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INTUITIONISTIC FUZZY WEAK CONGRUENCES ON A SEMIRING

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Abstract

We introduce the concept of intuitionistic fuzzy weak congruence on a semiring and obtain the relation between intuitionistic fuzzy weak congruence and intuitionistic fuzzy ideal of a semiring. Also we define and investigate intuitionistic fuzzy quotient semiring of a semiring over an intuitionistic fuzzy ideal or over an intuitionistic fuzzy weak congruence.

Key words : semiring, (intuitionistic fuzzy weak) congruence, (intuitionistic fuzzy) ideal, (intuitionistic fuzzy) k-ideal, intuitionistic fuzzy quotient semiring

0. Introduction

The concept of fuzzy set was formulated by Zadeh [24]. Since then, there has been a remarkable growth of fuzzy set theory. The notion of fuzzy relation on a set was defined by Zadeh [25]. Some researchers [8,19,21-23] applied the concept of fuzzy sets to congruence theory. In particular, Dutta and Biswas [8] investigated fuzzy congruence and quotient semiring of a semiring.

In 1986, Atanassov [1] introduced the notion of intuitionistic fuzzy sets as the generalization of fuzzy sets. After that time, Çoker [6], Lee and Lee [20], and Hur and his colleagues [12] applied the notion of intuitionistic fuzzy sets to topology. Also, several researchers [2,3,9-11,14] applied the notion of intuitionistic fuzzy sets to algebra. In particular, Bustince and Burillo [5], and Deschrijver and Kerre [7] applied the concept of intuitionistic fuzzy sets to relation. Also, Hur and his colleagues [15] investigated several properties of intuitionistic fuzzy equivalence relations. Moreover, Hur and his colleagues [16,17,18] introduced the notion of intuitionistic fuzzy congruence on a lattice, on a semigroup and on a near-ring module, respectively, and studied some of their properties.

In this paper, we introduce the concept of intuitionistic fuzzy weak congruence on a semiring. And we obtain the relation between intuitionistic fuzzy weak congruence and intuitionistic fuzzy ideal of a semiring. Also, we define and investigate intuitionistic fuzzy quotient semiring of a semiring over an intuitionistic fuzzy ideal or over an intuitionistic fuzzy weak congruence.

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1. Preliminaries

We recall some definitions and two results that are used in this paper.

For sets X, Y and $Z, f = (f_1, f_2) : X \to Y \times Z$ is called a *complex mapping* if $f_1 : X \to Y$ and $f_2 : X \to Z$ are mappings.

Throughout this paper, we will denote the unit interval $[0,1]$ as I and for any ordinary relation R on a set X, we will denote the characteristic mapping of R as χ_R.

Definition 1.1[1,6]. Let X be a nonempty set. A complex mapping $A = (\mu_A, \nu_A) : X \to I \times I$ is called an *intuitionistic fuzzy set* (in short, *IFS*) in X if for each $x \in X$ $\mu_A(x) + \nu_A(x) \leq 1$, where the mappings $\mu_A : X \to I$ and $\nu_A : X \to I$ denote the degree of membership (namely $\mu_A(x)$) and the degree of nonmembership (namely $\nu_A(x)$) of each $x \in X$ to A, respectively. In particular, $0_\sim$ and $1_\sim$ denote the *intuitionistic fuzzy empty set* and the *intuitionistic fuzzy whole set* in X defined by $0_\sim(x) = (0,1)$ and $1_\sim(x) = (1,0)$ for each $x \in X$, respectively.

We will denote the set of all IFSs in X as IFS(X).

Definitions 1.2[1]. Let X be a nonempty set and let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be IFSs in X. Then

(1) $A \subset B$ iff $\mu_A \leq \mu_B$ and $\nu_A \geq \nu_B$.

(2) $A = B$ iff $A \subset B$ and $B \subset A$.

(3) $A^c = (\nu_A, \mu_A)$.

(4) $A \cap B = (\mu_A \wedge \mu_B, \nu_A \vee \nu_B)$.

(5) $A \cup B = (\mu_A \vee \mu_B, \nu_A \wedge \nu_B)$.

Definition 1.3[6]. Let $\{A_i\}_{i \in J}$ be an arbitrary family of IFSs in X, where $A_i = (\mu_{A_i}, \nu_{A_i})$ for each $i \in J$. Then

(a) $\bigcap A_i = (\bigwedge \mu_{A_i}, \bigvee \nu_{A_i})$.

(b) $\bigcup A_i = (\bigvee \mu_{A_i}, \bigwedge \nu_{A_i})$.

Definition 1.4[9]. Let A be an IFS in a set X and let $\lambda, \mu \in I$ with $\lambda + \mu \leq 1$. Then the set $A^{(\lambda,\mu)} = \{x \in X : \mu_A(x) \geq \lambda$ and $\nu_A(x) \leq \mu\}$ is called a (λ, μ)-*level subset* of A.

Result 1.A[11, Proposition 2.2]. Let A be an IFS in a set X and let $(\lambda_1, \mu_1), (\lambda_2, \mu_2) \in \text{Im}A$. If $\lambda_1 \leq \lambda_2$ and $\mu_1 \geq \mu_2$, then $A^{(\lambda_2,\mu_2)} \subset A^{(\lambda_1,\mu_1)}$.

Definition 1.5[5,7]. Let X be a set. Then a complex mapping $R = (\mu_R, \ \nu_R) : X \times X \to I \times I$ is called an *intuitionistic fuzzy relation* (in short, *IFR*) on X if $\mu_R(x,y) + \nu_R(x,y) \leq 1$ for each $(x,y) \in X \times X$, i.e., $R \in$ IFS $(X \times X)$.

We will denote the set of all IFRs on a set X as IFR(X).

Definition 1.6[7] Let X be a set and let $R, Q \in$ IFR(X). Then the *composition* of R and Q, $Q \circ R$, is defined as follows : for any $x, y \in X$,

$$\mu_{Q \circ R}(x,y) = \bigvee_{z \in X}[\mu_R(x,z) \wedge \mu_Q(z,y)]$$

and

$$\nu_{Q \circ R}(x,y) = \bigwedge_{z \in X}[\nu_R(x,z) \vee \nu_Q(z,y)].$$

Definition 1.7[5,7]. An Intuitionistic fuzzy relation R on a set X is called an *intuitionistic fuzzy equivalence relation* (in short, IFER) on X if it satisfies the following conditions :

(i) it is *intuitionistic fuzzy reflexive*, i.e., $R(x,x) = (1,0)$ for each $x \in X$.

(ii) it is *intuitionistic fuzzy symmetric*, i.e., $R(x,y) = R(y,x)$ for any $x, y \in X$.

(iii) it is *intuitionistic fuzzy transitive*, i.e., $R \circ R \subset R$.

We will denote the set of all IFERs on X as IFE(X).

Let R be an intuitionistic fuzzy equivalence relation on a set X and let $a \in X$. We define a complex mapping $Ra : X \to I \times I$ as follows : for each $x \in X$

$$Ra(x) = R(a,x).$$

Then clearly $Ra \in$ IFS(X). The intuitionistic fuzzy set Ra in X is called an *intuitionistic fuzzy equivalence class* of R containing $a \in X$. The set $\{Ra : a \in X\}$ is called the *intuitionistic fuzzy quotient set of* X *by* R and denoted by X/R.

Definition 1.8[17]. An IFR R on a groupoid S is said to be:

(1) *intuitionistic fuzzy left compatible* if $\mu_R(x,y) \leq \mu_R(zx,zy)$ and $\nu_R(x,y) \geq \nu_R(zx,zy)$, for any $x, y, z \in S$.

(2) *intuitionistic fuzzy right compatible* if $\mu_R(x,y) \leq \mu_R(xz,yz)$ and $\nu_R(x,y) \geq \nu_R(xz,yz)$, for any $x, y, z \in S$.

(3) *intuitionistic fuzzy compatible* if $\mu_R(x,y) \wedge \mu_R(z,t) \leq \mu_R(xz,yt)$ and $\nu_R(x,y) \vee \nu_R(z,t) \geq \nu_R(xz,yt)$, for any $x, y, z, t \in S$.

Definition 1.9[17]. An IFER R on a groupoid S is called an:

(1) *intuitionistic fuzzy left congruence* (in short, $IFLC$) if it is intuitionistic fuzzy left compatible.

(2) *intuitionistic fuzzy right congruence* (in short, $IFRC$) if it is intuitionistic fuzzy right compatible.

(3) *intuitionistic fuzzy congruence* (in short, IFC) if it is intuitionistic fuzzy compatible.

We will denote the set of all IFCs [resp. IFLCs and IFRCs] on a groupoid S as IFC(S) [resp. IFLC(S) and IFRC(S)].

Result 1.B[17, Theorem 2.8]. Let R be a relation on a groupoid S. Then R is a congruence on S if and only if $(\chi_R, \chi_{R^c}) \in$ IFC(S).

Let R be an intuitionistic fuzzy congruence on a semigroup S and let $a \in S$. The intuitionistic fuzzy set Ra in S is called an *intuitionistic fuzzy congruence class of* R *containing* $a \in S$ and we will denote the set of all intuitionistic fuzzy congruence classes of R as S/R.

2. Intuitionistic fuzzy weak congruences

Definition 2.1. Let S be a set and let $0_\sim \neq R \in$ IFR(X). Then R is called an *intuitionistic fuzzy weak equivalence relation* (in short, *IFWER*) on X if

(1) R is *intuitionistic fuzzy weakly reflexive*,
i.e., for each $x \in X$,
$R(x,x) = (\bigvee_{y,z\in X} \mu_R(y,z),\ \bigwedge_{y,z\in X} \nu_R(y,z))$.
(2) R is *intuitionistic fuzzy symmetric*,
i.e., $R(x,y) = R(y,x)$ for any $x, y \in X$.
(3) R is *intuitionistic fuzzy transitive*,
i.e., $R \circ R \subset R$.

We will denote the set of all IFWERs on X as $\text{IFE}_W(X)$.

Proposition 2.2. Let X be a set and let $\{R_\alpha\}_{\alpha\in\Gamma}$ be a nonempty set of intuitionistic fuzzy weakly reflexive relation on X. Then $\bigcap_{\alpha\in\Gamma} R_\alpha$ is intuitionistic fuzzy weakly reflexive.

Proof. Let $R = \bigcap_{\alpha\in\Gamma} R_\alpha$ and let $x \in X$. Then

$$\begin{aligned}\mu_R(x,x) &= \textstyle\bigwedge_{\alpha\in\Gamma} \mu_{R_\alpha}(x,x)\\ &= \textstyle\bigwedge_{\alpha\in\Gamma}[\bigvee_{y,z\in X} \mu_{R_\alpha}(y,z)]\\ &= \textstyle\bigvee_{y,z\in X}[\bigwedge_{\alpha\in\Gamma} \mu_{R_\alpha}(y,z)]\\ &= \textstyle\bigvee_{y,z\in X} \mu_R(y,z)\end{aligned}$$

and

$$\begin{aligned}\nu_R(x,x) &= \textstyle\bigvee_{\alpha\in\Gamma} \nu_{R_\alpha}(x,x)\\ &= \textstyle\bigvee_{\alpha\in\Gamma}[\bigwedge_{y,z\in X} \nu_{R_\alpha}(y,z)]\\ &= \textstyle\bigwedge_{y,z\in X}[\bigvee_{\alpha\in\Gamma} \nu_{R_\alpha}(y,z)]\\ &= \textstyle\bigwedge_{y,z\in X} \nu_R(y,z).\end{aligned}$$

Hence R is intuitionistic fuzzy weakly reflexive. □

Definition 2.3[4]. A *semiring* is defined by an algebra $(S, +, \cdot)$ such that $(S, +)$ and $(S, \cdot)$ are semigroup connected by $a(b+c) = ab + ac$ and $(b+c)a = ba + ca$ for any $a, b, c \in S$. A semiring may have an identity 1 defined by $1a = a1 = a$ and a zero (which is an absorbing zero also) 0 defined by $a + 0 = 0 + a = a$ and $a0 = 0a = 0$ for each $a \in S$. A semiring S is said to be *additively commutative* if $(S, +)$ is commutative and *multiplicatively commutative* if $(S, \cdot)$ is commutative.

Definition 2.4. Let S be a semiring and let $R \in \text{IFR}(S)$. Then R is said to be:

(1) *intuitionistic fuzzy left compatible* if for any $a, b, t \in S$,
$\mu_R(t+a, t+b) \geq \mu_R(a,b)$,
$\nu_R(t+a, t+b) \leq \nu_R(a,b)$
and
$\mu_R(ta, tb) \geq \mu_R(a,b)$,
$\nu_R(ta, tb) \leq \nu_R(a,b)$.

(2) *intuitionistic fuzzy right compatible* if for any $a, b, t \in S$,
$\mu_R(a+t, b+t) \geq \mu_R(a,b)$,
$\nu_R(a+t, b+t) \leq \nu_R(a,b)$
and
$\mu_R(at, bt) \geq \mu_R(a,b)$,
$\nu_R(at, bt) \leq \nu_R(a,b)$.

(3) *intuitionistic fuzzy compatible* if it is both intuitionistic fuzzy left compatible and intuitionistic fuzzy right compatible.

The following is the immediate result of Definitions 2.3 and 2.4.

Proposition 2.5. Let S be a semiring and let $R \in \text{IFR}(S)$. Then R is intuitionistic fuzzy compatible if and only if for any $a, b, c, d \in S$,
$\mu_R(a+c, b+d) \geq \mu_R(a,b) \wedge \mu_R(c,d)$,
$\nu_R(a+c, b+d) \leq \nu_R(a,b) \vee \nu_R(c,d)$
and
$\mu_R(ac, bd) \geq \mu_R(a,b) \wedge \mu_R(c,d)$,
$\nu_R(ac, bd) \leq \nu_R(a,b) \vee \nu_R(c,d)$.

Definition 2.6. Let S be a semiring and let $R \in \text{IFR}(S)$. Then R is called:

(1) an *intuitionistic fuzzy congruence* (in short, *IFC*) on S if $R \in \text{IFE}(S)$ and R is intuitionistic fuzzy compatible.

(2) an *intuitionistic fuzzy weak congruence* (in short, *IFWC*) on S if $R \in \text{IFE}_W(S)$ and R is intuitionistic fuzzy compatible.

We will denote the set of all IFCs [resp. IFWCs on S as $\text{IFC}(S)$ [resp. $\text{IFC}_W(S)$]. It is clear that $\text{IFC}(S) \subset \text{IFC}_W(S)$.

Example 2.7. Let $\mathbf{N}$ be the additively commutative semiring of all nonnegative integers with respect to the usual addition and multiplication. Then $\mathbf{N}$ contains zero which is absorbing. We define a complex mapping $R = (\mu_R, \nu_R) : \mathbf{N} \times \mathbf{N} \to I \times I$ as follows: for any $x, y \in \mathbf{N}$,

$$R(x,y) = \begin{cases} (1,0) & \text{if } x = y,\\ (0.5, 0.4) & \text{if } x \neq y \text{ and both } x, y \text{ are}\\ & \text{even or both } x, y \text{ are odd,}\\ (0,1) & \text{otherwise.}\end{cases}$$

Then it can be easily verified $R \in \text{IFC}(\mathbf{N})$.

The following is the immediate result of Propositions 2.2 and 2.5 and Definition 2.6.

Proposition 2.8. Let S be a semiring and let $\{R_\alpha\}_{\alpha\in\Gamma}$ be a nonempty subset of $\text{IFC}(S)$ [resp. $\text{IFC}_W(S)$]. Then $\bigcap_{\alpha\in\Gamma} R_\alpha \in \text{IFC}(S)$ [resp. $\text{IFC}_W(S)$].

The following is the similar result as Result 1.B.

Theorem 2.9. Let R be a relation on a semiring S. Then R is a congruence on S if and only if $(\chi_R, \chi_{R^c}) \in \text{IFC}(S) \cap \text{IFC}_W(S)$.

Hur and his colleagues in Proposition 2.13 of [17] proved that if R is an IFC on a groupoid S, then for each

$(\lambda, \mu) \in I \times I$, $R^{(\lambda,\mu)}$ is a congruence on S. But our definition of intuitionistic fuzzy weak reflexivity enables as to establish both necessary and sufficient condition of the Theorem which is as follows.

Theorem 2.10. Let S be a semiring and let $R \in \text{IFR}(S)$. Then $R \in \text{IFC}_W(S)$ if and only if $R^{(\lambda,\mu)}$ is a congruence on S for each $(\lambda, \mu) \in \text{Im}R$.

Definition 2.11. Let A be a nonempty intuitionistic fuzzy set in a semiring S. Then A is called an *intuitionistic fuzzy ideal* (in short, *IFI*) of S if it satisfies the following conditions: For any $x, y \in S$,

(i) $\mu_A(x+y) \geq \mu_A(x) \wedge \mu_A(y)$ and $\nu_A(x+y) \leq \nu_A(x) \vee \nu_A(y)$,

(ii) $\mu_A(xy) \geq \mu_A(y)$ and $\nu_A(xy) \leq \nu_A(y)$,

(iii) $\mu_A(xy) \geq \mu_A(x)$ and $\nu_A(xy) \leq \nu_A(x)$.

It is clear that $\mu_A(0) \geq \mu_A(x)$ and $\nu_A(0) \leq \nu_A(x)$ for each $x \in S$. We will denote the set of all IFI_S of S as $\text{IFI}(S)$.

A *k-ideal* J of a semiring S is an ideal such that if $a \in J$ and $x \in S$ and $x + a$ or $a + x \in J$, then $x \in J$ (See [4]).

Definition 2.12. Let S be a semiring and let $A \in \text{IFI}(S)$. Then A is called an *intuitionistic fuzzy k-ideal* (in short, *IFKI*) of S if for any $x, y \in S$,

$$\mu_A(x) \geq [\mu_A(x+y) \vee \mu_A(y+x)] \wedge \mu_A(y)$$

and

$$\nu_A(x) \leq [\nu_A(x+y) \wedge \nu_A(y+x)] \vee \nu_A(y).$$

If S is additively commutative, then the condition reduces to

$$\mu_A(x) \geq \mu_A(x+y) \wedge \mu_A(y)$$

and

$$\nu_A(x) \leq \nu_A(x+y) \vee \nu_A(y)$$

for any $x, y \in S$.

We will denote the set of all IFKIs of S as $\text{IFKI}(S)$.

Proposition 2.13. Let S be a semiring and let $0_\sim \neq A \in \text{IFS}(X)$. Then $A \in \text{IFKI}(S)$ [resp. $\text{IFI}(S)$] if and only if $A^{(\lambda,\mu)}$ is a k-ideal [resp. an ideal] of S for each $(\lambda, \mu) \in \text{Im}A$.

Proof. (i) $(\Rightarrow)$: Suppose $A \in \text{IFI}(S)$ and let $(\lambda, \mu) \in \text{Im}A$. Let $a, b \in A^{(\lambda,\mu)}$ and let $x \in S$. Then $\mu_A(a) \geq \lambda$, $\nu_A(a) \leq \mu$ and $\mu_A(b) \geq \lambda$, $\nu_A(b) \leq \mu$. Thus

$$\mu_A(a+b) \geq \mu_a(a) \wedge \mu_A(b) \geq \lambda \wedge \lambda = \lambda$$

and

$$\nu_A(a+b) \leq \nu_a(a) \vee \nu_A(b) \leq \mu \vee \mu = \mu.$$

So $a + b \in A^{(\lambda,\mu)}$. On the other hand, $\mu_A(xa) \geq \mu_a(a) \geq \lambda$ and $\nu_A(xa) \leq \nu_a(a) \leq \mu$. Thus $xa \in A^{(\lambda,\mu)}$. Similarly, we have $ax \in A^{(\lambda,\mu)}$. Hence $A^{(\lambda,\mu)}$ is an ideal of S.

$(\Leftarrow)$: Suppose $A^{(\lambda,\mu)}$ is an ideal of S for each $(\lambda, \mu) \in \text{Im}A$. For any $x, y \in S$, let $A(x) = (\lambda_1, \mu_1)$ and $A(y) = (\lambda_2, \mu_2)$ such that $\lambda_1 < \lambda_2$ and $\mu_1 > \mu_2$. Then clearly $x \in A^{(\lambda_1,\mu_1)}$ and $y \in A^{(\lambda_2,\mu_2)}$. Since $A^{(\lambda_2,\mu_2)} \subset A^{(\lambda_1,\mu_1)}$, $y \in A^{(\lambda_1,\mu_1)}$. Since $A^{(\lambda_1,\mu_1)}$ is an ideal of S, $x + y \in A^{(\lambda_1,\mu_1)}$, and $xy \in A^{(\lambda_1,\mu_1)}$ and $yx \in A^{(\lambda_1,\mu_1)}$. Then

$$\mu_A(x+y) \geq \lambda_1 = \lambda_1 \wedge \lambda_2 = \mu_A(x) \wedge \mu_A(y),$$
$$\nu_A(x+y) \leq \nu_1 = \nu_1 \vee \nu_2 = \nu_A(x) \vee \nu_A(y),$$
$$\mu_A(xy) \geq \lambda_1 = \mu_A(x),\ \nu_A(xy) \leq \mu_1 = \nu_A(x),$$
$$\mu_A(yx) \geq \lambda_1 = \mu_A(x),\ \nu_A(yx) \leq \mu_1 = \nu_A(x).$$

Hence $A \in \text{IFI}(S)$.

(ii) $(\Rightarrow)$: Suppose $A \in \text{IFKI}(S)$ and let $(\lambda, \mu) \in \text{Im}A$. Then, by (i), A is an ideal of S. For each $a \in A^{(\lambda,\mu)}$ and each $x \in S$, suppose $x + a \in A^{(\lambda,\mu)}$ or $a + x \in A^{(\lambda,\mu)}$. Then $\mu_A(x+a) \geq \lambda$, $\nu_A(x+a) \leq \mu$ or $\mu_A(a+x) \geq \lambda$, $\nu_A(a+x) \leq \mu$. Since $A \in \text{IFKI}(S)$,

$$\mu_A(x) \geq [\mu_A(a+x) \vee \mu_A(x+a)] \wedge \mu_A(a) \geq \lambda$$

and

$$\nu_A(x) \leq [\nu_A(a+x) \wedge \nu_A(x+a)] \vee \nu_A(a) \leq \mu.$$

Thus $x \in A^{(\lambda,\mu)}$. Hence $A^{(\lambda,\mu)}$ is a k-ideal of S.

$(\Leftarrow)$: Suppose $A^{(\lambda,\mu)}$ is a k-ideal of S for each $(\lambda, \mu) \in \text{Im}A$. Then, by (i), $A \in \text{IFI}(S)$. For any $x, y \in S$, let $A(x) = (\lambda_1, \mu_1)$ and $A(y) = (\lambda_2, \mu_2)$ such that $\lambda_1 < \lambda_2$ and $\mu_1 > \mu_2$. Then $x \in A^{(\lambda_1,\mu_1)}$ and $y \in A^{(\lambda_2,\mu_2)}$. By Result 1.A, since $A^{(\lambda_2,\mu_2)} \subset A^{(\lambda_1,\mu_1)}$, $y \in A^{(\lambda_1,\mu_1)}$. Thus $x + y \in A^{(\lambda_1,\mu_1)}$ and $y + x \in A^{(\lambda,\mu)}$. So $[\mu_A(x+y) \vee \mu_A(y+x)] \wedge \mu_A(y) \geq (\lambda_1 \vee \lambda_1) \wedge \lambda_1 = \lambda_1$. □

Proposition 2.14. Let S be a semiring with zero 0 and let $R \in \text{IFC}_W(S)$. We define a complex mapping $A_R = (\mu_{A_R}, \nu_{A_R}) : S \to I \times I$ as follows: for each $a \in S$,

$$A_R(a) = R(a, 0).$$

Then $A_R \in \text{IFKI}(S)$. In this case, A_R is called the *intuitionistic fuzzy k-ideal induced by R*.

Proof. $A_R(0) = R(0,0)$

$$= (\textstyle\bigvee_{x,y \in S} \mu_R(x,y), \bigwedge_{x,y \in S} \nu_R(x,y))$$
$$\neq (0,1) \text{ since } R \neq 0_\sim.$$

Then $A_R \neq 0_\sim$. Let $a, b \in S$. Then

$$\mu_{A_R}(a+b) = \mu_R(a+b, 0)$$
$$\geq \mu_R(a,0) \wedge \mu_R(b,0)$$
$$(\text{Since } R \in \text{IFC}_W(S))$$
$$= \mu_{A_R}(a) \wedge \mu A_R(b)$$

and

$$\nu_{A_R}(a+b) = \mu_R(a+b, 0)$$
$$\leq \nu_R(a,0) \vee \nu_R(b,0)$$
$$= \nu_{A_R}(a) \vee \nu A_R(b).$$

Also,

$$\mu_{A_R}(ab) = \mu_R(ab, 0)$$
$$\geq \mu_R(a,0) \wedge \mu_R(b,0)$$
$$(\text{Since } R \in \text{IFC}_W(S))$$

$$= \mu_R(b,0)$$
$$= \mu A_R(b)$$

and

$$\nu_{A_R}(ab) = \nu_R(ab,0)$$
$$\leq \nu_R(a,0) \vee \nu_R(b,0)$$
$$= \nu_R(b,0)$$
$$= \nu A_R(b).$$

Similarly, we have $\mu_{A_R}(ab) \geq \mu_{A_R}(a)$ and $\nu_{A_R}(ab) \leq \nu_{A_R}(a)$. So $A_R \in \text{IFI}(S)$. On the other hand,

$$\mu_{A_R}(a) = \mu_R(a,0)$$
$$\geq \bigvee_{x\in S}[\mu_R(a,x) \wedge \mu_R(x,0)]$$
(Since $R \circ R \subset R$)
$$\geq \mu_R(a,a+b) \wedge \mu_R(a+b,0)$$
$$\geq [\mu_R(a,a) \wedge \mu_R(0,b)] \wedge \mu_R(a+b,0)$$
(Since R is intuitionistic fuzzy compatible)
$$= \mu_R(0,b) \wedge \mu_R(a+b,0)$$
$$= \mu_R(a+b,0) \wedge \mu_R(b,0)$$
$$= \mu_{A_R}(a+b) \wedge \mu_{A_R}(b)$$

and

$$\nu_{A_R}(a) = \nu_R(a,0)$$
$$\leq \bigwedge_{x\in S}[\nu_R(a,x) \vee \nu_R(x,0)]$$
$$\leq \nu_R(a,a+b) \vee \nu_R(a+b,0)$$
$$\leq [\nu_R(a,a) \vee \nu_R(0,b)] \vee \nu_R(a+b,0)$$
$$= \nu_R(0,b) \vee \nu_R(a+b,0)$$
$$= \nu_R(a+b,0) \vee \nu_R(b,0)$$
$$= \nu_{A_R}(a+b) \vee \nu_{A_R}(b).$$

Hence $A_R \in \text{IFKI}(S)$. This completes the proof. □

Proposition 2.15. Let S be a semiring with zero 0 and let $A \in \text{IFI}(S)$. We define a complex mapping $R_A = (\mu_{R_A}, \nu_{R_A}) : S \times S \to I \times I$ as follows: for each $(x,y) \in S \times S$,

$$R_A(x,y) = \Big(\bigvee_{\substack{x+a=y+b \\ a,b\in S}}[\mu_A(a) \wedge \mu_A(b)], \bigwedge_{\substack{x+a=y+b \\ a,b\in S}}[\nu_A(a) \vee \nu_A(b)]\Big).$$

Then $R_A \in \text{IFWC}(S)$. In this case R_A is called the *intuitionistic fuzzy weak congruence induced by A.*

Proof. Since $A \neq 0_\sim$, it is clear that $R_A \neq 0_\sim$. Let $x \in X$. Then

$$\mu_{R_A}(x,x) = \bigvee_{\substack{x+a=y+b \\ a,b\in S}}[\mu_A(a) \wedge \mu_A(b)]$$
$$\geq \mu_A(a) \wedge \mu_A(0)$$
(Since $x + 0 = x + 0$)
$$\geq \mu_A(u) \wedge \mu_A(v)$$
$$\text{for any } u,v \in S \tag{2.1}$$

and

$$\nu_{R_A}(x,y) = \bigwedge_{\substack{x+a=y+b \\ a,b\in S}}[\nu_A(a) \vee \nu_A(b)]$$
$$\leq \nu_A(a) \vee \nu_A(0)$$
$$\leq \nu_A(u) \vee \nu_A(v)$$
$$\text{for any } u,v \in S. \tag{2.2}$$

Since $\mu_{R_A}(y,z) = \bigvee_{\substack{x+u=y+v \\ u,v\in S}}[\mu_A(u) \wedge \mu_A(v)]$ and $\nu_{R_A}(y,z) = \bigwedge_{\substack{x+u=y+v \\ u,v\in S}}[\nu_A(u) \vee \nu_A(v)]$ for any $y,z \in S$, by (2.1) and (2.2), $\mu_{R_A}(x,x) \geq \mu_{R_A}(y,z)$ and $\nu_{R_A}(x,x) \leq \nu_{R_A}(y,z)$ for any $y,z \in S$. So $\mu_{R_A}(x,x) \geq \bigvee_{y,z\in S}\mu_{R_A}(y,z)$ and $\nu_{R_A}(x,x) \leq \bigwedge_{y,z\in S}\nu_{R_A}(y,z)$, i.e., $R_A(x,x) = (\bigvee_{y,z\in S}\mu_{R_A}(y,z), \bigwedge_{y,z\in S}\nu_{R_A}(y,z))$. Hence R_A is intuitionistic fuzzy weakly reflexive. It is clear that R_A is intuitionistic fuzzy symmetric. Now let $x,y \in S$. Then

$$\mu_{R_A}(x,y)$$
$$= \bigvee_{\substack{x+a=y+b \\ a,b\in S}}[\mu_A(a) \wedge \mu_A(b)]$$
$$\geq \bigvee_{\substack{x+a=z+c \\ a,c\in S}} \bigvee_{\substack{z+c=y+b \\ b,c\in S}}[(\mu_A(a) \wedge \mu_A(c)) \wedge (\mu_A(c) \wedge \mu_A(b))]$$
$$= (\bigvee_{\substack{x+a=z+c \\ a,c\in S}}[\mu_A(a) \wedge \mu_A(c)]) \wedge (\bigvee_{\substack{z+c=y+b \\ b,c\in S}}[\mu_A(c) \wedge \mu_A(b)])$$
$$= \mu_{R_A}(x,z) \wedge \mu_{R_A}(z,y)$$

and

$$\nu_{R_A}(x,y)$$
$$= \bigwedge_{\substack{x+a=y+b \\ a,b\in S}}[\nu_A(a) \vee \nu_A(b)]$$
$$\leq \bigwedge_{\substack{x+a=z+c \\ a,c\in S}} \bigwedge_{\substack{z+c=y+b \\ b,c\in S}}[(\nu_A(a) \vee \nu_A(c)) \vee (\nu_A(c) \vee \nu_A(b))]$$
$$= (\bigwedge_{\substack{x+a=z+c \\ a,c\in S}}[\nu_A(a) \vee \nu_A(c)]) \vee (\bigwedge_{\substack{z+c=y+b \\ b,c\in S}}[\nu_A(c) \vee \nu_A(b)])$$
$$= \nu_{R_A}(x,z) \vee \nu_{R_A}(z,y).$$

Thus

$$\mu_{R_A}(x,y) \geq \bigvee_{z\in S}[\mu_{R_A}(x,z) \wedge \mu_{R_A}(z,y)]$$
$$= \mu_{R_A \circ R_A}(x,y)$$

and

$$\nu_{R_A}(x,y) \leq \bigwedge_{z\in S}[\nu_{R_A}(x,z) \vee \nu_{R_A}(z,y)]$$
$$= \nu_{R_A \circ R_A}(x,y).$$

So R_A is intuitionistic fuzzy transitive. Hence $R_A \in \text{IFE}_W(S)$.

Now let $a,b,c,d \in S$. Let $\mu_{R_A}(a,b) > \mu_{R_A}(c,d)$ and $\nu_{R_A}(a,b) < \nu_{R_A}(c,d)$. Suppose $\mu_{R_A}(a+c,b+d) \geq \mu_{R_A}(a,b)$ and $\nu_{R_A}(a+c,b+d) \leq \nu_{R_A}(a,b)$. Then clearly $\mu_{R_A}(a+c,b+d) > \mu_{R_A}(a,b) \wedge \mu_{R_A}(c,d)$ and $\nu_{R_A}(a+c,b+d) < \nu_{R_A}(a,b) \vee \nu_{R_A}(c,d)$. Suppose $\mu_{R_A}(a+c,b+d) > \mu_{R_A}(a,b)$ and $\nu_{R_A}(a+c,b+d) < \nu_{R_A}(a,b)$. Then there exist $u,v \in S$ such that $a+u = b+v$ and

$$\mu_{R_A}(a+c,b+d) > \mu_A(u) \wedge \mu_A(v),$$
$$\nu_{R_A}(a+c,b+d) < \nu_A(u) \vee \nu_A(v). \tag{2.3}$$

Let $u,v \in S$ such that $c + u_1 = d + v_1$. Then

$$\mu_{R_A}(a+c,b+d)$$
$$\geq \mu_A(u+u_1) \wedge \mu_A(v+v_1)$$
(Since $a + c + u + u_1 = b + d + v + v_1$)
$$\geq \mu_A(u) \wedge \mu_A(u_1) \wedge \mu_A(v) + \mu_A(v_1)$$
$$= [\mu_A(u) \wedge \mu_A(v)] \wedge [\mu_A(u_1) \wedge \mu_A(v_1)]$$

and

$$\nu_{R_A}(a+c,b+d)$$
$$\leq \nu_A(u+u_1) \vee \nu_A(v+v_1)$$
$$\leq \nu_A(u) \vee \nu_A(u_1) \vee \nu_A(v) + \nu_A(v_1)$$
$$= [\nu_A(u) \vee \nu_A(v)] \vee [\nu_A(u_1) \vee \nu_A(v_1)].$$

By (2.3), $\mu_{R_A}(a+c,b+d) \geq \mu_A(u_1) \wedge \mu_A(v_1)$, $\nu_{R_A}(a+c,b+d) \leq \nu_A(u_1) \vee \nu_A(v_1)$.

Thus

$\mu_{R_A}(a+c,b+d)$
$\geq \bigvee_{\substack{c+u_1=d+v_1 \\ u_1,v_1\in S}} [\mu_A(u_1) \wedge \mu_A(v_1)]$
$= \mu_{R_A}(c,d)$
$= \mu_{R_A}(a,b) \wedge \mu_{R_A}(c,d)$

and

$\nu_{R_A}(a+c,b+d)$
$\leq \bigwedge_{\substack{c+u_1=d+v_1 \\ u_1,v_1\in S}} [\nu_A(u_1) \vee \nu_A(v_1)]$
$= \nu_{R_A}(c,d)$
$= \nu_{R_A}(a,b) \vee \nu_{R_A}(c,d)$.

Let $R_A(a,b) = R_A(c,d)$. Then we can show that:

$\mu_{R_A}(a+c,b+d) \geq \mu_{R_A}(a,b) \wedge \mu_{R_A}(c,d)$

and

$\nu_{R_A}(a+c,b+d) \leq \nu_{R_A}(a,b) \vee \nu_{R_A}(c,d)$.

By the similar arguments, we can see that:

$\mu_{R_A}(ac,bd) \geq \mu_{R_A}(a,b) \wedge \mu_{R_A}(c,d)$

and

$\nu_{R_A}(ac,bd) \leq \nu_{R_A}(a,b) \vee \nu_{R_A}(c,d)$.

Thus R_A is intuitionistic fuzzy compatible. Hence $R_A \in \text{IFC}_W(S)$. This is completes the proof. □

Corollary 2.16. In Proposition 2.15, if $A \in \text{IFKI}(S)$, then R_A is the smallest intuitionistic fuzzy weak congruence on S such that $R_A(x,0) = A(x)$.

Proof. Suppose $A \in \text{IFKI}(S)$. Let $a,b \in S$ such that $x+a = 0+b$. Then

$\mu_A(a) \wedge \mu_A(b) = \mu_A(a) \wedge \mu_A(x+a)$
$\leq \mu_A(x)$ (Since $A \in \text{IFKI}(S)$)

and

$\nu_A(a) \vee \nu_A(b) = \nu_A(a) \vee \nu_A(x+a)$
$\geq \nu_A(x)$.

Thus

$\mu_{R_A}(x,0) = \bigvee_{\substack{x+a=0+b \\ a,b\in S}} [\mu_A(a) \wedge \mu_A(b)]$
$\leq \mu_A(x)$

and

$\nu_{R_A}(x,0) = \bigwedge_{\substack{x+a=0+b \\ a,b\in S}} [\nu_A(a) \vee \nu_A(b)]$
$\geq \nu_A(x)$.

On the other hand,

$\mu_{R_A}(x,0) = \bigvee_{\substack{x+a=0+b \\ a,b\in S}} [\mu_A(a) \wedge \mu_A(b)]$
$\geq \mu_A(x) \wedge \mu_A(0)$ (Since $x+0=0+x$)
$= \mu_A(x)$

and

$\nu_{R_A}(x,0) = \bigwedge_{\substack{x+a=0+b \\ a,b\in S}} [\nu_A(a) \vee \nu_A(b)]$
$\leq \nu_A(x) \vee \nu_A(0)$
$= \nu_A(x)$.

So $R_A(x,0) = A(x)$.

Let $Q \in \text{IFC}_W(S)$ such that $Q(x,0) = A(x)$. Let $(x,y) \in S \times S$ and let $x+a = y+b$, $a,b \in S$. Then

$\mu_Q(x,y) \geq \mu_Q(x,x+a) \wedge \mu_Q(x+a,y)$
(Since$Q$ is intuitionistic fuzzy transitive)
$\geq \mu_Q(x,x) \wedge \mu_Q(0,a) \wedge \mu_Q(y+b,y)$
(Since Q is intuitionistic fuzzy transitive)
$\geq \mu_Q(0,a) \wedge \mu_Q(b,0)$
(Since Q is intuitionistic fuzzy transitive)
$= \mu_Q(a,0) \wedge \mu_Q(b,0)$
(Since Q is intuitionistic fuzzy transitive)
$= \mu_A(a) \wedge \mu_A(b)$

and

$\nu_Q(x,y) \leq \nu_Q(x,x+a) \vee \nu_Q(x+a,y)$
$\leq \nu_Q(x,x) \vee \nu_Q(0,a) \vee \nu_Q(y+b,y)$
$\leq \nu_Q(0,a) \vee \nu_Q(b,0)$
$= \nu_Q(a,0) \vee \nu_Q(b,0)$
$= \nu_A(a) \vee \nu_A(b)$.

Thus

$\mu_Q(x,y) \geq \bigvee_{\substack{x+a=y+b \\ a,b\in S}} [\mu_A(a) \wedge \mu_A(b)]$
$= \mu_{R_A}(x,y)$

and

$\nu_Q(x,y) \leq \bigwedge_{\substack{x+a=y+b \\ a,b\in S}} [\nu_A(a) \vee \nu_A(b)]$
$= \nu_{R_A}(x,y)$.

So $R_A \subset Q$. Hence R_A is the smallest intuitionistic fuzzy weak congruence on S such that $R_A(x,0) = A(x)$ for each $x \in S$. This completes the proof. □

Theorem 2.17. Let S be a semiring with zero 0. Then, there exists an inclusion preserving injection from $\text{IFKI}(S)$ to $\text{IFC}_W(S)$.

Proof. We define a mapping $f : \text{IFC}_W(S) \to \text{IFKI}(S)$ and a mapping $g : \text{IFKI}(S) \to \text{IFC}_W(S)$ as follows, respectively: for each $R \in \text{IFC}_W(S)$ and each $A \in \text{IFKI}(S)$,

$$f(R) = A_R \text{ and } g(A) = R_A.$$

Then, by Proposition 2.15 and Corollary 2.16, f and g are will-defined. Moreover, $(f \circ g)(A) = f(g(A)) = f(R_A) = A_{R_A}$ for each $A \in \text{IFKI}(S)$ and $A_{R_A}(a) = R_A(a,0) = A(a)$ for each $a \in S$. Thus $(f \circ g)(A) = A = id_{\text{IFKI}(S)}(A)$ for each $A \in \text{IFKI}(S)$. So g is injective. Now let $A, B \in \text{IFKI}(S)$ such that $A \subset B$ and let $x,y \in S$. Then

$\mu_{R_B}(x,y) = \bigvee_{\substack{x+a=y+b \\ a,b\in S}} [\mu_B(a) \wedge \mu_B(b)]$
$\geq \mu_B(a) \wedge \mu_B(b)$
$\geq \mu_A(a) \wedge \mu_B(b)$

and

$\nu_{R_B}(x,y) = \bigwedge_{\substack{x+a=y+b \\ a,b\in S}} [\nu_B(a) \vee \nu_B(b)]$
$\leq \nu_B(a) \vee \nu_B(b)$
$\leq \nu_A(a) \vee \nu_B(b)$.

Thus

$\mu_{R_B}(x,y) = \bigvee_{\substack{x+a=y+b \\ a,b\in S}} [\mu_A(a) \wedge \mu_A(b)]$
$= \mu_{R_A}(x,y)$

and

$\nu_{R_B}(x,y) = \bigwedge_{\substack{x+a=y+b \\ a,b\in S}} [\nu_A(a) \vee \nu_A(b)]$
$= \nu_{R_A}(x,y)$.

So $R_A \subset R_B$, i.e., $g(A) \subset g(B)$. Hence g is an inclusion preserving injection. This completes the proof. □

Proposition 2.18. Let S be a semiring with zero 0. Let $R \in \mathrm{IFC}_W(S)$ and let A_R be the intuitionistic fuzzy k-ideal induced by R. Then $A_R^{(\lambda,\mu)} = \{x \in S : x \equiv 0(R^{(\lambda,\mu)})\}$ for each $(\lambda, \mu) \in \mathrm{Im}R$.

Proof. Let $(\lambda, \mu) \in I \times I$ and let $a \in S$. Then
$a \in A_R^{(\lambda,\mu)}$
if and only if $\mu_{A_R}(a) \geq \lambda$ and $\nu_{A_R}(a) \leq \mu$
if and only if $\mu_R(a, 0) \geq \lambda$ and $\nu_R(a, 0) \leq \mu$
if and only if $(a, 0) \in R^{(\lambda,\mu)}$
if and only if $a \equiv 0(R^{(\lambda,\mu)})$
if and only if $a \in \{x \in S : x \equiv 0(R^{(\lambda,\mu)})\}$.
□

Definition 2.19[9]. Let A be an intuitionistic fuzzy set in a semigroup S. Then A is said to have the *sup-property* if for any subset T of S, there exists $t_0 \in T$ such that $A(t_0) = \bigcup_{t \in T} A(t)$, i.e., $\mu_A(t_0) = \bigvee_{t \in T} \mu_A(t)$ and $\nu_A(t_0) = \bigwedge_{t \in T} \nu_A(t)$.

Proposition 2.20. Let S be a semigroup and let $A \in \mathrm{IFKI}(S)$. Let R_A be the intuitionistic fuzzy weak congruence on S induced by A. If R has the sup-property, then $R_A^{(\lambda,\mu)}$ is a congruence on S induced by $A^{(\lambda,\mu)}$ for each $(\lambda, \mu) \in \mathrm{Im}A$.

Proof. Let $(\lambda, \mu) \in \mathrm{Im}A$ and let Q be the congruence on S induced by $A^{(\lambda,\mu)}$, i.e., $(x, y) \in Q$ if and only if there exist $i_1, i_2 \in A^{(\lambda,\mu)}$ such that $x + i_1 = y + i_2$ (See p.908 in [4]). Let $(x, y) \in R_A^{(\lambda,\mu)}$. Since A has the sup-property, there exist $a_1, a_2 \in S$ such that $x + a_1 = y + b_1$, and

$$\begin{aligned}\mu_{R_A}(x, y) &= \bigvee_{\substack{x+a=y+b \\ a,b \in S}} [\mu_A(a) \wedge \mu_A(b)] \\ &= \mu_A(a_1) \wedge \mu_A(b_1) \\ &\geq \lambda\end{aligned}$$

and

$$\begin{aligned}\nu_{R_A}(x, y) &= \bigwedge_{\substack{x+a=y+b \\ a,b \in S}} [\nu_A(a) \vee \nu_A(b)] \\ &= \nu_A(a_1) \vee \nu_A(b_1) \\ &\leq \mu.\end{aligned}$$

Then $\mu_A(a_1) \geq \lambda$, $\nu_A(a_1) \leq \mu$ and $\mu_A(b_1) \geq \lambda$, $\nu_A(b_1) \leq \mu$. Thus $a_1, b_1 \in A^{(\lambda,\mu)}$. So $(x, y) \in Q$. Hence $R_A^{(\lambda,\mu)} \subset Q$. By reversing the above argument, we have $Q \subset R_A^{(\lambda,\mu)}$. Therefore $Q = R_A^{(\lambda,\mu)}$. □

3. Intuitionistic fuzzy cosets

Definition 3.1. Let S be a semigroup, let $A \in \mathrm{IFI}(S)$ and let $x \in S$. We define a complex mapping $Ax = (\mu_{Ax}, \nu_{Ax}) : S \to I \times I$ as follows: For each $r \in S$,

$$Ax(r) = \Big(\bigvee_{\substack{x+u=r+v \\ u,v \in S}} [\mu_A(u) \wedge \mu_A(v)], \bigwedge_{\substack{x+u=r+v \\ u,v \in S}} [\nu_A(u) \vee \nu_A(v)]\Big).$$

Then Ax is called the *intuitionistic fuzzy coset determined by A and x*.

It is clear that $Ax \in \mathrm{IFS}(S)$.

Remark 3.2. Let A be an k-ideal of a semiring S and let $x \in S$. Then $(\chi_A, \chi_{A^c})_x = (\chi_{Ax}, \chi_{Ax^c})$.

Proposition 3.3. Let S be a ring, let $A \in \mathrm{IFI}(S)$ and let $x \in S$. Then
$Ax(r) = A(x - r) = A(r - x)$ for each $r \in S$.

Proof. Let $r \in S$. Then

$$\begin{aligned}\mu_{Ax}(r) &= \bigvee_{\substack{x+a=r+b \\ a,b \in S}} [\mu_A(a) \wedge \mu_A(b)] \\ &\leq \bigvee_{\substack{x+a=r+b \\ a,b \in S}} [\mu_A(b - a)] \text{ (Since } A \in \mathrm{IFI}(S)) \\ &= \mu_A(x - r) \quad \text{(Since } b - a = x - r) \\ &= \mu_A(r - x) \quad \text{(Since } R \text{ is a ring)}\end{aligned}$$

and

$$\begin{aligned}\nu_{Ax}(r) &= \bigwedge_{\substack{x+a=r+b \\ a,b \in S}} [\nu_A(a) \vee \nu_A(b)] \\ &\geq \bigwedge_{\substack{x+a=r+b \\ a,b \in S}} [\nu_A(b - a)] \\ &= \nu_A(x - r) = \nu_A(r - x).\end{aligned}$$

On the other hand,

$$\begin{aligned}\mu_{Ax}(r) &= \bigvee_{\substack{x+a=r+b \\ a,b \in S}} [\mu_A(a) \wedge \mu_A(b)] \\ &\leq \mu_A(r - x) \wedge \mu_A(0) \\ &\quad \text{(Since } x + (r - x) = r + 0) \\ &= \mu_A(r - x) \\ &= \mu_A(x - r)\end{aligned}$$

and

$$\begin{aligned}\nu_{Ax}(r) &= \bigwedge_{\substack{x+a=r+b \\ a,b \in S}} [\nu_A(a) \vee \nu_A(b)] \\ &\geq \nu_A(r - x) \vee \nu_A(0) \\ &= \nu_A(r - x) = \nu_A(x - r).\end{aligned}$$

Hence $Ax(r) = A(x - r) = A(r - x)$. □

From Proposition 3.3, we can define the intuitionistic fuzzy coset in a ring as follows.

Definition 3.4. Let R be a ring, let $A \in \mathrm{IFI}(S)$ and let $x \in R$. Then the *intuitionistic fuzzy coset determined by A and x*, denoted by Ax, is defined by $Ax(r) = A(x - r)$ for each $x \in R$.

Theorem 3.5. Let S be a semigroup, let $A \in \mathrm{IFI}(S)$ and let S/A be the set of all intuitionistic fuzzy coset of A in S. We define two binary operations $+$ and $\cdot$ on S/A as follows, respectively: for any $x, y \in S$,
$Ax + Ay = Ax + y$ and $Ax \cdot Ay = Axy$.
Then $+$ and $\cdot$ are well-defined. Hence $(S/A, +, \cdot)$ is a semiring. In this case, $(S/A, +, \cdot)$ is called the *quotient semiring over A*.

Proof. For any $x, y, p, q \in S$, suppose $Ax = Ap$ and $Ay = Aq$. Let $r \in S$. Then $Ax(r) = Ap(r)$ and

$Ay(r) = Aq(r)$. By Proposition 2.15 and the definition Ax, $R_A(x,r) = R_A(p,r)$ and $R_A(y,r) = R_A(q,r)$. Thus

$$R_A(x,p) = R_A(p,p) = (\bigvee_{u,v\in S}\mu_{R_A}(u,v), \bigwedge_{u,v\in S}\nu_{R_A}(u,v)) \quad (3.1)$$

and

$$R_A(y,q) = R_A(q,q) = (\bigvee_{u,v\in S}\mu_{R_A}(u,v), \bigwedge_{u,v\in S}\nu_{R_A}(u,v)). \quad (3.2)$$

On the other hand,

$$\begin{aligned}\mu_{Ax+Ay}(r) &= \mu_{Ax+y}(r)\\ &= \mu_{R_A}(x+y,r)\\ &\geq \mu_{R_A}(x+y,p+q)\wedge\mu_{R_A}(p+q,r)\\ &\text{(Since } R_A \text{ is intuitionistic fuzzy transitive)}\\ &\geq \mu_{R_A}(x,p)\wedge\mu_{R_A}(y,q)\wedge\mu_{R_A}(p+q,r)\\ &\text{(Since } R_A \text{ is intuitionistic fuzzy compatible)}\\ &= \mu_{R_A}(p+q,r) \text{ (By (3.1) and (3.2))}\\ &= \mu_{Ap+q}(r)\\ &= \mu_{Ap+Aq}(r)\end{aligned}$$

and

$$\begin{aligned}\nu_{Ax+Ay}(r) &= \nu_{Ax+y}(r) = \nu_{R_A}(x+y,r)\\ &\leq \nu_{R_A}(x+y,p+q)\vee\nu_{R_A}(p+q,r)\\ &\leq \nu_{R_A}(x,p)\vee\nu_{R_A}(y,q)\vee\nu_{R_A}(p+q,r)\\ &= \nu_{R_A}(p+q,r)\\ &= \nu_{Ap+q}(r)\\ &= \nu_{Ap+Aq}(r).\end{aligned}$$

Then $Ap+Aq \subset Ax+Ay$. By the similar arguments, we have $Ax+Ay \subset Ap+Aq$. So $Ax+Ay = Ap+Aq$. Also,

$$\begin{aligned}\mu_{AxAy}(r) &= \mu_{Axy}(r)\\ &= \mu_{R_A}(xy,r)\\ &\geq \mu_{R_A}(xy,pq)\wedge\mu_{R_A}(pq,r)\\ &\text{(Since } R_A \text{ is intuitionistic fuzzy transitive)}\\ &\geq \mu_{R_A}(x,p)\wedge\mu_{R_A}(y,q)\wedge\mu_{R_A}(pq,r)\\ &\text{(Since } R_A \text{ is intuitionistic fuzzy compatible)}\\ &= \mu_{R_A}(pq,r) \text{ (By (3.1) and (3.2))}\\ &= \mu_{Apq}(r)\\ &= \mu_{ApAq}(r)\end{aligned}$$

and

$$\begin{aligned}\nu_{AxAy}(r) &= \nu_{Axy}(r) = \nu_{R_A}(xy,r)\\ &\leq \nu_{R_A}(xy,pq)\vee\nu_{R_A}(pq,r)\\ &\leq \nu_{R_A}(x,p)\vee\nu_{R_A}(y,q)\vee\nu_{R_A}(pq,r)\\ &= \nu_{R_A}(pq,r)\\ &= \nu_{Apq}(r)\\ &= \nu_{ApAq}(r).\end{aligned}$$

Thus $ApAq \subset AxAy$. By the similar arguments, we have $AxAy \subset ApAq$. So $AxAy = ApAq$. Hence $+$ and $\cdot$ are well-defined. It can by easily seen that $(S/A, +, \cdot)$ is a semiring. This completes the proof. □

Remark 3.6. (1) In the definition of S/A, if S is a semiring with zero 0 and $A \in \mathrm{IFKI}(S)$, then $A = A_0$.

(2) Let S be a semiring, let $R \in \mathrm{IFC}_W(S)$ and let $x \in S$. We can define the intuitionistic fuzzy coset Rx by $Rx(r) = R(x,r)$ for each $r \in S$. Then $S/R = \{Rx : x \in S\}$ forms a semiring as above. But if $A \in \mathrm{IFI}(S)$, then $S/A = S/R_A$.

Then following is easily seen.

Proposition 3.7. Let S be a semiring and let $A \in \mathrm{IFI}(S)$. We define a mapping $f : S \to S/A$ by $f(x) = Ax$ for each $x \in S$. Then f is a homomorphism.

Definition 3.8. Let S be a semiring and let $R, Q \in \mathrm{IFC}_W(S)$. Then Q is said to be *R-invariant* if $R(x,y) = R(u,v)$ implies that $Q(x,y) = Q(u,v)$ for any $(x,y),(u,v) \in S\times S$.

Remark 3.9. Let R and Q be congruences on a semiring S. If (χ_Q, χ_{Q^c}) is (χ_R, χ_{R^c})-invariant, then $R \subset Q$.

Lemma 3.10. Let S be a semiring and let $A \in \mathrm{IFI}(S)$. Let R be the intuitionistic fuzzy weak congruence on S induced by A. We define a complex mapping $R/R = (\mu_{R/R}, \nu_{R/R}) : S/A \times S/A \to I \times I$ as follows:

$$R/R(Ax, Ay) = R(x,y) \text{ for any } x,y \in S.$$

Then $R/R \in \mathrm{IFC}_W(S/A)$.

Proof. It is clear that R/R is well-defined. Moreover, by the definition of R/R, $R/R \in \mathrm{IFR}(S/A)$. The rest of the proof is a routine matter of verification. So we omit it. □

Theorem 3.11. Let S be a semiring, and let $A \in \mathrm{IFI}(S)$ and let R be the intuitionistic fuzzy weak congruence on S induced by A. Then there exists a one-to-one correspondence between $\mathrm{IFC}_R(S)$ and $\mathrm{IFC}_{R/R}(S/A)$, where $\mathrm{IFC}_R(S)$ [resp. $\mathrm{IFC}_{R/R}(S/A)$] denotes the set of all intuitionistic fuzzy R-invariant [resp. R/R-invariant] weak congruences on S [resp. on S/A].

Proof. Let $Q \in \mathrm{IFC}_R(S)$. We define a complex mapping $Q/R : S/A \times S/A \to I \times I$ by $Q/R(Ax, Ay) = Q(x,y)$ for any $x,y \in S$. For any $x,y,p,q \in S$, suppose $Ax = Ap$ and $Ay = Aq$. Let $r \in S$. Then $Ax(r) = Ap(r)$ and $Ay(r) = Aq(r)$. Thus $R(x,r) = R(p,r)$ and $R(y,r) = R(q,r)$. So $R(x,y) = R(p,y)$ and $R(y,p) = R(q,p)$. Since Q is R-invariant, $Q(x,y) = Q(p,y) = Q(p,q)$. Hence Q/R is well-defined.

It can be easily shown that $Q/R \in \mathrm{IFC}_W(S/A)$. Now we define a mapping $f : \mathrm{IFC}_R(S) \to \mathrm{IFC}_{R/R}(S/A)$ by $f(Q) = Q/R$. Let $Q_1, Q_2 \in \mathrm{IFC}_R(S)$ such that $Q_1 \neq Q_2$. Then there exists $(x,y) \in S\times S$ such that $Q_1(x,y) \neq Q_2(x,y)$. Thus $Q_1/R(Ax,Ay) = Q_1(x,y)Q_2(x,y) = Q_2/R(Ax,Ay)$. So f is injective. Let $Q' \in \mathrm{IFC}_{R/R}(S/A)$. We define a complex mapping $Q = (\mu_Q, \nu_Q) : S\times S \to I\times I$ as follows: for any $x,y \in S$,

$$Q(x,y) = Q'(Ax,Ay).$$

Then clearly $Q \in \text{IFR}(S)$ from the definition of Q. Let $x \in S$. Then

$$Q(x,x) = Q'(Ax,Ay) = (\bigvee_{Au,Av\in S/A}\mu_{Q'}(Au,Av),\ \bigwedge_{Au,Av\in S/A}\nu_{Q'}(Au,Av)) = (\bigvee_{u,v\in S}\mu_Q(u,v),\ \bigwedge_{u,v\in S}\nu_Q(u,v)).$$

Thus Q is intuitionistic fuzzy weakly reflexive. We can easily see that Q is intuitionistic fuzzy symmetric and intuitionistic fuzzy transitive. So $Q \in \text{IFE}_W(S)$. Now let $x, y, a, b \in S$. Then

$$\mu_Q(x+a, y+b) = \mu_{Q'}(Ax+a, Ay+b) = \mu_{Q'}(Ax+Aa,\ Ay+Ab) \geq \mu_{Q'}(AxAy) \wedge \mu_{Q'}(Aa+Ab)$$

(Since Q' is intuitionistic fuzzy compatible)

$$= \mu_Q(x,y) \wedge \mu_Q(a,b)$$

and

$$\nu_Q(x+a, y+b) = \nu_{Q'}(Ax+a, Ay+b) = \nu_{Q'}(Ax+Aa,\ Ay+Ab) \leq \nu_{Q'}(AxAy) \vee \nu_{Q'}(Aa+Ab) = \nu_Q(x,y) \vee \nu_Q(a,b).$$

By the similar arguments, we have

$$\mu_Q(xa, yb) \geq \mu_Q(x,y) \wedge \mu_Q(a,b)$$

and

$$\nu_Q(xa, yb) \leq \nu_Q(x,y) \vee \nu_Q(a,b).$$

So $Q \in \text{IFC}_W(S)$. For any $x, y, u, v \in S$, suppose $R(x,y) = R(u,v)$. Then, by the definition of R/R, $R/R(Ax,Ay) = R/R(Au,Av)$. Since $Q' \in \text{IFC}_{R/R}(S/A)$, $Q'(Ax,Ay) = Q'(Au,Av)$. Thus $Q(x,y) = Q(u,v)$. So $Q \in \text{IFC}_R(S)$. On the other hand, $Q/R(Ax,Ay) = Q(x,y) = Q'(Ax,Ay)$. Then $Q' = Q/R = f(Q)$. So f is surjective. Hence f is bijective. This completes the proof. □

Theorem 3.12. Let S be a semiring, let $A \in \text{IFI}(S)$ and let R be the intuitionistic fuzzy weak congruence on S induced by A. If $(\lambda_0, \mu_0) = (\bigvee_{u,v\in S}\mu_R(u,v),\ \bigwedge_{u,v\in S}\nu_R(u,v))$, then $S/A \cong S/R(\lambda_0, \mu_0)$.

Proof. We define mapping $f : S/A \to S/R(\lambda_0, \mu_0)$ by $f(Ax) = xR^{(\lambda_0,\mu_0)}$, where $xR^{(\lambda_0,\mu_0)}$ denotes the congruence class containing x of the congruence $R^{(\lambda_0,\mu_0)}$. For each $x, y \in S$, suppose $Ax = Ay$. Then

$Ax(r) = Ay(r)$ for each $r \in S$

$$\Rightarrow R(x,y) = R(y,r)$$
$$\Rightarrow R(x,y) = R(y,y) = (\bigvee_{u,v\in S}\mu_R(u,v),\ \bigwedge_{u,v\in S}\nu_R(u,v)) = (\lambda_0, \mu_0)$$
$$\Rightarrow (x,y) \in R^{(\lambda_0,\mu_0)}$$
$$\Rightarrow xR^{(\lambda_0,\mu_0)} = yR^{(\lambda_0,\mu_0)}$$
$$\Rightarrow f(Ax) = f(Ay).$$

So f is well-defined. Let $x, y \in S$. Then

$$f(Ax + Ay) = f(Ax + y) = (x+y)R^{(\lambda_0,\mu_0)} = xR^{(\lambda_0,\mu_0)} + yR^{(\lambda_0,\mu_0)} = f(Ax) + f(Ay)$$

and

$$f(AxAy) = f(Axy) = (xy)R^{(\lambda_0,\mu_0)} = (xR^{(\lambda_0,\mu_0)})(yR^{(\lambda_0,\mu_0)}) = f(Ax)f(Ay).$$

Thus f is a homomorphism. For any $x, y \in S$, suppose $f(Ax) = f(Ay)$. Then $xR^{(\lambda_0,\mu_0)} = yR^{(\lambda_0,\mu_0)}$. Thus $(x,y) \in R^{(\lambda_0,\mu_0)}$, i.e., $R(x,y) = (\lambda_0, \mu_0)$. So,

$$\mu_R(x,r) \geq \mu_R(x,y) \wedge \mu_R(y,r)$$

(Since R is intuitionistic fuzzy transitive)

$= \mu_R(y,r)$ for each $r \in S$

and

$$\nu_R(x,r) \leq \nu_R(x,y) \vee \nu_R(y,r)$$

$= \nu_R(y,r)$ for each $r \in S$.

Similarly, we have $\mu_R(y,r) \geq \mu_R(x,r)$ and $\nu_R(y,r) \leq \nu_R(x,r)$ for each $r \in S$. Thus $R(x,r) = R(y,r)$ for each $r \in S$. So $Ax(r) = Ay(r)$ for each $r \in S$, i.e., $Ax = Ay$. Hence f is injective. It is clear that f is surjective.

Therefore f is an isomorphism. This completes the proof. □

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