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JOINT ASYMPTOTIC DISTRIBUTIONS OF SAMPLE AUTOCORRELATIONS FOR TIME SERIES OF MARTINGALE DIFFERENCES†

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ABSTRACT

It is well known fact for the *iid* data that the limiting standard errors of sample autocorrelations are all unity for all time lags and they are asymptotically independent for different lags (Brockwell and Davis, 1991). It is also usual practice in time series modeling that this fact continues to be valid for white noise series which is a sequence of uncorrelated random variables. This paper contradicts this usual practice for white noise. We consider a sequence of martingale differences which belongs to white noise time series and derive *exact* joint asymptotic distributions of sample autocorrelations. Some implications of the result are illustrated for conditionally heteroscedastic time series.

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1. INTRODUCTION

The sample autocorrelation function (ACF) for a given time series plays a crucial role in identifying an appropriate model for the data. To evaluate the adequacy of the model, the "residual" is recommended to be carefully examined. Here the term residual is defined as the difference of observations and the fitted values obtained after fitting an appropriate model. As discussed by Box and Pierce (1970), the ACF based on residuals can be used as a useful diagnostic tool. See also Hwang *et al.* (1994). For the case of *iid* data, it is a well known fact that

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the asymptotic variances of sample ACF's are all unity, and sample ACF's for different lags are asymptotically independent (Brockwell and Davis, 1991; Fuller 1996). Also, it has been an usual practice in time series modeling that this fact is blindly employed for any white noise time series. Here the term "white noise" refers to a sequence of uncorrelated random variates. In this short paper we provide a contradiction to this (blind) practice employed for any white noise. We consider a sequence of martingale differences (MD) with finite fourth moment. It is noted that MD belongs to white noise time series. Exact joint asymptotic distributions of sample autocorrelations from MD is derived. It is shown that asymptotic standard errors of sample ACF is far from unity especially for lower lags and they are asymptotically *dependent* for different lags. These findings are illustrated for conditionally heteroscadstic time series.

2. Asymptotic Distributions of Sample ACF's from MD

Consider the zero mean time series $\{\varepsilon_t\}$, and denote by F_{t-1} the σ-field generated by $\varepsilon_{t-1}, \varepsilon_{t-2}, \ldots$. It will be assumed that $\{\varepsilon_t\}$ forms a MD, *i.e.*, for all t,

$$E(\varepsilon_t | F_{t-1}) = 0. \tag{2.1}$$

It is further assumed that

(C1) $\{\varepsilon_t\}$ is strictly stationary and ergodic time series with variance σ_ε^2. Further, fourth order stationary moment exists, *i.e.*, $E(\varepsilon_t^4) < \infty$.

If $\{\varepsilon_t\}$ retains independence structure then $\{\varepsilon_t\}$ reduces to *iid* sequence. A typical example of $\{\varepsilon_t\}$ featuring non-iid structure is GARCH (generalized ARCH) time series. See, for instance, Engle (1982) and Bollerslev (1986).

Based on the sample (of size n) $\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n\}$, the sample ACF of lag k denoted by $\gamma_\varepsilon(k)$ is given by

$$\gamma_\varepsilon(k) = C_k / C_0, \ k = 0, 1, \ldots, l, \tag{2.2}$$

where C_k represents sample autocovariance of lag k defined by

$$C_k = \sum_{t=1}^{n-k} (\varepsilon_t - \bar{\varepsilon})(\varepsilon_{t+k} - \bar{\varepsilon})/n \tag{2.3}$$

with $\bar{\varepsilon}$ denoting the average of the sample. Fix l and define

$$T_n = \sqrt{n}(\gamma_\varepsilon(1), \ldots, \gamma_\varepsilon(l))'. \tag{2.4}$$

We now present asymptotic distribution of T_n.

THEOREM 2.1. *Under (C1), as $n \to \infty$, we have*

$$T_n \xrightarrow{d} N(0, \Sigma),$$

where $l \times l$ matrix Σ is given by, for $i, j = 1, \cdots, l$,

$$\Sigma_{ii} = E(\varepsilon_t^2 \varepsilon_{t-i}^2)/\sigma_\varepsilon^4$$

and

$$\Sigma_{ij} = E(\varepsilon_{t-i}\varepsilon_{t-j}\varepsilon_t^2)/\sigma_\varepsilon^4, \ i \neq j.$$

PROOF. The sample ACF for $\{\varepsilon_1, \ldots, \varepsilon_n\}$ is given by, for $k = 1, \ldots, l$,

$$\sqrt{n}\gamma_\varepsilon(k) = \frac{\frac{1}{\sqrt{n}} \sum_{t=1}^{n-k} (\varepsilon_t - \bar{\varepsilon})(\varepsilon_{t+k} - \bar{\varepsilon})}{\frac{1}{n} \sum_{t=1}^{n} (\varepsilon_t - \bar{\varepsilon})^2}.$$

Since $\sum_{t=1}^n (\varepsilon_t - \bar{\varepsilon})^2/n$ and $\bar{\varepsilon}$ converge in probability to σ_ε^2 and zero respectively, one can write

$$\begin{aligned} \sqrt{n}\gamma_\varepsilon(k) &= \sigma_\varepsilon^{-2} \frac{1}{\sqrt{n}} \sum_{t=1}^{n-k} \varepsilon_t \varepsilon_{t+k} + o_p(1) \\ &= \sigma_\varepsilon^{-2} \frac{1}{\sqrt{n}} \sum_{t=1}^{n} \varepsilon_t \varepsilon_{t-k} + o_p(1), \end{aligned}$$

where in the second equality the summation actually runs from $t = k + 1$ to n. However we will use $\sum_{t=1}^n$ without affecting the asymptotics. Under (C1), it can be shown that $E(\varepsilon_t \varepsilon_{t-k} | F_{t-1}) = 0$, for $k \geq 1$ and $E(\varepsilon_t^2 \varepsilon_{t-k}^2) < \infty$, thus by using the central limit theorem for stationary martingale differences (Billingsley, 1961), we have

$$\sqrt{n}\gamma_\varepsilon(k) \xrightarrow{d} N(0, E(\varepsilon_t^2 \varepsilon_{t-k}^2)/\sigma_\varepsilon^4). \tag{2.5}$$

Now, we derive the joint asymptotic distribution of $T_n = \sqrt{n}(\gamma_\varepsilon(1), \ldots, \gamma_\varepsilon(l))'$. Consider for $\mathbf{a} = (a_1, a_2, \ldots, a_l)'$, a given real vector, it can be verified via some algebra

$$\sqrt{n}(a_1\gamma_\varepsilon(1) + \cdots + a_l\gamma_\varepsilon(l)) = \sigma_\varepsilon^{-2} \frac{1}{\sqrt{n}} \sum_{k=1}^{l} a_k \sum_{t=1}^{n-k} \varepsilon_t \varepsilon_{t+k} + o_p(1).$$

Also, the RHS of the above equation can be rearranged as

$$\frac{1}{\sqrt{n}}\sum_{k=1}^{l} a_k \sum_{t=1}^{n-k} \varepsilon_t\varepsilon_{t+k} = \frac{1}{\sqrt{n}}\sum_{t=1}^{n} \varepsilon_t \sum_{k=1}^{l} a_k\varepsilon_{t-k} + o_p(1).$$

Note that $\{\varepsilon_t \sum_{k=1}^{l} a_k\varepsilon_{t-k}\}$ is again a sequence of zero mean martingale differences. We then have

$$\frac{1}{\sqrt{n}}\sum_{t=1}^{n} \varepsilon_t \sum_{k=1}^{l} a_k\varepsilon_{t-k} \xrightarrow{d} N(0, \mathbf{a}'\Gamma\mathbf{a}), \tag{2.6}$$

where $\Gamma = (\Gamma_{ij})$ is the $l \times l$ matrix with

$$\Gamma_{ij} = E(\varepsilon_{t-i}\varepsilon_{t-j}\varepsilon_t^2). \tag{2.7}$$

Therefore, using the Cramer-Wold device (Serfling, 1980), we finally obtain

$$T_n = \sqrt{n}(\gamma_\varepsilon(1), \ldots, \gamma_\varepsilon(l))' \xrightarrow{d} N(0, \Sigma),$$

where Σ is the $l \times l$ matrix with $\Sigma_{ij} = \sigma_\varepsilon^{-4}\Gamma_{ij}$, completing the proof. □

REMARK 2.1. For the special case when $\{\varepsilon_t\}$ is *iid*, Σ_{ii} turns out to be unity and $\Sigma_{ij}, i \neq j$, reduces to zero. For the diagonal term Σ_{ii}, a connection to the squared process $\{\varepsilon_t^2\}$ can be made. It can be verified that

$$Corr(\varepsilon_t^2, \varepsilon_{t-i}^2) = (\Sigma_{ii} - 1)\left[\frac{\{E(\varepsilon_t^2)\}^2}{Var(\varepsilon_t^2)}\right]. \tag{2.8}$$

Consequently, $\Sigma_{ii} > 1$, $\Sigma_{ii} = 1$ and $\Sigma_{ii} < 1$ according to $Corr(\varepsilon_t^2, \varepsilon_{t-i}^2)$ is positive, zero and negative, respectively. For the standard GARCH(p, q) (Bollerslev, 1986), it is seen that the squared process follows ARMA and thus facilitating the identification of Σ_{ii}.

3. APPLICATIONS TO GARCH PROCESSES

Consider the following conditionally heteroscedastic time series $\{\varepsilon_t\}$ specified by

$$\varepsilon_t = \sqrt{h_t} \cdot e_t, \tag{3.1}$$

where $\{e_t\}$ is a sequence of *iid* random variates with mean zero and unit variance. Here, h_t is reserved for the conditional variance ε_t of given the past, *i.e.*,

$$h_t = Var(\varepsilon_t | F_{t-1}). \tag{3.2}$$

We take, for simplicity, ARCH(1) structure for h_t. Engle's ARCH(1) model is specified by

$$h_t = \alpha_0 + \alpha_1 \varepsilon_{t-1}^2. \tag{3.3}$$

For checking (C1), we note that ARCH(1) is stationary if $\alpha_1 < 1$ and $E(\varepsilon_t^4) < \infty$ if $3\alpha_1^2 < 1$ when $\{e_t\}$ is Gaussian. Utilizing the fact that the squared process $\{\varepsilon_t^2\}$ follows the standard AR(1) model with autoregressive coefficient α_1, the asymptotic variance of sample ACF of lag i is shown to be

$$\Sigma_{ii} = 1 + \Big(\frac{2\alpha^i}{1 - 3\alpha_1^2}\Big). \tag{3.4}$$

It is noted for ARCH(1) that Σ_{ii} converges exponentially to 1 as the lag increases and hence Σ_{ii} for lower lags rather than higher lags should be carefully handled for model diagnostic purposes. This argument can be carried over to the GARCH time series.

Consider the following GARCH(p, q) process given by

$$h_t - \gamma_1 h_{t-1} - \cdots - \gamma_p h_{t-p} = \alpha_0 + \alpha_1 \varepsilon_{t-1}^2 + \cdots + \alpha_q \varepsilon_{t-q}^2. \tag{3.5}$$

The order q is taken to be greater than or equal to p, without loss of generality. Define $\eta_t = \varepsilon_t^2 - h_t$. It is noted that $\{\eta_t\}$ forms a sequence of martingale differences with respect to the increasing σ-field $\{F_t\}$. Then, (3.5) can be written in terms of ARMA(q, p) as

$$\varepsilon_t^2 - \sum_{i=1}^{q} (\alpha_i + \gamma_i)\varepsilon_{t-i}^2 = \alpha_0 + \eta_t - \sum_{j=1}^{p} \gamma_j \eta_{t-j}. \tag{3.6}$$

Provided $E\eta_t^2 < \infty$, or equivalently, $E\varepsilon_t^4 < \infty$, (3.6) exhibits a stationary ARMA(q, p) process. This notion is quite useful in determing the GARCH-order, say p and q, via a stationary ARMA(q, p) theory. Consequently, it has been a common practice in GARCH modeling for (3.5) to investigate the sample ACF of the residual $\hat{\eta}_t$ after fitting the model (3.6) assuming that the asymptotic variance-covariance matrix, say, Δ of the sample ACF's of $\hat{\eta}_t$ be identity matrix. However, our theorem applied to the MD $\{\eta_t = \varepsilon_t^2 - h_t\}$ tells us that Δ may not be identity matrix and thus Δ can be refined according to the Theorem 2.1.

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