_n\}, \{y'_n\}, \{Ay_n\}$ in $L^2(0, T; V')$, the boundedness of $\{q_n\}$ in \mathbf{R}^{2L+4} and the inequality $|\sin k_{in}y_n| \leq |k_{in}||y_n|$ to the first equation in (3.6). Hence we can extract a subsequence of $\{y_n\}$, denoting it by $\{y_n\}$ again, and choose $z \in W_V(0, T)$ such that

$$(3.7) \quad \begin{cases} y_n \rightarrow z \text{ weakly in } L^2(0, T; V), \\ y'_n \rightarrow z' \text{ weakly in } L^2(0, T; V), \\ y''_n \rightarrow z'' \text{ weakly in } L^2(0, T; V'), \\ z(0) = y_0, \quad z'(0) = y_1. \end{cases}$$

Since the embedding $V \hookrightarrow H$ is compact, the embedding $L^2(0, T; V) \cap W^{1,2}(0, T; H) \hookrightarrow L^2(0, T; H)$ is also compact. Since $\{y_n\} \subset L^2(0, T; V) \cap W^{1,2}(0, T; H)$, we see by the first one in (3.7) that

$$(3.8) \quad y_n \rightarrow z \text{ strongly in } L^2(0, T; H),$$

which yields

$$(3.9) \quad \delta_n y_n \rightarrow \delta z \text{ strongly in } L^2(0, T; H).$$

Also the nonlinear term is estimated by

$$(3.10) \quad \begin{aligned} & |\gamma_{i_n} \sin \kappa_{i_n} y_n(t) - \gamma_i \sin \kappa_i z(t)| \\ & \leq |\gamma_{i_n} \kappa_{i_n} - \gamma_i \kappa_i| |y_n(t)| + |\gamma_i| |\kappa_i| |y_n(t) - z(t)|. \end{aligned}$$

Since $\gamma_{i_n} \rightarrow \gamma_i$, $\kappa_{i_n} \rightarrow \kappa_i$ and $\{y_n\}$ is bounded in $L^\infty(0, T; H)$, we have from (3.8) and (3.10) that

$$(3.11) \quad \sum_{i=1}^L \gamma_{i_n} \sin \kappa_{i_n} y_n \rightarrow \sum_{i=1}^L \gamma_i \sin \kappa_i z \text{ strongly in } L^2(0, T; H).$$

Finally we take the limit $n \rightarrow \infty$ on the weak form of (3.6) by using $\nu_n \rightarrow \nu$, conditions (3.7), (3.9) and (3.11). Then z is a weak solution of (3.12)

$$\begin{cases} z'' + (\alpha^2 + \alpha_0)Az' + (\beta^2 + \beta_0)Az + \sum_{i=1}^L \gamma_i \sin \kappa_i z + \delta z = \nu f, \\ z(0) = y_0, \quad z'(0) = y_1. \end{cases}$$

Hence by the uniqueness of weak solutions, we have $z = y(q)$. These prove that $y(q_n) \rightarrow y(q)$ weakly in $W_V(0, T)$ without extracting a sub-sequence $\{q_n\}$ again by the uniqueness of weak solutions. \square

Theorem 3.2 follows immediately from Theorem 3.1 and the lower semi-continuity of norms.

THEOREM 3.2. *If $\mathcal{P}_{ad} \subset \mathcal{P} = \mathbf{R}^{2L+4}$ is compact or M is a positive and symmetric on \mathbf{R}^{2L+4} , then there exists at least one optimal parameter $q^* \in \mathcal{P}_{ad}$ for the cost (3.3).*

Proof. If M is a positive and symmetric matrix on \mathbf{R}^{2L+4} , then we see easily that the minimizing sequence $\{q_n\}$ such that $\lim_{n \rightarrow \infty} J(q_n) = \inf_{q \in \mathcal{P}_{ad}} J(q)$ is bounded in \mathbf{R}^{2L+4} . If \mathcal{P}_{ad} is compact, it is trivial that the minimizing sequence $\{q_n\}$ is bounded in \mathcal{P} . Since the cost is lower semi-continuous with respect to the weak topology of $W_V(0, T)$, this theorem follows immediately from Theorem 3.1. \square

3.2. Necessary conditions

For the proof of necessary conditions for optimality we utilize the Gâteaux differential of $y(q)$ with respect to the parameter $q \in \mathcal{P}$. So it need to estimate quotients $z_\lambda = (y(q_\lambda) - y(q^*))/\lambda$ in $W_V(0, T)$, where $q_\lambda = q^* + \lambda(q - q^*)$, $\lambda \in [-1, 1]$ and $q, q^* \in \mathcal{P}$. We set $y_\lambda = y(q_\lambda)$ and $y^* = y(q^*)$ for simplicity. Let us begin to prove that the weak Gâteaux differential of $y(q)$ at q^* in the direction $q - q^*$ exists in $W_V(0, T)$ and it is the solution of a related differential system.

THEOREM 3.3. *The map $q \rightarrow y(q)$ of \mathcal{P} into $W_V(0, T)$ is weakly Gâteaux differentiable. That is, for fixed $q = (\alpha, \beta, \gamma_i, \delta, \kappa_i, \nu)$ and $q^* = (\alpha^*, \beta^*, \gamma_i^*, \delta^*, \kappa_i^*, \nu^*)$ in \mathcal{P} the weak Gâteaux derivative $z = Dy(q^*)(q - q^*)$ of $y(q)$ at $q = q^*$ in the direction $q - q^*$ exists in $W_V(0, T)$ and it is a unique weak solution of the system*

$$(3.13) \quad \begin{cases} z'' + (\alpha^{*2} + \alpha_0)Az' + (\beta^{*2} + \beta_0)Az \\ \quad + \sum_{i=1}^L (\gamma_i^* \kappa_i^* \cos \kappa_i^* y^*)z + \delta^* z = f_0, \\ z(0) = z'(0) = 0, \end{cases}$$

where $y^* = y(q^*)$ and

$$f_0 = 2\alpha^*(\alpha^* - \alpha)Ay^{*'} + 2\beta^*(\beta^* - \beta)Ay^* + (\gamma^* - \gamma) \sin y^* + (\delta^* - \delta)y^* \\ + \sum_{i=1}^L (\gamma_i^* - \gamma_i) \sin \kappa_i^* y^* + \sum_{i=1}^L (\gamma_i^* \cos \kappa_i^* y^*)(\kappa_i^* - \kappa_i)y^* + (\nu^* - \nu)f.$$

Proof. For fixed q we set $q_\lambda = q^* + \lambda(q - q^*) = (\alpha_\lambda, \beta_\lambda, \gamma_{i_\lambda}, \delta_\lambda, \kappa_{i_\lambda}, \nu_\lambda)$, $\lambda \in [-1, 1]$. We recall the simplified notations $y_\lambda = y(q_\lambda)$ and $y^* = y(q^*)$, which are the weak solutions to (2.5) for given parameters q_λ and q^* , respectively. Then $q_\lambda \in \mathcal{P}$ and $|q_\lambda - q^*| = |\lambda||q - q^*| \rightarrow 0$ as $\lambda \rightarrow 0$. Hence by Theorem 3.1 we have

$$(3.14) \quad y_\lambda \rightarrow y^* \text{ weakly in } W_V(0, T) \text{ as } \lambda \rightarrow 0,$$

which also implies

$$(3.15) \quad y_\lambda \rightarrow y^* \text{ strongly in } L^2(0, T; H) \text{ as } \lambda \rightarrow 0.$$

Since y_λ is the weak solution, by (2.8) we have

$$(3.16) \quad C_3 \triangleq \sup\{\|y_\lambda(t)\|^2 + |y'_\lambda(t)|^2 : (t, \lambda) \in [0, T] \times [-1, 1]\} < \infty.$$

By integral mean value theorem the quotient $z_\lambda = (y_\lambda - y^*)/\lambda, \lambda \neq 0$ satisfies

$$(3.17) \quad \begin{cases} z_\lambda'' + (\alpha^{*2} + \alpha_0)Az_\lambda' + (\beta^{*2} + \beta_0)Az_\lambda + B(\lambda, t)z_\lambda + \delta^* z_\lambda = f_\lambda, \\ z_\lambda(0) = z_\lambda'(0) = 0, \end{cases}$$

where $B(\lambda, \cdot) = \sum_{i=1}^L \gamma_i^* \left(\int_0^1 \kappa_i^* \cos \kappa_i^*(\theta y_\lambda + (1 - \theta)y^*) d\theta \right)$ and

$$\begin{aligned} f_\lambda &= [2\alpha^*(\alpha^* - \alpha) - \lambda(\alpha - \alpha^*)^2]Ay_\lambda' \\ &\quad + [2\beta^*(\beta^* - \beta) - \lambda(\beta - \beta^*)^2]Ay_\lambda \\ &\quad + (\delta^* - \delta)y_\lambda + \sum_{i=1}^L (\gamma_i^* - \gamma_i) \sin \kappa_{i\lambda} y_\lambda \\ &\quad + \sum_{i=1}^L \gamma_i^* \int_0^1 \cos(\theta \kappa_{i\lambda} y_\lambda + (1 - \theta)\kappa_i^* y_\lambda) d\theta (\kappa_i^* - \kappa_i) y_\lambda + (\nu - \nu^*)f. \end{aligned}$$

Now shall show that $\{z_\lambda\}$ is bounded in $W_V(0, T)$ by applying Corollary 2.3. It is verified readily that

$$(3.18) \quad \|B(\lambda, t)\|_{\mathcal{L}(H)} = \sum_{i=1}^L |\gamma_i^* \kappa_i^*|.$$

Also f_λ is estimated as follows.

$$(3.19) \quad \begin{aligned} &\|f_\lambda(t)\|_* \\ &\leq [2|\alpha^*| |\alpha^* - \alpha| + \lambda(\alpha - \alpha^*)^2] \|y_\lambda'(t)\| + [2|\beta^*| |\beta^* - \beta| \\ &\quad + \lambda(\beta - \beta^*)^2] \|y_\lambda(t)\| + |\delta^* - \delta| \|y_\lambda(t)\| + |\nu - \nu^*| \|f(t)\|_* \\ &+ \sum_{i=1}^L [|\gamma_i^* - \gamma_i| \kappa_i^* + \lambda(\kappa_i - \kappa_i^*)] \|y_\lambda(t)\| + |\gamma_i^* \kappa_i^*| |\kappa_i^* - \kappa_i| \|y_\lambda(t)\|, \end{aligned}$$

where we used the inequality $\|Ay_\lambda'(t)\|_* \leq \|y_\lambda'(t)\|$. Since $\{y_\lambda\}$ is bounded in $W_V(0, T)$, the above estimate implies that $\{f_\lambda\}_\lambda$ is uniformly bounded in $L^2(0, T; V')$. Applying (3.18) and (3.19) to Corollary 2.3 yields that $\{z_\lambda\}$ is bounded in $W_V(0, T)$. Hence we can choose a subsequence of $\{z_\lambda\}$, denoting it again by $\{z_\lambda\}$, and choose $z \in W_V(0, T)$ such that

$$(3.20) \quad \begin{cases} z_\lambda \rightarrow z \text{ weakly in } L^2(0, T; V), \\ z_\lambda' \rightarrow z' \text{ weakly in } L^2(0, T; V), \\ z_\lambda'' \rightarrow z'' \text{ weakly in } L^2(0, T; V'), \\ z(0) = y_0, z'(0) = y_1. \end{cases}$$

Now let us prove that

$$(3.21) \quad \lim_{\lambda \rightarrow 0} f_\lambda = f_0 \text{ weakly in } L^2(0, T; V').$$

It is clear from (3.15) that

$$(3.22) \quad \sin \kappa_{i\lambda} y_\lambda \rightarrow \sin \kappa_i y^* \text{ strongly in } L^2(0, T; H) \text{ as } \lambda \rightarrow 0.$$

Here we set $G_i(y_\lambda) \equiv \gamma_i^* \int_0^1 \kappa_i^* \cos(\theta \kappa_{i\lambda} y_\lambda + (1 - \theta) \kappa_i^* y_\lambda) d\theta (\kappa_i^* - \kappa_i) y_\lambda$. In order to show (3.21) from (3.20) and (3.22), it is enough to show

$$(3.23) \quad \lim_{\lambda \rightarrow 0} G_i(y_\lambda) = (\gamma_i^* \cos \kappa_i^* y^*) (\kappa_i^* - \kappa_i) y^* \text{ strongly in } L^2(0, T; H).$$

Put $G_i(y_\lambda) - (\gamma_i^* \cos \kappa_i^* y^*) (\kappa_i^* - \kappa_i) y^* = E_i^1(y_\lambda) + E_i^2(y_\lambda)$, where

$$\begin{aligned} & E_i^1(\lambda; \cdot) \\ &= \gamma_i^* \int_0^1 \kappa_i^* \cos(\theta \kappa_{i\lambda} y_\lambda + (1 - \theta) \kappa_i^* y_\lambda) d\theta (\kappa_i^* - \kappa_i) (y_\lambda - y^*), \\ & E_i^2(\lambda; \cdot) \\ &= \gamma_i^* \int_0^1 \kappa_i^* (\cos(\theta \kappa_{i\lambda} y_\lambda + (1 - \theta) \kappa_i^* y_\lambda) - \cos \kappa_i^* y^*) d\theta (\kappa_i^* - \kappa_i) y^*. \end{aligned}$$

It is easily verified by (3.15) and the boundedness of the cosine operators in H that

$$(3.24) \quad \lim_{\lambda \rightarrow 0} E_i^1(\lambda; \cdot) = 0 \text{ strongly in } L^2(0, T; H).$$

We consider the convergence of E_i^2 . Since $y_\lambda \rightarrow y^*$ strongly in $L^2(0, T; H)$,

$$\theta \kappa_{i\lambda} y_\lambda + (1 - \theta) \kappa_i^* y_\lambda \rightarrow \kappa_i^* y^* \text{ strongly in } L^2(0, T; H)$$

as $\lambda \rightarrow 0$ for all $\theta \in [0, 1]$. Since the cosine operator is continuous on $L^2(0, T; H)$,

$$\cos(\theta \kappa_{i\lambda} y_\lambda + (1 - \theta) \kappa_i^* y_\lambda) \rightarrow \cos \kappa_i^* y^* \text{ strongly in } L^2(0, T; H).$$

By the uniform boundedness of a sequence $\{\cos(\theta \kappa_{i\lambda} y_\lambda + (1 - \theta) \kappa_i^* y_\lambda)\}$ in θ , we apply Lebesgue dominated convergence theorem to obtain

$$\int_0^1 \cos(\theta \kappa_{i\lambda} y_\lambda + (1 - \theta) \kappa_i^* y_\lambda) d\theta \rightarrow \cos \kappa_i^* y^* \text{ strongly in } L^2(0, T; H).$$

Hence we show $\lim_{\lambda \rightarrow 0} E_i^2(\lambda; \cdot) = 0$ strongly in $L^2(0, T; H)$, which proves (3.21).

Next we consider the convergence including z_λ terms in (3.17). By using the classical compactness theorem again, we suppose, by taking subsequence if necessary, that

$$(3.25) \quad z_\lambda \rightarrow z \text{ strongly in } L^2(0, T; H) \text{ as } \lambda \rightarrow 0.$$

Then by using (3.25), as in the similar way as above, we can prove the strong convergence

$$(3.26) \quad \gamma_i^* \left(\int_0^1 \kappa_i^* \cos(\kappa_i^*(\theta y_\lambda + (1-\theta)y_\lambda)) d\theta \right) z_\lambda \rightarrow (\gamma_i^* \kappa_i^* \cos \kappa_i^* y^*) z$$

in $L^2(0, T; H)$.

Finally we take the limit $\lambda \rightarrow 0$ in (3.17) by using (3.20), (3.21) and (3.26). Then the limit z satisfies the equation (3.13). By applying Corollary 2.3 with $B(t) = \sum_{i=1}^L \gamma_i^* \kappa_i^* \cos \kappa_i^* y^*(t)$, we see that the equation (3.13) has a unique weak solution $z \in W_V(0, T)$. Hence, without choosing subsequences, z_λ converges weakly to z in $W_V(0, T)$, so that z is shown to be a weak Gâteaux derivative $Dy(q^*)(q - q^*)$ in $W_V(0, T)$. This completes the proof. \square

Since the map $q \rightarrow y(q) : \mathcal{P} \rightarrow W_V(0, T)$ is weakly Gâteaux differentiable at q^* in the direction $q - q^*$, $J(q)$ is Gâteaux differentiable at q^* and the inequality (3.5) implies

$$(3.27) \quad \langle Cy(q^*) - z_d, Cz \rangle_{K', K} \geq 0, \quad \forall q \in \mathcal{P}_{ad},$$

where z is the solution of (3.13) (cf. Ahmed [1, p.46]). To avoid the identification problem from complicating we shall study the problem according to the following four types of simple observations, which are possible due to (2.7).

1. Observe the distributed state $Cy(q) = y(q) \in L^2(0, T; H)$ and take $K = L^2(0, T; H)$;
2. Observe the distributed velocity $Cy(q) = y'(q) \in L^2(0, T; H)$ and take $K = L^2(0, T; H)$;
3. Observe the time terminal state $Cy(q) = y(q; T) \in H$ and take $K = H$;
4. Observe the time terminal velocity $Cy(q) = y'(q; T) \in H$ and take $K = H$.

1. Case of $Cy(q) = y(q) \in L^2(0, T; H)$

In this case we give the cost functional by

$$(3.28) \quad J(q) = r \|y(q) - z_d\|_{L^2(0, T; H)}^2 + (Mq, q),$$

where $z_d \in L^2(0, T; H)$ and $r > 0$. Then the necessary condition (3.27) with respect to (3.28) is written by

$$(3.29) \quad r(y^* - z_d, z)_{L^2(0, T; H)} + (Mq^*, q - q^*) \geq 0, \quad \forall q \in \mathcal{P}_{ad}.$$

We introduce an adjoint state p given by the system

$$(3.30) \quad \begin{cases} p'' - (\alpha^{*2} + \alpha_0)Ap' + (\beta^{*2} + \beta_0)Ap \\ \quad + \sum_{i=1}^L (\gamma_i^* \kappa_i^* \cos \kappa_i^* y^*)p + \delta^* p = r(y^* - z_d), \\ p(T) = p'(T) = 0. \end{cases}$$

We can easily show the existence and uniqueness of weak solutions for (3.30) if we take the time reversion $t \rightarrow T - t$ and apply Corollary 2.3. Multiplying (3.30) by z and integrating it over $[0, T]$ by using (3.13) the necessary condition corresponding to (3.29) is characterized by

$$\begin{aligned} & \int_0^T \langle p, 2\alpha^*(\alpha^* - \alpha)Ay^{*'} + 2\beta^*(\beta^* - \beta)Ay^* + (\delta^* - \delta)y^* \\ & + \sum_{i=1}^L (\gamma_i^* - \gamma_i) \sin \kappa_i^* y^* + \sum_{i=1}^L (\gamma_i^* \cos \kappa_i^* y^*)(\kappa_i^* - \kappa_i)y^* + (\nu^* - \nu)f \rangle dt \\ & + (Mq^*, q^* - q) \geq 0 \quad \text{for all } q \in \mathcal{P}_{ad}. \end{aligned}$$

Summarizing these we have the following theorem.

THEOREM 3.4. *The optimal parameter q^* for the cost (3.28) is characterized by the two states $y = y(q^*), p = p(q^*)$ of the system*

$$\begin{cases} y'' + (\alpha_0 + \alpha^{*2})y' + (\beta_0 + \beta^{*2})Ay + \sum_{i=1}^L \gamma_i^* \sin \kappa_i^* y + \delta^* y = \nu^* f, \\ y(0) = y_0, \quad y'(0) = y_1, \end{cases}$$

$$\begin{cases} p'' - (\alpha^{*2} + \alpha_0)Ap' + (\beta^{*2} + \beta_0)Ap \\ \quad + \sum_{i=1}^L (\gamma_i^* \kappa_i^* \cos \kappa_i^* y)p + \delta^* p = r(y - z_d), \\ p(T) = p'(T) = 0 \end{cases}$$

and one inequality

$$\begin{aligned} & \int_0^T \langle p, 2\alpha^*(\alpha^* - \alpha)Ay' + 2\beta^*(\beta^* - \beta)Ay + (\delta^* - \delta)y \\ & + \sum_{i=1}^L (\gamma_i^* - \gamma_i) \sin \kappa_i^* y + \sum_{i=1}^L (\gamma_i^* \cos \kappa_i^* y)(\kappa_i^* - \kappa_i)y \\ & + (\nu^* - \nu)f \rangle dt + (Mq^*, q^* - q) \geq 0, \quad \forall q \in \mathcal{P}_{ad}. \end{aligned}$$

2. Case of $Cy(q) = y'(q) \in L^2(0, T; H)$

In this case we consider the cost functional given by

$$(3.31) \quad J(q) = r\|y'(q) - z_d\|_{L^2(0,T;H)}^2 + (Mq, q),$$

where $z_d \in L^2(0, T; H)$ and $r > 0$. Then the necessary condition (3.27) with respect to (3.31) is written by

$$(3.32) \quad r(y^{*'} - z_d, z')_{L^2(0,T;H)} + (Mq^*, q - q^*) \geq 0, \quad \forall q \in \mathcal{P}_{ad}.$$

We introduce an adjoint state p defined by the system

$$(3.33) \quad \begin{cases} p'' - (\alpha^{*2} + \alpha_0)Ap' + (\beta^{*2} + \beta_0)Ap \\ \quad + \sum_{i=1}^L \int_t^T (\gamma_i^* \kappa_i^* \cos \kappa_i^* y^*)p \, ds + \delta^*p = r(y^{*'} - z_d), \\ p(T) = p'(T) = 0. \end{cases}$$

Through the approach as similarly as we do in Theorem 2.3, we can prove the existence, uniqueness and regularity of a weak solution $p \in W_V(0, T)$ of (3.33). Let us multiply z' on the both side hands of (3.33) and integrate it on $[0, T]$ by using (3.13). Then by (3.32) a necessary condition on q^* is given by

$$\begin{aligned} & \int_0^T \langle p', 2\alpha^*(\alpha^* - \alpha)Ay^{*'} + 2\beta^*(\beta^* - \beta)Ay^* + (\delta^* - \delta)y^* \\ & + \sum_{i=1}^L (\gamma_i^* - \gamma_i) \sin \kappa_i^* y^* + \sum_{i=1}^L (\gamma_i^* \cos \kappa_i^* y^*)(\kappa_i^* - \kappa_i)y^* \\ & + (\nu^* - \nu)f \rangle dt + (Mq^*, q - q^*) \leq 0, \quad \forall q \in \mathcal{P}_{ad}. \end{aligned}$$

Summarizing these we have the following theorem.

THEOREM 3.5. *The optimal parameter q^* for the cost (3.31) is characterized by two states $y = y(q^*), p = p(q^*)$ of the system*

$$\begin{cases} y'' + (\alpha_0 + \alpha^{*2})y' + (\beta_0 + \beta^{*2})Ay + \sum_{i=1}^L \gamma_i^* \sin \kappa_i^* y + \delta^* y = \nu^* f, \\ y(0) = y_0, \quad y'(0) = y_1, \\ \begin{cases} p'' - (\alpha^{*2} + \alpha_0)Ap' + (\beta^{*2} + \beta_0)Ap \\ \quad + \sum_{i=1}^L \int_t^T (\gamma_i^* \kappa_i^* \cos \kappa_i^* y) p \, ds + \delta^* p = r(y' - z_d), \\ p(T) = p'(T) = 0 \end{cases} \end{cases}$$

and one inequality

$$\begin{aligned} & \int_0^T \langle p', 2\alpha^*(\alpha^* - \alpha)Ay' + 2\beta^*(\beta^* - \beta)Ay + (\delta^* - \delta)y \\ & + \sum_{i=1}^L (\gamma_i^* - \gamma_i) \sin \kappa_i^* y + \sum_{i=1}^L (\gamma_i^* \cos \kappa_i^* y)(\kappa_i^* - \kappa_i)y + (\nu^* - \nu)f \rangle dt \\ & + (Mq^*, q - q^*) \leq 0, \quad \forall q \in \mathcal{P}_{ad}. \end{aligned}$$

3. Case of $Cy(q) = y(q; T) \in H$

In this case the cost functional is given by

$$(3.34) \quad J(q) = r|y(q; T) - z_d|^2 + (Mq, q),$$

where $z_d \in H$ and $r > 0$. Then the necessary condition (3.27) with respect to (3.34) is written by

$$(3.35) \quad r(y^*(T) - z_d, z(T)) + (Mq^*, q - q^*) \geq 0, \quad \forall q \in \mathcal{P}_{ad}.$$

We introduce an adjoint state p given by the system

$$(3.36) \quad \begin{cases} p'' - (\alpha^{*2} + \alpha_0)Ap' + (\beta^{*2} + \beta_0)Ap + \sum_{i=1}^L (\gamma_i^* \kappa_i^* \cos \kappa_i^* y^*) p \, ds + \delta^* p = 0, \\ p(T) = 0, \quad p'(T) = -r(y^*(T) - z_d). \end{cases}$$

Since $r(y^*(T) - z_d) \in H$, by Corollary 2.3, there is a unique weak solution $p \in W_V(0, T)$ of (3.36). Similar to the cases 1 and 2 the necessary

condition (3.35) is characterized by

$$\int_0^T \langle p, 2\alpha^*(\alpha^* - \alpha)Ay' + 2\beta^*(\beta^* - \beta)Ay + (\delta^* - \delta)y + \sum_{i=1}^L (\gamma_i^* - \gamma_i) \sin \kappa_i^* y + \sum_{i=1}^L (\gamma_i^* \cos \kappa_i^* y)(\kappa_i^* - \kappa_i)y + (\nu^* - \nu)f \rangle dt + (Mq^*, q - q^*) \geq 0, \quad \forall q \in \mathcal{P}_{ad}.$$

Summarizing these we have the following theorem.

THEOREM 3.6. *The optimal parameter q^* for the cost (3.34) is characterized by two states $y = y(q^*), p = p(q^*)$ of the system*

$$\begin{cases} y'' + (\alpha_0 + \alpha^{*2})y' + (\beta_0 + \beta^{*2})Ay + \sum_{i=1}^L \gamma_i^* \sin \kappa_i^* y + \delta^* y = \nu^* f, \\ y(0) = y_0, \quad y'(0) = y_1, \end{cases}$$

$$\begin{cases} p'' - (\alpha^{*2} + \alpha_0)Ap' + (\beta^{*2} + \beta_0)Ap + \sum_{i=1}^L (\gamma_i^* \kappa_i^* \cos \kappa_i^* y)p + \delta^* p = 0, \\ p(T) = 0, \quad p'(T) = -r(y(T) - z_d) \end{cases}$$

and one inequality

$$\int_0^T \langle p, 2\alpha^*(\alpha^* - \alpha)Ay' + 2\beta^*(\beta^* - \beta)Ay + (\delta^* - \delta)y + \sum_{i=1}^L (\gamma_i^* - \gamma_i) \sin \kappa_i^* y + \sum_{i=1}^L (\gamma_i^* \cos \kappa_i^* y)(\kappa_i^* - \kappa_i)y + (\nu^* - \nu)f \rangle dt + (Mq^*, q - q^*) \geq 0, \quad \forall q \in \mathcal{P}_{ad}.$$

4. Case of $Cy'(q) = y'(q; T) \in H$

In this case the cost functional is given by

$$(3.37) \quad J(q) = |y'(q; T) - z_d|^2 + (Mq, q),$$

where $z_d \in H$ and $r > 0$. Then the necessary condition (3.27) with respect to (3.37) is written by

$$(3.38) \quad r(y^{*'}(T) - z_d, z'(T)) + (Mq^*, q - q^*) \geq 0, \quad \forall q \in \mathcal{P}_{ad}.$$

We consider the adjoint state p given by system

$$(3.39) \quad \begin{cases} p'' - (\alpha^{*2} + \alpha_0)Ap' + (\beta^{*2} + \beta_0)Ap + \sum_{i=1}^L \gamma_i^* \kappa_i^* \cos \kappa_i^* y^* p + \delta^* p = 0, \\ p(T) = r(y^{*'}(T) - z_d), \quad p'(T) = r(\alpha^{*2} + \alpha_0)(y^{*'}(T) - z_d). \end{cases}$$

In general $y^{*'}(T) - z_d \notin V$ in spite of $y^{*'}(T) - z_d \in H$, we can not give any information of solutions for the equation (3.39). Thus, in this case we assume the additional regularity $y^{*'}(T) - z_d \in V$ and hence have the unique solution $p \in W_V(0, T)$ of (3.39). As doing as the previous cases the necessary condition (3.38) is characterized by

$$\begin{aligned} & \int_0^T \langle p, 2\alpha^*(\alpha^* - \alpha)Ay' + 2\beta^*(\beta^* - \beta)Ay + (\delta^* - \delta)y \\ & + \sum_{i=1}^L (\gamma_i^* - \gamma_i) \sin \kappa_i^* y + \sum_{i=1}^L (\gamma_i^* \cos \kappa_i^* y)(\kappa_i^* - \kappa_i)y + (\nu^* - \nu)f \rangle dt \\ & + (Mq^*, q - q^*) \geq 0, \quad \forall q \in \mathcal{P}_{ad}. \end{aligned}$$

Summarizing these we have the following theorem.

THEOREM 3.7. *Assume that $y^{*'}(T) - z_d \in V$. Then the optimal parameter q^* for the cost (3.37) is characterized by two states $y = y(q^*), p = p(q^*)$ of the system*

$$\begin{cases} y'' + (\alpha_0 + \alpha^{*2})y' + (\beta_0 + \beta^{*2})Ay + \sum_{i=1}^L \gamma_i^* \sin \kappa_i^* y + \delta^* y = \nu^* f, \\ y(0) = y_0, \quad y'(0) = y_1, \end{cases}$$

$$\begin{cases} p'' - (\alpha^{*2} + \alpha_0)Ap' + (\beta^{*2} + \beta_0)Ap + \sum_{i=1}^L (\gamma_i^* \kappa_i^* \cos \kappa_i^* y)p ds + \delta^* p = 0, \\ p(T) = r(y'(T) - z_d), \quad p'(T) = r(\alpha^{*2} + \alpha_0)(y'(T) - z_d) \end{cases}$$

and one inequality

$$\begin{aligned} & \int_0^T \langle p, 2\alpha^*(\alpha^* - \alpha)Ay' + 2\beta^*(\beta^* - \beta)Ay + (\delta^* - \delta)y \\ & + \sum_{i=1}^L (\gamma_i^* - \gamma_i) \sin \kappa_i^* y + \sum_{i=1}^L (\gamma_i^* \cos \kappa_i^* y)(\kappa_i^* - \kappa_i)y + (\nu^* - \nu)f \rangle dt \\ & + (Mq^*, q - q^*) \geq 0, \quad \forall q \in \mathcal{P}_{ad}. \end{aligned}$$

EXAMPLE 3.8. Let us deduce the bang-bang principle for a case where M is the null operator and \mathcal{P}_{ad} is compact. For simplicity we consider the case of $Cy(q) = y(q) \in L^2(0, T; H)$. Assume that \mathcal{P}_{ad} is given by

$$\mathcal{P}_{ad} = [0, \alpha_1] \times [0, \beta_1] \times \prod_{i=1}^L [\gamma_{i1}, \gamma_{i2}] \times [\delta_1, \delta_2] \times \prod_{i=1}^L [\kappa_{i1}, \kappa_{i2}] \times [\nu_1, \nu_2].$$

In this case the necessary condition in Theorem 3.4 is equivalent to

$$(3.40) \quad \int_0^T \langle \alpha^*(\alpha^* - \alpha)Ay'(t), p(t) \rangle dt \geq 0, \quad \forall \alpha \in [0, \alpha_1],$$

$$(3.41) \quad \int_0^T \langle \beta^*(\beta^* - \beta)Ay(t), p(t) \rangle dt \geq 0, \quad \forall \beta \in [0, \beta_1],$$

$$(3.42) \quad \int_0^T ((\gamma_i^* - \gamma_i) \sin \kappa_i^* y(t), p(t)) dt \geq 0, \quad \forall \gamma_i \in [\gamma_{i1}, \gamma_{i2}],$$

$$(3.43) \quad \int_0^T ((\delta^* - \delta)y(t), p(t)) dt \geq 0, \quad \forall \delta \in [\delta_1, \delta_2],$$

$$(3.44) \quad \int_0^T ((\kappa_i^* - \kappa_i)(\gamma_i^* \cos \kappa_i^* y(t))y(t), p(t)) dt \geq 0, \quad \forall \kappa_i \in [\kappa_{i1}, \kappa_{i2}],$$

$$(3.45) \quad \int_0^T ((\nu - \nu^*)f(t), p(t)) dt \geq 0, \quad \forall \nu \in [\nu_1, \nu_2].$$

First let us analyze (3.40). Put $a = \int_Q \nabla \frac{\partial y}{\partial t}(t, x) \nabla p(t, x) dxdt$ and assume that $a \neq 0$. Then (3.40) is rewritten simply by

$$\alpha^*(\alpha^* - \alpha)a \geq 0, \quad \forall \alpha \in [0, \alpha_1].$$

Consequently it is easily verified that α^* is given by

$$\alpha^* = \frac{1}{2} \{ \text{sign}(a) + 1 \} \alpha_1 \quad \text{or} \quad \alpha^* = 0.$$

Next we consider (3.41). Also put $b = \int_Q \nabla y(x, t) \cdot \nabla p(x, t) dxdt$ and assume $b \neq 0$. Then similarly as above, β^* is given by

$$\beta^* = \frac{1}{2} \{ \text{sign}(b) + 1 \} \beta_1 \quad \text{or} \quad \beta^* = 0.$$

Put $d = \int_Q y(t, x)p(x, t) dxdt$ and assume $d \neq 0$. Then the condition (3.43) is rewritten by $(\delta^* - \delta)d \geq 0, \quad \forall \delta \in [\delta_1, \delta_2]$, which imply $\delta^* = \frac{1}{2} \{ \text{sign}(d) + 1 \} \delta_2 - \frac{1}{2} \{ \text{sign}(d) - 1 \} \delta_1$. Similarly, it follows from (3.42), (3.44) and (3.45) that $\gamma_i^* = \frac{1}{2} \{ \text{sign}(c_i) + 1 \} \gamma_{i2} - \frac{1}{2} \{ \text{sign}(c_i) - 1 \} \gamma_{i1}$,

$\kappa_i^* = \frac{1}{2}\{\text{sign}(k_i) + 1\}\kappa_{i2} - \frac{1}{2}\{\text{sign}(k_i) - 1\}\kappa_{i1}$ and $\nu^* = \frac{1}{2}\{\text{sign}(n) + 1\}\nu_2 - \frac{1}{2}\{\text{sign}(n) - 1\}\nu_1$ provided that $c_i = \int_Q (\sin \kappa_i^* y(t, x))p(t, x) dxdt \neq 0$, $k_i = \int_Q (\gamma_i^* \cos \kappa_i^* y(t, x))y(t, x)p(t, x) dxdt \neq 0$ and $n = \int_Q f(t, x)p(t, x) dxdt \neq 0$. These are the so called bang-bang principle for the optimal parameter $q^* = (\alpha^*, \beta^*, \gamma_i^*, \delta^*, \kappa_i^*, \nu^*)$.

EXAMPLE 3.9. We consider the case where $\mathcal{P}_{ad} = \mathcal{P}$ and M is the identity operator on \mathcal{P} . As in Example 3.8 we consider the case of $Cy(q) = y(q) \in L^2(0, T; H)$. Then by Theorem 3.4 we have $\delta^* = -\int_Q y(t, x)p(t, x) dxdt$, $\nu^* = -\int_Q f(t, x)p(t, x) dxdt$, $\gamma_i^* = -\int_Q (\sin \kappa_i^* y(t, x))p(t, x) dxdt$ and $\kappa_i^* = -\gamma_i^* \int_Q y(t, x)(\cos \kappa_i^* y(t, x))p(t, x) dxdt$. If $2 \int_Q \nabla \frac{\partial y}{\partial t}(t, x) \cdot \nabla p(t, x) dxdt + 1 \neq 0$, and $2 \int_Q \nabla y(t, t) \cdot \nabla p(t, x) dxdt + 1 \neq 0$, then $\alpha^* = 0$ and $\beta^* = 0$.

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