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A REFINEMENT OF GRÜSS TYPE INEQUALITY FOR THE BOCHNER INTEGRAL OF VECTOR-VALUED FUNCTIONS IN HILBERT SPACES AND APPLICATIONS

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Abstract. A refinement of Grüss type inequality for the Bochner integral of vector-valued functions in real or complex Hilbert spaces is given. Related results are obtained. Application for finite Fourier transforms of vector-valued functions and some particular inequalities are provided.

1. Introduction

In 1934, G. Grüss [5] showed that

$$|T(f,g)| \leq \frac{1}{4}(M-m)(N-n), \tag{1.1}$$

provided m, M, n, N are real numbers with the property

$$-\infty < m \leq f \leq M < \infty, \quad -\infty < n \leq g \leq N < \infty \quad \text{a.e. on } [a,b]$$

and $T(f,g)$ is the *Čebyšev functional*

$$T(f,g) := \frac{1}{b-a}\int_a^b f(t)\,g(t)\,dt - \frac{1}{b-a}\int_a^b f(t)\,dt \cdot \frac{1}{b-a}\int_a^b g(t)\,dt.$$

The constant $\frac{1}{4}$ is best possible in (1.1) in the sense that it cannot be replaced by a smaller one.

An extension of this classical result to real or complex inner product spaces has been obtained by S. S. Dragomir in [2]:

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THEOREM 1. *Let* $(H;\langle\cdot,\cdot\rangle)$ *be an inner product space over the real or complex number field* $\mathbb{K}$ *and* $e\in H$, $\|e\|=1$. *If* $\varphi,\phi,\gamma,\Gamma\in\mathbb{R}$ *and* $x,y\in H$ *are such that*

$$\operatorname{Re}\langle\phi e-x,x-\varphi e\rangle\geq 0 \quad\text{and}\quad \operatorname{Re}\langle\Gamma e-y,y-\gamma e\rangle\geq 0 \tag{1.2}$$

or, equivalently (see [4]*)*

$$\left\|x-\frac{\varphi+\phi}{2}e\right\|\leq\frac{1}{2}|\phi-\varphi| \quad\text{and}\quad \left\|y-\frac{\gamma+\Gamma}{2}e\right\|\leq\frac{1}{2}|\Gamma-\gamma|, \tag{1.3}$$

then

$$|\langle x,y\rangle-\langle x,e\rangle\langle e,y\rangle|\leq\frac{1}{4}|\phi-\varphi||\Gamma-\gamma|. \tag{1.4}$$

The constant $\frac{1}{4}$ *is best possible in* (1.4).

A further extension for Bochner integrals of vector-valued functions in real or complex Hilbert spaces was obtained by S. S. Dragomir in [3].

THEOREM 2. *Let* $(H;\langle\cdot,\cdot\rangle)$ *be a real or complex Hilbert space,* $\Omega\subset\mathbb{R}^n$ *be a Lebesgue measurable set and* $\rho:\Omega\to[0,\infty)$ *a Lebesgue measurable function with* $\int_\Omega\rho(s)\,ds=1$. *We denote by* $L_{2,\rho}(\Omega,H)$ *the set of all Bochner measurable functions* f *on* Ω *such that* $\|f\|_{2,\rho}^2:=\int_\Omega\rho(s)\|f(s)\|^2ds<\infty$. *If* f,g *belong to* $L_{2,\rho}(\Omega,H)$ *and there exist the vectors* $x,X,y,Y\in H$ *such that*

$$\begin{aligned}&\int_\Omega\rho(t)\operatorname{Re}\langle X-f(t),f(t)-x\rangle\,dt\geq 0,\\&\int_\Omega\rho(t)\operatorname{Re}\langle Y-g(t),g(t)-y\rangle\,dt\geq 0,\end{aligned}\tag{1.5}$$

then we have the inequality

$$\begin{aligned}&\left|\int_\Omega\rho(t)\langle f(t),g(t)\rangle\,dt-\left\langle\int_\Omega\rho(t)f(t)\,dt,\int_\Omega\rho(t)g(t)\,dt\right\rangle\right|\\&\leq\frac{1}{4}\|X-x\|\,\|Y-y\|.\end{aligned}\tag{1.6}$$

The constant $\frac{1}{4}$ *is sharp in the sense mentioned above.*

REMARK 1. We must state that the functions under the integrals (1.5) and (1.6) are Bochner integrable on Ω since they are Bochner measurable and we can state the following obvious results

$$\begin{aligned}&\rho(t)|\operatorname{Re}\langle X-f(t),f(t)-x\rangle|\\\leq\;&\rho(t)|\langle X-f(t),f(t)-x\rangle|\\\leq\;&\rho(t)\|f(t)\|^2+(\|X\|+\|x\|)\rho(t)\|f(t)\|+|\langle X,x\rangle|\rho(t)\end{aligned}$$

for a.e. $t \in \Omega$;

$$\int_{\Omega} \rho(t) \|f(t)\| \, dt \leq \|f\|_{2,\rho}$$

and

$$\int_{\Omega} \rho(t) |\langle f(t), g(t) \rangle| \, dt \leq \|f\|_{2,\rho} \|g\|_{2,\rho}.$$

REMARK 2. A practical sufficient condition for (1.5) to hold is

$$\operatorname{Re} \langle X - f(t), f(t) - x \rangle \geq 0, \qquad \operatorname{Re} \langle Y - g(t), g(t) - y \rangle \geq 0$$

or, equivalently

$$\left\| f(t) - \frac{X + x}{2} \right\| \leq \frac{1}{2} \|X - x\| \quad \text{and} \quad \left\| g(t) - \frac{Y + y}{2} \right\| \leq \frac{1}{2} \|Y - y\|,$$

for a.e. $t \in \Omega$.

An interesting particular inequality that has not been mentioned in [3] can be obtained by considering $H = \mathbb{C}$, $\langle x, y \rangle := x \cdot \bar{y}$ and $g = \bar{f}$, to give

$$\left| \int_{\Omega} \rho(s) f^2(s) \, ds - \left(\int_{\Omega} \rho(s) f(s) \, ds \right)^2 \right| \leq \frac{1}{4} |A - a|^2, \tag{1.7}$$

provided

$$\int_{\Omega} \rho(s) \operatorname{Re} \left[(A - f(s)) \left(\overline{f(s)} - \bar{a} \right) \right] ds \geq 0 \tag{1.8}$$

or, sufficiently,

$$\operatorname{Re} \left[(A - f(s)) \left(\overline{f(s)} - \bar{a} \right) \right] \geq 0 \tag{1.9}$$

for a.e. $s \in \Omega$.

Note that the alternative result

$$0 \leq \int_{\Omega} \rho(s) |f(s)|^2 \, ds - \left| \int_{\Omega} \rho(s) f(s) \, ds \right|^2 \leq \frac{1}{4} |A - a|^2, \tag{1.10}$$

provided (1.8) or (1.9) hold, has been stated in [3].

The main aim of this paper is to obtain an improvement of the Grüss inequality (1.6) and establish some Grüss type results in providing upper bounds for the quantities

$$\left| \int_{\Omega} \rho(t) \langle f(t), g(t) \rangle \, dt - \left\langle \int_{\Omega} \rho(t) f(t) \, dt, \int_{\Omega} \rho(t) g(t) \, dt \right\rangle \right|$$

and

$$\left\| \int_\Omega \rho(t)\,\alpha(t)\,f(t)\,dt - \int_\Omega \rho(t)\,\alpha(t)\,dt \cdot \int_\Omega \rho(t)\,f(t)\,dt \right\|,$$

under various assumptions for ρ, $\alpha \in L_{2,\rho}(\Omega, \mathbb{K})$ and $f \in L_{2,\rho}(\Omega, H)$.

Applications in approximating the finite Fourier transform of vector-valued functions in Hilbert spaces are provided. Inequalities for some particular vector-valued functions are given as well.

2. Some inequalities of Grüss type

The following lemma holds.

LEMMA 1. *Assume that $f \in L_{2,\rho}(\Omega, H)$ and there exist the vectors $x, X \in H$ such that*

$$\int_\Omega \rho(t)\,\mathrm{Re}\,\langle X - f(t), f(t) - x\rangle\,dt \geq 0 \tag{2.1}$$

or, equivalently,

$$\int_\Omega \rho(t) \left\| f(t) - \frac{X+x}{2} \right\|^2 dt \leq \frac{1}{4}\|X - x\|^2. \tag{2.2}$$

Then we have the inequality

$$\begin{aligned} 0 &\leq \int_\Omega \rho(t)\,\|f(t)\|^2\,dt - \left\| \int_\Omega \rho(t)\,f(t)\,dt \right\|^2 \\ &\leq \frac{1}{4}\|X - x\|^2 - \int_\Omega \rho(t)\,\mathrm{Re}\,\langle X - f(t), f(t) - x\rangle\,dt \\ &\leq \frac{1}{4}\|X - x\|^2. \end{aligned} \tag{2.3}$$

The constant $\frac{1}{4}$ in the second and third inequalities cannot be replaced by a smaller quantity.

Proof. Since, for any $y, x, X \in H$

$$\left\| y - \frac{X+x}{2} \right\|^2 - \frac{1}{4}\|X - x\|^2 = \mathrm{Re}\,\langle y - X, y - x\rangle,$$

hence

$$\begin{aligned}&\int_{\Omega}\rho(t)\operatorname{Re}\langle X-f(t),f(t)-x\rangle\,dt\\&=\int_{\Omega}\rho(t)\left[\frac{1}{4}\|X-x\|^{2}-\left\|f(t)-\frac{X+x}{2}\right\|^{2}\right]dt\\&=\frac{1}{4}\|X-x\|^{2}-\int_{\Omega}\rho(t)\left\|f(t)-\frac{X+x}{2}\right\|^{2}dt\end{aligned}$$

showing that, indeed, (2.1) and (2.2) are equivalent.

Define (see also [3])

$$I_1:=\left\langle X-\int_{\Omega}\rho(t)f(t)\,dt,\int_{\Omega}\rho(t)f(t)\,dt-x\right\rangle$$

and

$$I_2:=\int_{\Omega}\rho(t)\langle X-f(t),f(t)-x\rangle\,dt.$$

Then, obviously

$$I_1=\int_{\Omega}\rho(t)\left[\langle X,f(t)\rangle+\langle f(t),x\rangle\right]dt-\langle X,x\rangle-\left\|\int_{\Omega}\rho(t)f(t)\,dt\right\|^{2}$$

and

$$I_2=\int_{\Omega}\rho(t)\left[\langle X,f(t)\rangle+\langle f(t),x\rangle\right]dt-\langle X,x\rangle-\int_{\Omega}\rho(t)\|f(t)\|^{2}\,dt.$$

Consequently,

$$I_1-I_2=\int_{\Omega}\rho(t)\|f(t)\|^{2}\,dt-\left\|\int_{\Omega}\rho(t)f(t)\,dt\right\|^{2}. \tag{2.4}$$

Taking the real value in (2.4), we can state the following identity as well [3]

$$\begin{aligned}&\int_{\Omega}\rho(t)\|f(t)\|^{2}\,dt-\left\|\int_{\Omega}\rho(t)f(t)\,dt\right\|^{2}\\&=\operatorname{Re}\left\langle X-\int_{\Omega}\rho(t)f(t)\,dt,\int_{\Omega}\rho(t)f(t)\,dt-x\right\rangle\\&\quad-\int_{\Omega}\rho(t)\operatorname{Re}\langle X-f(t),f(t)-x\rangle\,dt\end{aligned} \tag{2.5}$$

that is of interest in itself.

Using the well known inequality in inner product spaces

$$\operatorname{Re}\langle z,y\rangle \le \left\|\frac{z+y}{2}\right\|^2 \tag{2.6}$$

with equality if and only if $z=y$, we may state that

$$\operatorname{Re}\left\langle X-\int_\Omega \rho(t) f(t)\,dt, \int_\Omega \rho(t) f(t)\,dt - x\right\rangle \le \frac{1}{4}\|X-x\|^2$$

and by the identity (2.5), we deduce the second inequality in (2.3).

The third inequality follows by the assumption (2.1).

Now, assume that (2.3) holds with the constants $C,D>0$. That is,

$$\begin{aligned} 0 &\le \int_\Omega \rho(t)\|f(t)\|^2 dt - \left\|\int_\Omega \rho(t) f(t)\,dt\right\|^2 \\ &\le C\|X-x\|^2 - \int_\Omega \rho(t)\operatorname{Re}\langle X-f(t), f(t)-x\rangle\,dt \\ &\le D\|X-x\|^2. \end{aligned} \tag{2.7}$$

If we choose $\Omega=[a,b]\subset\mathbb{R}$, $H=\mathbb{R}$, $f:[a,b]\to\mathbb{R}$,

$$f(t)=\begin{cases} -1 & \text{if } t\in\left[a,\frac{a+b}{2}\right] \\ 1 & \text{if } t\in\left(\frac{a+b}{2},b\right] \end{cases}$$

then for $X=1$, $x=-1$ and $\rho:[a,b]\to\mathbb{R}$, $\rho(t)=1$, $t\in[a,b]$ the condition (2.1) holds and by (2.7) we deduce

$$1\le 4C\le 4D$$

giving $C\ge\frac{1}{4}$ and $D\ge\frac{1}{4}$, and the lemma is proved. □

The following refinement of the Grüss inequality holds.

THEOREM 3. *Assume that $f,g\in L_{2,\rho}(\Omega,H)$ and there exist the vectors $x,X,y,Y\in H$ such that*

$$\begin{aligned} &\int_\Omega \rho(t)\operatorname{Re}\langle X-f(t), f(t)-x\rangle\,dt \ge 0, \\ &\int_\Omega \rho(t)\operatorname{Re}\langle Y-g(t), g(t)-y\rangle\,dt \ge 0 \end{aligned} \tag{2.8}$$

or, equivalently,

$$\int_{\Omega} \rho(t) \left\| f(t) - \frac{X+x}{2} \right\|^2 dt \le \frac{1}{4} \|X - x\|^2, \tag{2.9}$$
$$\int_{\Omega} \rho(t) \left\| g(t) - \frac{Y+y}{2} \right\|^2 dt \le \frac{1}{4} \|Y - y\|^2.$$

Then we have the inequality

$$\begin{aligned} &\left| \int_{\Omega} \rho(t) \langle f(t), g(t) \rangle dt - \left\langle \int_{\Omega} \rho(t) f(t) dt, \int_{\Omega} \rho(t) g(t) dt \right\rangle \right| \\ &\le \frac{1}{4} \|X - x\| \|Y - y\| - \left[\int_{\Omega} \rho(t) \operatorname{Re} \langle X - f(t), f(t) - x \rangle dt \right. \\ &\quad \left. \times \int_{\Omega} \rho(t) \operatorname{Re} \langle Y - g(t), g(t) - y \rangle dt \right]^{1/2} \\ &\le \frac{1}{4} \|X - x\| \|Y - y\|. \end{aligned} \tag{2.10}$$

The constant $\frac{1}{4}$ in both inequalities is sharp in the sense that it cannot be replaced by a smaller quantity.

Proof. We start with the following Korkine type identity (see also [3])

$$\begin{aligned} &\int_{\Omega} \rho(t) \langle f(t), g(t) \rangle dt - \left\langle \int_{\Omega} \rho(t) f(t) dt, \int_{\Omega} \rho(t) g(t) dt \right\rangle \\ &= \frac{1}{2} \int_{\Omega} \int_{\Omega} f(t) \rho(s) \langle f(t) - f(s), g(t) - g(s) \rangle dt ds. \end{aligned} \tag{2.11}$$

Taking the modulus and using the Schwarz inequality in inner product spaces, we have

$$\begin{aligned} &\left| \int_{\Omega} \rho(t) \langle f(t), g(t) \rangle dt - \left\langle \int_{\Omega} \rho(t) f(t) dt, \int_{\Omega} \rho(t) g(t) dt \right\rangle \right| \\ &\le \frac{1}{2} \int_{\Omega} \int_{\Omega} \rho(t) \rho(s) \|f(t) - f(s)\| \|g(t) - g(s)\| dt ds. \end{aligned} \tag{2.12}$$

Using the Cauchy-Bunyakovski-Schwarz inequality for double integrals, we have

$$\frac{1}{2} \int_{\Omega} \int_{\Omega} \rho(t) \rho(s) \|f(t) - f(s)\| \|g(t) - g(s)\| dt ds \tag{2.13}$$

$$\leq \left(\frac{1}{2}\int_{\Omega}\int_{\Omega}\rho(t)\rho(s)\|f(t)-f(s)\|^{2}dtds\right)^{\frac{1}{2}} \times \left(\frac{1}{2}\int_{\Omega}\int_{\Omega}\rho(t)\rho(s)\|g(t)-g(s)\|^{2}dtds\right)^{\frac{1}{2}}.$$

Since a simple computation shows that

$$\frac{1}{2}\int_{\Omega}\int_{\Omega}\rho(t)\rho(s)\|f(t)-f(s)\|^{2}dtds = \int_{\Omega}\rho(t)\|f(t)\|^{2}dt-\left\|\int_{\Omega}\rho(t)f(t)dt\right\|^{2} \tag{2.14}$$

and

$$\frac{1}{2}\int_{\Omega}\int_{\Omega}\rho(t)\rho(s)\|g(t)-g(s)\|^{2}dtds = \int_{\Omega}\rho(t)\|g(t)\|^{2}dt-\left\|\int_{\Omega}\rho(t)g(t)dt\right\|^{2}, \tag{2.15}$$

then by (2.11)-(2.15), we deduce

$$\left|\int_{\Omega}\rho(t)\langle f(t),g(t)\rangle dt-\left\langle\int_{\Omega}\rho(t)f(t)dt,\int_{\Omega}\rho(t)g(t)dt\right\rangle\right|^{2} \leq \left(\int_{\Omega}\rho(t)\|f(t)\|^{2}dt-\left\|\int_{\Omega}\rho(t)f(t)dt\right\|^{2}\right) \times \left(\int_{\Omega}\rho(t)\|g(t)\|^{2}dt-\left\|\int_{\Omega}\rho(t)g(t)dt\right\|^{2}\right)=:M. \tag{2.16}$$

Using Lemma 1, we may deduce

$$M \leq \left(\frac{1}{4}\|X-x\|^{2}-\int_{\Omega}\rho(t)\operatorname{Re}\langle X-f(t),f(t)-x\rangle dt\right) \times \left(\frac{1}{4}\|Y-y\|^{2}-\int_{\Omega}\rho(t)\operatorname{Re}\langle Y-g(t),g(t)-y\rangle dt\right)=:N. \tag{2.17}$$

By the elementary inequality

$$\left(m^{2}-n^{2}\right)\left(p^{2}-q^{2}\right)\leq(mp-nq)^{2}, \quad m,n,p,q\in\mathbb{R},$$

we may state that

(2.18)

$$N \leq \left(\frac{1}{4} \|X - x\| \|Y - y\| - \left[\int_\Omega \rho(t) \operatorname{Re} \langle X - f(t), f(t) - x \rangle \, dt \right. \right. \\ \left. \left. \times \int_\Omega \rho(t) \operatorname{Re} \langle Y - g(t), g(t) - y \rangle \, dt \right]^{\frac{1}{2}} \right)^2$$

and thus by (2.16)-(2.18) we can conclude that

$$\left| \int_\Omega \rho(t) \langle f(t), g(t) \rangle \, dt - \left\langle \int_\Omega \rho(t) f(t) \, dt, \int_\Omega \rho(t) g(t) \, dt \right\rangle \right| \\ \leq \left| \frac{1}{4} \|X - x\| \|Y - y\| - \left[\int_\Omega \rho(t) \operatorname{Re} \langle X - f(t), f(t) - x \rangle \, dt \right. \right. \\ \left. \left. \times \int_\Omega \rho(t) \operatorname{Re} \langle Y - g(t), g(t) - y \rangle \, dt \right]^{\frac{1}{2}} \right| \tag{2.19}$$

and since, by Lemma 1,

$$\left[\int_\Omega \rho(t) \operatorname{Re} \langle X - f(t), f(t) - x \rangle \, dt \right. \\ \left. \cdot \int_\Omega \rho(t) \operatorname{Re} \langle Y - g(t), g(t) - y \rangle \, dt \right]^{\frac{1}{2}} \\ \leq \frac{1}{4} \|X - x\| \|Y - y\|,$$

hence from (2.19) we deduce (2.10).

The sharpness of the constant $\frac{1}{4}$ follows by Lemma 1, and we omit the details. □

REMARK 3. The inequality (2.10) is obviously a refinement of (1.6), which has been obtained in [3].

The following result of Grüss type also holds.

THEOREM 4. *Assume that* $\alpha \in L_{2,\rho}(\Omega, H)$, $f \in L_{2,\rho}(\Omega, H)$ *and there exist the scalars* $a, A \in \mathbb{K}$ $(\mathbb{K} = \mathbb{C}, \mathbb{R})$ *and the vectors* $x, X \in H$ *such that*

$$\int_\Omega \rho(t) \operatorname{Re} \left[(A - \alpha(t)) \left(\overline{\alpha(t)} - \overline{a} \right) \right] dt \geq 0 \quad \text{and} \\ \int_\Omega \rho(t) \operatorname{Re} \langle X - f(t), f(t) - x \rangle \, dt \geq 0 \tag{2.20}$$

or, equivalently,

$$\int_{\Omega} \rho(t) \left| \alpha(t) - \frac{A+a}{2} \right|^2 dt \le \frac{1}{4} |A-a|^2, \tag{2.21}$$
$$\int_{\Omega} \rho(t) \left\| f(t) - \frac{X+x}{2} \right\|^2 dt \le \frac{1}{4} \|X-x\|^2.$$

Then we have the inequality

$$\begin{aligned} &\left\| \int_{\Omega} \rho(t) \alpha(t) f(t) dt - \int_{\Omega} \rho(t) \alpha(t) dt \cdot \int_{\Omega} \rho(t) f(t) dt \right\| \\ &\le \frac{1}{4} |A-a| \|X-x\| - \left(\int_{\Omega} \rho(t) \operatorname{Re}\left[(A - \alpha(t)) \left(\overline{\alpha(t)} - \overline{a} \right) \right] dt \right. \\ &\quad \left. \times \int_{\Omega} \rho(t) \operatorname{Re} \langle X - f(t), f(t) - x \rangle dt \right)^{1/2} \\ &\le \frac{1}{4} |A-a| \|X-x\|. \end{aligned} \tag{2.22}$$

The constant $\frac{1}{4}$ in both inequalities is sharp in the sense that it cannot be replaced by a smaller quantity.

Proof. We observe that the following Korkine type identity holds

$$\begin{aligned} &\int_{\Omega} \rho(t) \alpha(t) f(t) dt - \int_{\Omega} \rho(t) \alpha(t) dt \cdot \int_{\Omega} \rho(t) f(t) dt \\ &= \frac{1}{2} \int_{\Omega} \int_{\Omega} \rho(t) \rho(s) (\alpha(t) - \alpha(s)) (f(t) - f(s)) dtds. \end{aligned}$$

Using a similar approach to the one in Theorem 3, we have successively

$$\begin{aligned} &\left\| \int_{\Omega} \rho(t) \alpha(t) f(t) dt - \int_{\Omega} \rho(t) \alpha(t) dt \cdot \int_{\Omega} \rho(t) f(t) dt \right\| \\ &\le \frac{1}{2} \int_{\Omega} \int_{\Omega} \rho(t) \rho(s) |\alpha(t) - \alpha(s)| \|f(t) - f(s)\| dtds \\ &\le \left[\frac{1}{2} \int_{\Omega} \int_{\Omega} \rho(t) \rho(s) |\alpha(t) - \alpha(s)|^2 dtds \right. \end{aligned} \tag{2.23}$$

$$\times \frac{1}{2}\int_{\Omega}\int_{\Omega}\rho(t)\rho(s)\|f(t)-f(s)\|^{2}dtds\Bigg]^{\frac{1}{2}}$$

$$= \left[\int_{\Omega}\rho(t)|\alpha(t)|^{2}dt-\left|\int_{\Omega}\rho(t)\alpha(t)dt\right|^{2}\right]^{\frac{1}{2}}$$

$$\times \left[\int_{\Omega}\rho(t)\|f(t)\|^{2}dt-\left\|\int_{\Omega}\rho(t)f(t)dt\right\|^{2}\right]^{\frac{1}{2}}$$

$$\leq \left(\frac{1}{4}|A-a|^{2}-\int_{\Omega}\rho(t)\operatorname{Re}\left[(A-\alpha(t))\left(\overline{\alpha(t)}-\overline{a}\right)\right]dt\right)^{\frac{1}{2}}$$

$$\times \left(\frac{1}{4}\|X-x\|^{2}-\int_{\Omega}\rho(t)\operatorname{Re}\langle X-f(t),f(t)-x\rangle dt\right)^{\frac{1}{2}}$$

$$\leq \left|\frac{1}{4}|A-a|\,\|X-x\| - \left(\int_{\Omega}\rho(t)\operatorname{Re}\left[(A-\alpha(t))\left(\overline{\alpha(t)}-\overline{a}\right)\right]dt\right)^{\frac{1}{2}}\right.$$

$$\left.\times \left(\int_{\Omega}\rho(t)\operatorname{Re}\langle X-f(t),f(t)-x\rangle dt\right)^{\frac{1}{2}}\right|$$

$$= \frac{1}{4}|A-a|\,\|X-x\| - \left(\int_{\Omega}\rho(t)\operatorname{Re}\left[(A-\alpha(t))\left(\overline{\alpha(t)}-\overline{a}\right)\right]dt\right.$$

$$\left.\times \int_{\Omega}\rho(t)\operatorname{Re}\langle X-f(t),f(t)-x\rangle dt\right)^{\frac{1}{2}}$$

and the first inequality in (2.22) is proved.

The second inequality and the sharpness of the constant $\frac{1}{4}$ are obvious and we omit the details. □

3. Pre-Grüss type inequalities

The following result provides an inequality of *pre-Grüss type* that may be useful in applications when one of the factors is known and some bounds for the second factor are provided.

THEOREM 5. *Assume that $f,g \in L_{2,\rho}(\Omega,H)$ and there exist the vectors $x,X \in H$ such that either* (2.1) *or* (2.2) *holds true. Then we have the inequality:*

$$\left|\int_{\Omega}\rho(t)\langle f(t),g(t)\rangle dt-\left\langle\int_{\Omega}\rho(t)f(t)dt,\int_{\Omega}\rho(t)g(t)dt\right\rangle\right| \tag{3.1}$$

$$\leq \left(\frac{1}{4}\|X-x\|^2 - \int_\Omega \rho(t)\operatorname{Re}\langle X - f(t), f(t) - x\rangle dt\right)^{\frac{1}{2}}$$
$$\times \left(\int_\Omega \rho(t)\|g(t)\|^2 dt - \left\|\int_\Omega \rho(t) g(t)\, dt\right\|^2\right)^{\frac{1}{2}}$$
$$\leq \frac{1}{2}\|X-x\| \left(\int_\Omega \rho(t)\|g(t)\|^2 dt - \left\|\int_\Omega \rho(t) g(t)\, dt\right\|^2\right)^{\frac{1}{2}}.$$

The proof follows by Lemma 1 and the inequality (2.16) and we omit the details.

Similarly, we can state the following pre-Grüss inequality related to the case when our function is scalar.

THEOREM 6. *Assume that* $\alpha \in L_{2,\rho}(\Omega, H)$, $f \in L_{2,\rho}(\Omega, H)$ *and there exist the vectors* $x, X \in H$ *such that* (2.1) *or, equivalently* (2.2) *holds true. Then we have the inequality*

$$\left\|\int_\Omega \rho(t)\alpha(t) f(t)\, dt - \int_\Omega \rho(t)\alpha(t)\, dt \cdot \int_\Omega \rho(t) f(t)\, dt\right\| \tag{3.2}$$
$$\leq \left(\frac{1}{4}\|X-x\|^2 - \int_\Omega \rho(t)\operatorname{Re}\langle X - f(t), f(t) - x\rangle dt\right)^{\frac{1}{2}}$$
$$\times \left(\int_\Omega \rho(t)|\alpha(t)|^2 dt - \left|\int_\Omega \rho(t)\alpha(t)\, dt\right|^2\right)^{\frac{1}{2}}$$
$$\leq \frac{1}{2}\|X-x\| \left(\int_\Omega \rho(t)|\alpha(t)|^2 dt - \left|\int_\Omega \rho(t)\alpha(t)\, dt\right|^2\right)^{\frac{1}{2}}.$$

The proof follows by Lemma 1 and the inequality (2.23) and we omit the details.

REMARK 4. Assume that $\Omega = [a, b] \subset \mathbb{R}$ and $\rho(t) = \frac{1}{b-a}$. Then, from (3.2) we get

$$\left\|\frac{1}{b-a}\int_\Omega \alpha(t) f(t)\, dt - \frac{1}{b-a}\int_\Omega \alpha(t)\, dt \cdot \frac{1}{b-a}\int_\Omega f(t)\, dt\right\| \tag{3.3}$$
$$\leq \left[\frac{1}{4}\|X-x\|^2 - \frac{1}{b-a}\int_\Omega \operatorname{Re}\langle X - f(t), f(t) - x\rangle dt\right]^{\frac{1}{2}}$$
$$\times \left[\frac{1}{b-a}\int_\Omega |\alpha(t)|^2 dt - \left|\frac{1}{b-a}\int_\Omega \alpha(t)\, dt\right|^2\right]^{\frac{1}{2}},$$

provided $\alpha \in L_2([a,b],\mathbb{K})$, $f \in L_2([a,b],H)$ and

$$\int_\Omega \operatorname{Re}\langle X - f(t), f(t) - x\rangle \, dt \geq 0 \tag{3.4}$$

or, equivalently,

$$\int_\Omega \left\| f(t) - \frac{X+x}{2} \right\|^2 dt \leq \frac{1}{4}\|X-x\|^2 . \tag{3.5}$$

We observe that, in practical applications the conditions (3.4) and (3.5) may be replaced with the more convenient sufficient conditions

$$\operatorname{Re}\langle X - f(t), f(t) - x\rangle \geq 0 \quad \text{for a.e.} \quad t \in [a,b], \tag{3.6}$$

or, equivalently,

$$\left\| f(t) - \frac{X+x}{2} \right\| \leq \frac{1}{2}\|X-x\| \quad \text{for a.e.} \quad t \in [a,b]. \tag{3.7}$$

4. Inequalities for the finite Fourier transform

Let $(H;\langle\cdot,\cdot\rangle)$ be a real or complex Hilbert space and $g:[a,b]\to H$ be a Bochner integrable function on $[a,b]$. Define its *finite Fourier transform* by

$$\mathcal{F}(g)(t) := \int_a^b e^{-2\pi i t s} g(s)\, ds. \tag{4.1}$$

We also consider the *exponential mean* of two complex numbers (see also [6])

$$E(z,w) := \begin{cases} \dfrac{e^z - e^w}{z-w} & \text{if } z \neq w \\ \exp(w) & \text{if } z = w \end{cases}, \quad z,w \in \mathbb{C}.$$

The following result may be stated.

THEOREM 7. *Assume that $f \in L_2([a,b],H)$ satisfies either (3.4) or, equivalently, (3.5). Then we have the inequality*

$$\begin{aligned} &\left\| \mathcal{F}(f)(t) - E(-2\pi i t a, -2\pi i t b)\int_a^b f(s)\, ds \right\| \\ &\leq \frac{1}{2}\|X-x\| \left[1 - \frac{\sin^2[\pi t(b-a)]}{(b-a)^2\pi^2 t^2}\right]^{\frac{1}{2}} (b-a) \\ &\leq \frac{b-a}{2}\|X-x\| \end{aligned} \tag{4.2}$$

for each $t \in [a,b]$ $(t \neq 0)$.

Proof. We apply the pre-Grüss inequality (3.3) to get

(4.3)

$$\left\| \frac{1}{b-a} \int_a^b e^{-2\pi its} f(s)\, ds - \frac{1}{b-a} \int_a^b e^{-2\pi its} ds \cdot \frac{1}{b-a} \int_a^b f(s)\, ds \right\|$$

$$\leq \frac{1}{2} \|X - x\| \left[\frac{1}{b-a} \int_a^b \left| e^{-2\pi its} \right|^2 ds - \left| \frac{1}{b-a} \int_a^b e^{-2\pi its} ds \right|^2 \right]^{\frac{1}{2}}.$$

However,

$$\int_a^b e^{-2\pi its} ds = (b-a)\, E(-2\pi ita, -2\pi itb),$$

$$\left| e^{-2\pi its} \right|^2 = 1,$$

$$\int_a^b e^{2\pi its} ds = \frac{e^{2\pi itb} - e^{2\pi ita}}{2\pi it},$$

and

$$\begin{aligned} \left| \int_a^b e^{-2\pi its} ds \right|^2 &= \left| \int_a^b e^{2\pi its} ds \right|^2 \\ &= \frac{1}{4\pi^2 t^2} \left[\left| e^{2\pi itb} \right|^2 - 2 \operatorname{Re}\left[e^{2\pi itb} \cdot e^{-2\pi ita} \right] + \left| e^{2\pi ita} \right|^2 \right] \\ &= \frac{1}{2\pi^2 t^2} \left[1 - \cos\left[2\pi t (b-a) \right] \right] \\ &= \frac{\sin^2 \left[\pi t (b-a) \right]}{\pi^2 t^2}. \end{aligned}$$

Utilising (4.3), we deduce the desired inequality (4.2). □

REMARK 5. The above inequality (4.2) extends for vector-valued functions the corresponding result from [6].

From Theorem 5 for $\Omega = [a,b]$ and $\rho(t) = \frac{1}{b-a}$, we may deduce the following inequality that will be utilised in Theorem 8 to point out another type of inequality for Fourier transforms:

$$\text{(4.4)} \quad \left| \frac{1}{b-a} \int_a^b \langle f(t)\, g(t) \rangle\, dt - \left\langle \frac{1}{b-a} \int_a^b f(t)\, dt, \frac{1}{b-a} \int_a^b g(t)\, dt \right\rangle \right|$$

$$\leq \left(\frac{1}{4}\|X-x\|^2 - \frac{1}{b-a}\int_a^b \operatorname{Re}\langle X-f(t), f(t)-x\rangle\, dt\right)^{\frac{1}{2}}$$
$$\times \left(\frac{1}{b-a}\int_a^b \|g(t)\|^2\, dt - \left\|\frac{1}{b-a}\int_a^b g(t)\, dt\right\|^2\right)^{\frac{1}{2}}$$
$$\leq \frac{1}{2}\|X-x\| \left[\frac{1}{b-a}\int_a^b \|g(t)\|^2\, dt - \left\|\frac{1}{b-a}\int_a^b g(t)\, dt\right\|^2\right]^{\frac{1}{2}},$$

provided (3.4) or (3.5) holds true.

In the following we use the notation $\langle f, g\rangle$ for the function $\ell : [a,b] \to \mathbb{K}$, $\ell(t) := \langle f(t), g(t)\rangle$, $t \in [a,b]$, where $f, g \in L_2([a,b], H)$.

The following result may be stated as well.

THEOREM 8. *Let $f, h \in L_2([a,b], H)$. If f satisfies either* (3.4) *or, equivalently,* (3.5), *then we have the inequality:*

$$(4.5)\quad \left\|\mathcal{F}(\langle f, h\rangle)(t) - \left\langle \frac{1}{b-a}\int_a^b f(s)\, ds, \tilde{\mathcal{F}}(h)(t)\right\rangle\right\|$$
$$\leq \frac{1}{2}\|X-x\| \left[\frac{1}{b-a}\int_\Omega \|h(s)\|^2\, ds - \left\|\frac{1}{b-a}\tilde{\mathcal{F}}(h)(t)\right\|^2\right]^{\frac{1}{2}} (b-a),$$

for any $t \in [a,b]$, where

$$\tilde{\mathcal{F}}(h)(t) := \int_a^b e^{2\pi its} h(s)\, ds,$$

is the inverse Fourier transform.

Proof. If we apply the inequality (4.4) to $g(s) = e^{2\pi its}h(s)$, $t \in [a,b]$, then we get

$$\left|\frac{1}{b-a}\int_a^b e^{-2\pi its}\langle f(s)\, h(s)\rangle\, ds\right.$$
$$\left. - \left\langle \frac{1}{b-a}\int_a^b f(s)\, ds, \frac{1}{b-a}\int_a^b e^{2\pi its} h(s)\, ds\right\rangle\right|$$
$$\leq \frac{1}{2}\|X-x\| \left[\frac{1}{b-a}\int_\Omega \|h(s)\|^2\, ds - \left\|\frac{1}{b-a}\int_a^b e^{2\pi its} h(s)\, ds\right\|^2\right]^{\frac{1}{2}},$$

which is obviously equivalent to (4.5). □

5. Inequalities for particular vector-valued functions

Let H be a real or complex Hilbert space and $\mathcal{L}(H)$ be the linear space of all linear and bounded operators acting on H. The norms of vectors in H and of operators in $\mathcal{L}(H)$ will be denoted by $||\cdot||$.

1. Choose $\Omega = [0,1]$, $\rho(t) = 1$ and $f(t) = e^{tA}z$ for $t \in \Omega$, A being an invertible bounded linear operator acting on H and z be fixed in H. Since, for each $t \in [0,1]$ one has

$$||e^{tA}z|| \leq e^{t||A||}||z|| \leq e^{||A||}||z||,$$

then it follows that

$$||e^{tA}z - \frac{1}{2}e^{||A||}z|| \leq e^{t||A||}||z|| + \frac{1}{2}e^{||A||}||z|| \leq \frac{3}{2}e^{||A||}||z||.$$

Let $X := 2e^{||A||}z$ and $x := -e^{||A||}z$. An application of inequalities (2.3) gives:

$$0 \leq \int_0^1 ||e^{tA}z||^2 dt - \left\| \int_0^1 e^{tA}z dt \right\|^2 \leq \frac{9}{4}e^{2||A||}||z||^2. \tag{5.1}$$

On the other hand

$$\int_0^1 e^{tA}z dt = A^{-1}(e^A - I)z,$$

and so in view of (5.1) we get

$$\begin{aligned} ||A^{-1}(e^A - I)z||^2 &\leq \int_0^1 ||e^{tA}z||^2 dt \\ &\leq ||A^{-1}(e^A - I)||^2||z||^2 + \frac{9}{4}e^{2||A||}||z||^2, \end{aligned}$$

and, moreover:

$$\sup_{||z||\leq 1} \int_0^1 ||e^{tA}x||^2 dt \leq ||A^{-1}(e^A - I)||^2 + \frac{9}{4}e^{2||A||}.$$

2. Let Ω, ρ and z be as above. Consider $f(t) = e^{(1-t)B}(B-A)e^{tB}z$ for each $t \in \Omega$, where A and B belong to $\mathcal{L}(H)$. After a simple calculation, [1], we obtain:

$$\int_0^1 f(t)dt = \frac{1}{2}\left[e^B - e^A\right]z.$$

On the other hand it is clear that $||f(t)|| \leq g(t)$, where

$$g(t) := e^{(1-t)||B||}||B - A||e^{t||A||}||z||.$$

Consider here only the case when $||A|| \geq ||B||$. In this case the map g is non-decreasing and so

$$||f(t)|| \leq g(1) = ||B - A||e^{||A||}||z||$$

for all $t \in \Omega$. The inequality (2.1) holds for

$$X := 2||B - A||e^{||A||}z \text{ and } x := -||B - A||e^{||A||}z.$$

From the inequality (2.3) it then follows that

$$\frac{1}{4}||(e^B - e^A)z||^2 \leq \int_0^1 ||e^{(1-t)B}(B - A)e^{tA}z||^2 dt$$
$$\leq \frac{1}{4}[||e^B - e^A||^2 + 9||B - A||^2 e^{2||A||}]||z||^2.$$

In particular, for $B = I + A$ and $||A|| \geq ||I + A||$ we get:

$$\left(e^2 + 2e - 3\right)||e^A z||^2 \leq 9e^{2||A||}||z||^2,$$

or equivalently

$$\sqrt{e^2 + 2e - 3}||e^A|| \leq 3e^{||A||}.$$

3. Let $\mathbf{T} = \{T(t)\}_{t\in\mathbb{R}}$ be a strongly continuous group of linear and bounded operators acting on a Hilbert space H and let $A : D(A) \subset H \to H$ be its infinitesimal generator. We suppose that $\mathbf{T}$ is exponentially stable, that is, there exist the positive constants N and ν such that $||T(t)z|| \leq Ne^{-\nu|t|}||z||$ for all $t \in \mathbb{R}$. Then it is well-known that A has an inverse in $\mathcal{L}(H)$. Consider $\Omega = \mathbb{R}$, $\rho(t) =: \nu e^{-\nu|t|}$ and $f(t) := e^{\nu|t|}T(t)z$ for a fixed $z \in D(A)$ and $t \in \mathbb{R}$. An application of the inequality (2.3) for $X := 2Nz$ and $x := -Nz$ gives the inequality:

$$0 \leq \int_{-\infty}^{\infty} e^{\nu|t|}||T(t)z||^2 dt \leq \frac{9N^2}{4\nu}||z||^2,$$

where the fact that

$$\int_{-\infty}^{\infty} T(t)zdt = A^{-1}T(t)z|_{-\infty}^{\infty} = 0$$

has been used. In particular, if A is a real or complex quadratic n-dimensional matrix and

$$\nu_0 := \sup\{\Re(\lambda) : \det(\lambda I_n - A) = 0\} < \nu < 0,$$

then there exist a positive constant N such that

$$\int_{-\infty}^{\infty} e^{\nu|t|}||e^{tA}||^2 dt \leq \frac{9N^2}{4\nu}.$$

Here n is a positive integer, I_n is the n-dimensional identity matrix and we can take

$$N := \sup\{e^{-\nu|t|}||e^{tA}|| : \quad t \in \mathbb{R}\}.$$

4. Let $\mathbf{S} = \{S(t)\}_{t\geq 0}$ be a strongly continuous semigroup of linear and bounded operators acting on a Hilbert space H and let $G : D(G) \subset H \to H$ be its infinitesimal generator. We suppose that $\mathbf{S}$ is exponentially stable, that is, there exist the positive constants K and α such that $||S(t)|| \leq Ke^{-\alpha t}$ for all $t \geq 0$. Then it is well-known that G has an inverse in $\mathcal{L}(H)$. Consider $\Omega = [0, \infty)$, $\rho(t) := \alpha e^{-\alpha t}$ and $f(t) := e^{\alpha t}S(t)z$ for fixed $z \in D(G)$ and $t \geq 0$. An application of the inequality (2.3) for $X := 2Kz$ and $x := -Kz$ yields:

$$||G^{-1}z||^2 \leq \frac{1}{\alpha}\int_0^{\infty} e^{\alpha t}||S(t)z||^2 dt \leq ||G^{-1}z||^2 + \frac{9N^2}{4\alpha^2}.$$

5. A densely defined linear operator A on a Hilbert space H is said to be sectorial if $(0, \infty)$ resides in the resolvent set of A and there exist $M > 0$ such that

$$(t+1)||R(t, A)|| \leq M \text{ for all } t > 0,$$

where $R(t, A) := (tI - A)^{-1}$ is the resolvent operator of A.

Consider $\Omega := [0, \infty)$, $\rho(t) := (t+1)^{-2}$ and $f(t) = (t+1)^2 R(t, A)^2 z$ for a fixed $z \in H$ and every $t \geq 0$. In order to find suitable X and x we remark that:

$$||f(t)|| \leq (t+1)^2||R(t, A)||||R(t, A)z|| \leq M^2||z||.$$

An application of the inequality (2.3) for $X := 2M^2 z$ and $x := -M^2 z$ yields:

$$||A^{-1}z||^2 \leq \int_0^{\infty}(t+1)^2||R(t, A)^2 z||^2 dt \leq ||A^{-1}z||^2 + \frac{9}{4}M^4||z||^2,$$

where the identity

$$\int_0^{\infty} \rho(t)f(t)dt = -R(t, A)z\Big|_0^{\infty} = A^{-1}z$$

was used.

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