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Existence and Uniqueness of Solutions for the Semilinear Fuzzy Integrodifferential Equations with Nonlocal Conditions and Forcing Term with Memory

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Abstract

Many authors have studied several concepts of fuzzy systems. Balasubramaniam and Muralisankar (2004) proved the existence and uniqueness of fuzzy solutions for the semilinear fuzzy integrodifferential equation with nonlocal initial condition. Recently, Park, Park and Kwun (2006) find the sufficient condition of nonlocal controllability for the semilinear fuzzy integrodifferential equation with nonlocal initial condition. In this paper, we study the existence and uniqueness of solutions for the semilinear fuzzy integrodifferential equations with nonlocal condition and forcing term with memory in E_N by using the concept of fuzzy number whose values are normal, convex, upper semicontinuous and compactly supported interval in E_N.

Key words : Fuzzy number, semilinear, integrodifferential equation, nonlocal.

1. Introduction

Many authors have studied several concepts of fuzzy systems. Kaleva [3] studied the existence and uniqueness of solution for the fuzzy differential equation on E^n where E^n is normal, convex, upper semicontinuous and compactly supported fuzzy sets in R^n. Seikkala [7] proved the existence and uniqueness of fuzzy solution for the following equation:

$$\dot{x}(t) = f(t, x(t)), \ \ x(0) = x_0,$$

where f is a continuous mapping from $R^+ \times R$ into R and x_0 is a fuzzy number in E^1. Diamond and Kloeden [2] proved the fuzzy optimal control for the following system:

$$\dot{x}(t) = a(t)x(t) + u(t), \ \ x(0) = x_0,$$

where $x(\cdot)$ and $u(\cdot)$ are nonempty compact interval-valued functions on E^1. Kwun and Park [4] proved the existence of fuzzy optimal control for the nonlinear fuzzy differential system with nonlocal initial condition in E_N^1 using by Kuhn-Tucker theorems. Balasubramaniam and Muralisankar [1] proved the existence and uniqueness of fuzzy solutions for the semilinear fuzzy integrodifferential equation with nonlocal initial condition. Recently, Park, Park and Kwun [6] find the sufficient condition of nonlocal controllability for the semilinear fuzzy integrodifferential equation with nonlocal initial condition.

In this paper, we study the existence and uniqueness of solutions for the semilinear fuzzy integrodifferential equations with nonlocal initial conditions and forcing term with memory.

$$\frac{dx(t)}{dt} = A\left[x(t) + \int_0^t G(t-s)x(s)ds\right] \quad (1)$$
$$+f(t, x, \int_0^t k(t, s, x(s))ds), \ t \in I = [0, T],$$
$$x(0) + g(x) = x_0 \in E_N, \quad (2)$$

where $A : I \to E_N$ is a fuzzy coefficient, E_N is the set of all upper semicontinuous convex normal fuzzy numbers with bounded α-level intervals, $f : I \times E_N \times E_N \to E_N$ and $k : I \times I \times E_N \to E_N$ are nonlinear continuous functions, $G(t)$ is $n \times n$ continuous matrix such that $\frac{dG(t)x}{dt}$ is continuous for $x \in E_N$ and $t \in I$ with $\|G(t)\| \le k$, $k > 0$, and $g : E_N \to E_N$ is a nonlinear continuous function.

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2. Preliminaries

A fuzzy subset of R^n is defined in terms of membership function which assigns to each point $x \in R^n$ a grade of membership in the fuzzy set. Such a membership function $m : R^n \to [0,1]$ is used synonymously to denote the corresponding fuzzy set. We shall restrict attention here to the normal fuzzy sets which satisfy

Assumption 1. m maps R^n onto $[0,1]$.

Assumption 2. $[m]^0$ is a bounded subset of R^n.

Assumption 3. m is upper semicontinuous.

Assumption 4. m is fuzzy convex.

We denote by E^n the space of all fuzzy subsets m of R^n which satisfy assumptions 1-4; that is, normal, fuzzy convex and upper semicontinuous fuzzy sets with bounded supports. In particular, we denoted by E^1 the space of all fuzzy subsets m of R which satisfy assumptions 1-4 [2].

A fuzzy number a in real line R is a fuzzy set characterized by a membership function m_a as $m_a : R \to [0,1]$. A fuzzy number a is expressed as $a = \int_{x\in R} m_a(x)/x$, with the understanding that $m_a(x) \in [0,1]$ represents the grade of membership of x in a and $\int$ denotes the union of $m_a(x)/x$'s [5].

Let E_N be the set of all upper semicontinuous convex normal fuzzy number with bounded α-level intervals. This means that if $a \in E_N$ then the α-level set

$$[a]^\alpha = \{x \in R : m_a(x) \geq \alpha,\ 0 < \alpha \leq 1\}$$

is a closed bounded interval which we denote by

$$[a]^\alpha = [a_l^\alpha, a_r^\alpha]$$

and there exists a $t_0 \in R$ such that $a(t_0) = 1$ [4].

The support Γ_a of a fuzzy number a is defined, as a special case of level set, as the following

$$\Gamma_a = \{x \in R : m_a(x) > 0\}.$$

Two fuzzy numbers a and b are called equal, denoted by $a = b$, if $m_a(x) = m_b(x)$ for all $x \in R$. It follows that

$$a = b \Leftrightarrow [a]^\alpha = [b]^\alpha \text{ for all } \alpha \in (0,1].$$

A fuzzy number a may be decomposed into its level sets through the resolution identity

$$a = \int_0^1 \alpha[a]^\alpha,$$

where $\alpha[a]^\alpha$ is the product of a scalar α with the set $[a]^\alpha$ and $\int$ is the union of $[a]^\alpha$'s with α ranging from 0 to 1.

We denote the suprimum metric d_∞ on E^n and the suprimum metric H_1 on $C(I : E^n)$.

Definition 2.1. Let $a, b \in E^n$.

$$d_\infty(a,b) = \sup\{d_H([a]^\alpha, [b]^\alpha) : \alpha \in (0,1]\},$$

where d_H is the Hausdorff distance.

Definition 2.2. Let $x, y \in C(I : E^n)$

$$H_1(x,y) = \sup\{d_\infty(x(t), y(t)) : t \in I\}.$$

Let I be a real interval. A mapping $x : I \to E_N$ is called a fuzzy process. We denote

$$[x(t)]^\alpha = [x_l^\alpha(t), x_r^\alpha(t)],\ t \in I,\ 0 < \alpha \leq 1.$$

The derivative $x'(t)$ of a fuzzy process x is defined by

$$[x'(t)]^\alpha = [(x_l^\alpha)'(t), (x_r^\alpha)'(t)],\ \ 0 < \alpha \leq 1$$

provided that is equation defines a fuzzy $x'(t) \in E_N$.

The fuzzy integral

$$\int_a^b x(t)dt, \quad a, b \in I$$

is defined by

$$\left[\int_a^b x(t)dt\right]^\alpha = \left[\int_a^b x_l^\alpha(t)dt, \int_a^b x_r^\alpha(t)dt\right]$$

provided that the Lebesgue integrals on the right exist.

Definition 2.3. [1] The fuzzy process $x : I \to E_N$ is a solution of equations (1) and (2) without the inhomogeneous term if and only if

$$\begin{aligned}(\dot{x}_l^\alpha)(t) &= \min\Big\{A_l^\alpha(t)\big[x_j^\alpha(t) \\ &\quad + \int_0^t G(t-s)x_j^\alpha(s)ds\big],\ i,j = l,r\Big\}, \\ (\dot{x}_r^\alpha)(t) &= \max\Big\{A_r^\alpha(t)\big[x_j^\alpha(t) \\ &\quad + \int_0^t G(t-s)x_j^\alpha(s)ds\big],\ i,j = l,r\Big\},\end{aligned}$$

and

$$\begin{aligned}(x_l^\alpha)(0) &= x_{0l}^\alpha - g_l^\alpha(t_1, t_2, \cdots, t_p, x(\cdot)), \\ (x_r^\alpha)(0) &= x_{0r}^\alpha - g_r^\alpha(t_1, t_2, \cdots, t_p, x(\cdot)).\end{aligned}$$

Now we assume the following:

(H1) If the nonlinear function $f : [0,T] \times E_N \times E_N \to E_N$ satisfies a global Lipschitz condition, then there exists a finite constants $k_1, k_2 > 0$ such that

$$\begin{aligned}&d_H([f(s,\xi_1(s),\eta_1(s))]^\alpha,\ [f(s,\xi_2(s),\eta_2(s))]^\alpha) \\ &\leq k_1 d_H([\xi_1(s)]^\alpha, [\xi_2(s)]^\alpha) + k_2\, d_H([\eta_1(s)]^\alpha, [\eta_2(s)]^\alpha)\end{aligned}$$

for all $\xi_1(s), \xi_2(s), \eta_1(s), \eta_2(s) \in E_N$.

(H2) If the nonlinear function $k : [0,T] \times [0,T] \times E_N \to E_N$ satisfies a global Lipschitz condition, then there exists a finite constant $M > 0$ such that

$$d_H([k(t,s,\psi_1(s)]^\alpha, [k(t,s,\psi_2(s))]^\alpha) \le M d_H([\psi_1(s)]^\alpha, [\psi_2(s)]^\alpha)$$

for all $\psi_1(s), \psi_2(s) \in E_N$.

(H3) The nonlinear function $g : E_N \to E_N$ satisfies following inequality

$$d_H([g(\xi_1)]^\alpha, [g(\xi_2)]^\alpha) \le L d_H([\xi_1(\cdot)]^\alpha, [\xi_2(\cdot)]^\alpha),$$

where constant $L > 0$.

(H4) $S(t)$ is a fuzzy number satisfying, for $y \in E_N$ and $S'(t)y \in C^1(I : E_N) \cap C(I : E_N)$, the equation

$$\frac{d}{dt}S(t)y = A\left[S(t)y + \int_0^t G(t-s)S(s)yds\right] = S(t)Ay + \int_0^t S(t-s)AG(s)yds, \ \ t \in I,$$

such that

$$[S(t)]^\alpha = [S_l^\alpha(t), S_r^\alpha(t)],$$

and $S_i^\alpha(t)$ $(i = l, r)$ is continuous. That is, there exists a constant $c > 0$ such that $|S_i^\alpha(t)| \le c$ for all $t \in I$.

(H5) $c(L + k_1 T + k_2 M T^2) < 1$.

3. Existence and Uniqueness

In this section, we consider the existence and uniqueness of fuzzy solution for the equations (1) and (2).

The equations (1) and (2) is related to the following fuzzy integral equation:

$$x(t) = S(t)(x_0 - g(x)) + \int_0^t S(t-s)f(s,x(s), \int_0^s k(s,\tau,x(\tau))d\tau)ds, \tag{3}$$

where $S(t)$ satisfies (H4).

Theorem 3.1. Let $T > 0$, and hypotheses (H1)-(H5) hold. Then, for every $x_0 \in E_N$, the equation (3) has a unique fuzzy solution $x \in C([0,T] : E_N)$.

Proof. For each $\xi(t) \in E_N$ and $t \in [0,T]$, define

$$(G_0\xi)(t) = S(t)(x_0 - g(t_1,t_2,\cdots,t_p,\xi(\cdot))) + \int_0^t S(t-s)f(s,\xi(s), \int_0^s k(s,\tau,\xi(\tau))d\tau)ds.$$

Thus, $G_0\xi : [0,T] \to E_N$ is continuous and $G_0 : C([0,T] : E_N) \to C([0,T] : E_N)$. For $\xi_1, \xi_2 \in C([0,T] : E_N)$, we have

$$\begin{aligned}
&d_H([(G_0\xi_1)(t)]^\alpha, [(G_0\xi_2)(t)]^\alpha) \\
&= d_H\Big([S(t)g(\xi_1)]^\alpha + [\int_0^t S(t-s)f(s,\xi_1(s), \int_0^s k(s,\tau,\xi_1(\tau))d\tau)ds]^\alpha, \\
&\quad [S(t)g(\xi_2)]^\alpha + [\int_0^t S(t-s)f(s,\xi_2(s), \int_0^s k(s,\tau,\xi_2(\tau))d\tau)ds]^\alpha\Big) \\
&\le d_H\Big([S(t)g(\xi_1)]^\alpha, [S(t)g(\xi_2)]^\alpha\Big) \\
&\quad + \int_0^t d_H([S(t-s)f(s,\xi_1(s), \int_0^s k(s,\tau,\xi_1(\tau))d\tau)]^\alpha, \\
&\quad [S(t-s)f(s,\xi_2(s), \int_0^s k(s,\tau,\xi_2(\tau))d\tau)]^\alpha)ds \\
&\le cLd_H([\xi_1(\cdot)]^\alpha, [\xi_2(\cdot)]^\alpha) \\
&\quad + ck_1 \int_0^t d_H([\xi_1(s)]^\alpha, [\xi_2(s)]^\alpha)ds \\
&\quad + ck_2 M \int_0^t (\int_0^s d_H([\xi_1(s)]^\alpha, [\xi_2(s)]^\alpha)d\tau)ds.
\end{aligned}$$

Hence we get

$$\begin{aligned}
&d_\infty((G_0\xi_1)(t), (G_0\xi_2)(t)) \\
&= \sup_{\alpha\in(0,1]} d_H([(G_0\xi_1)(t)]^\alpha, [(G_0\xi_2)(t)]^\alpha) \\
&\le cL \sup_{\alpha\in(0,1]} d_H([\xi_1(\cdot)]^\alpha, [\xi_2(\cdot)]^\alpha) \\
&\quad + ck_1 \int_0^t \sup_{\alpha\in(0,1]} d_H([\xi_1(s)]^\alpha, [\xi_2(s)]^\alpha)ds \\
&\quad + ck_2 M \int_0^t (\int_0^s \sup_{\alpha\in(0,1]} d_H([\xi_1(s)]^\alpha, [\xi_2(s)]^\alpha)d\tau)ds \\
&= cLd_\infty(\xi_1(\cdot), \xi_2(\cdot)) + ck_1 \int_0^t d_\infty(\xi_1(s), \xi_2(s))ds \\
&\quad + ck_2 M \int_0^t (\int_0^s d_\infty([\xi_1(s)]^\alpha, [\xi_2(s)]^\alpha)d\tau)ds.
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
&H_1(G_0\xi_1, G_0\xi_2) \\
&= \sup_{t\in[0,T]} d_\infty((G_0\xi_1)(t), (G_0\xi_2)(t)) \\
&\le c(L + k_1 T + k_2 M T^2)\, H_1(\xi_1, \xi_2).
\end{aligned}$$

By hypothesis (H5), $c(L + k_1 T + k_2 M T^2) < 1$. Hence G_0 is a contraction mapping. Thus, by the Banach fixed point theorem, (3) has a unique fixed point $x \in C([0,T] : E_N)$. □

Example 3.2. Consider the semilinear one dimensional heat equation on a connected domain (0,1) for a material with memory, boundary condition $x(t,0) = x(t,1) = 0$ and with initial condition $x(0,z) = x_0(z)$, $\sum_{k=1}^{p} C_k x(t_k, z) = g(x)$, where $x_0(z) \in E_N$. Let $x(t,z)$ be the internal energy and $f(t, x(t,z), \int_0^t k(t,s,x(t,z))ds) = \tilde{2}tx(t,z)^2 + \int_0^t (t-s)x(s)ds$ be the external heat with memory .

Let $A = \tilde{2}\frac{\partial^2}{\partial z^2}$ and $G(t-s) = e^{-(t-s)}$, then the balance equation becomes

$$\frac{dx(t)}{dt} = \tilde{2}[x(t) - \int_0^t e^{-(t-s)}x(s)ds] + \tilde{2}tx(t)^2 + \int_0^t (t-s)x(s)ds, \; t \in I, \tag{4}$$

$$x(0) = x_0 - \sum_{k=1}^{p} c_k x(t_k, z). \tag{5}$$

Since α-level set of fuzzy number $\tilde{2}$ is $[2]^\alpha = [\alpha + 1, 3 - \alpha]$ for all $\alpha \in [0,1]$, α-level set of $f(t, x(t), \int_0^t k(t,s,x(s))ds)$ is

$$\begin{aligned}
&[f(t,x(t), \int_0^t k(t,s,x(s))ds)]^\alpha \\
&= [t(\alpha+1)(x_l^\alpha(t))^2 + \int_0^t (t-s)x_l^\alpha(t), \\
&\quad t(3-\alpha)(x_r^\alpha(t))^2 + \int_0^t (t-s)x_r^\alpha(t)].
\end{aligned}$$

Further, we have

$$\begin{aligned}
&d_H([f(t,x(t), \int_0^t k(t,s,x(s))ds)]^\alpha, \\
&\quad [f(t,y(t), \int_0^t k(t,s,y(s))ds)]^\alpha) \\
&= d_H\Big([t(\alpha+1)(x_l^\alpha(t))^2 + \int_0^t (t-s)x_l^\alpha(t), \\
&\quad t(3-\alpha)(x_r^\alpha(t))^2 + \int_0^t (t-s)x_r^\alpha(t)], \\
&\quad [t(\alpha+1)(y_l^\alpha(t))^2 + \int_0^t (t-s)y_l^\alpha(t), \\
&\quad t(3-\alpha)(y_r^\alpha(t))^2 + \int_0^t (t-s)y_r^\alpha(t)]\Big) \\
&= t\max\{(\alpha+1)|(x_l^\alpha(t))^2 - (y_l^\alpha(t))^2|, \\
&\quad (3-\alpha)|(x_r^\alpha(t))^2 - (y_r^\alpha(t))^2|\} \\
&\quad + \int_0^t (t-s)d_H([x_l^\alpha(s), x_r^\alpha(s)], [y_l^\alpha(s), y_r^\alpha(s)]) \\
&\leq 3T|x_r^\alpha(t) + y_r^\alpha(t)| \\
&\quad \times \max\{|x_l^\alpha(t) - y_l^\alpha(t)|, |x_r^\alpha(t) - y_r^\alpha(t)|\} \\
&\quad + \frac{T^2}{2}\max\{|x_l^\alpha(t) - y_l^\alpha(t)|, |x_r^\alpha(t) - y_r^\alpha(t)|\} \\
&= k_1 d_H([x(t)]^\alpha, [y(t)]^\alpha) + k_2 d_H([x(t)]^\alpha, [y(t)]^\alpha),
\end{aligned}$$

where k_1 and k_2 satisfy the inequality in hypotheses (H1) and (H2), and also we have

$$\begin{aligned}
&d_H([g(x)]^\alpha, [g(y)]^\alpha) \\
&\leq |\sum_{k=1}^{p} C_k| \max_k d_H([x(t_k)]^\alpha, [y(t_k)]^\alpha) \\
&= L d_H([x(t_k)]^\alpha, [y(t_k)]^\alpha),
\end{aligned}$$

where L satisfies the inequality in hypothesis (H3). Then all the conditions stated in Theorem 3.1 are satisfied, so the problems (4) and (5) has a unique fuzzy solution.

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