ON THE SCALAR AND DUAL FORMULATIONS OF THE CURVATURE THEORY OF LINE TRAJECTORIES IN THE LORENTZIAN SPACE

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ABSTRACT. This paper develops in detail the differential geometry of ruled surfaces from two perspectives, and presents the underlying relations which unite them. Both scalar and dual curvature functions which define the shape of a ruled surface are derived. Explicit formulas are presented for the computation of these functions in both formulations of the differential geometry of ruled surfaces. Also presented is a detailed analysis of the ruled surface which characterizes the shape of a general ruled surface in the same way that osculating circle characterizes locally the shape of a non-null Lorentzian curve.

1. Introduction

Dual numbers were introduced by W. K. Clifford (1849 − 79) as a tool for his geometrical investigations. After him E. Study used dual numbers and dual vectors in his research on the geometry of lines and kinematics. He devoted special attention to the representation of directed lines by dual unit vectors and defined the mapping that is known by his name. There exist one-to-one correspondence between the vectors of dual unit sphere S^2 and the directed lines of space of lines $\mathbb{R}^3$ [3].

If we take the Minkowski 3-space $\mathbb{R}^3_1$ instead of $\mathbb{R}^3$ the E. Study mapping can be stated as follows: The dual timelike and spacelike unit vectors of dual hyperbolic and Lorentzian unit spheres H^2_0 and S^2_1 at the dual Lorentzian space $\mathbb{D}^3_1$ are in one-to-one correspondence with the directed timelike and spacelike lines of the space of Lorentzian lines $\mathbb{R}^3_1$,

respectively [8, 10]. Then a differentiable curve on H_0^2 corresponds to a timelike ruled surface at $\mathbb{R}_1^3$. Similarly the timelike (resp. spacelike) curve on S_1^2 corresponds to any spacelike (resp. timelike) ruled surface at $\mathbb{R}_1^3$.

The curvature theory of line trajectories seeks to characterize the shape of the trajectory ruled surface and relate it to the motion of body carrying the line that generates it.

Since a ruled surface is a special case of a smooth surface, its differential geometry can be developed using traditional techniques of vector calculus. McCarthy and Roth [5] used this approach to obtain a scalar curvature theory of line trajectories for spatial kinematics. Ruled surfaces have a unique feature not shared by general surfaces, the presence of a uniquely defined curve, its striction curve. This curve coupled with a reference frame formed from the direction of the rulings and the surface normal combine to yield differential properties of the ruled surface which completely define it. This formulation is as simple as the Frenet formulas which define the shape of a space curve, see McCarthy and Roth [5].

In this paper, using the same way as in McCarthy [4], we derive the scalar and dual Lorentzian formulations of the curvature theory of line trajectories and expose the fundamental curvature functions that characterize the shape of a ruled surface in the Lorentzian space.

2. Preliminaries

If a and a^* are real numbers and $\xi^2 = 0$, the combination $A = a + \xi a^*$ is called a dual number, where ξ is dual unit.

The set of all dual numbers forms a commutative ring over the real number field and is denoted by $\mathbb{D}$. Then the set

$$\mathbb{D}^3 = \{\hat{a} = (A_1, A_2, A_3) \mid A_i \in \mathbb{D},\ 1 \leq i \leq 3\}$$

is a module over the ring $\mathbb{D}$ which is called a $\mathbb{D}$–module or dual space. The elements of $\mathbb{D}^3$ are called dual vectors. Thus a dual vector $\hat{a}$ can be written

$$\hat{a} = a + \xi a^*,$$

where a and a^* are real vectors at $\mathbb{R}^3$ [9].

The Lorentzian inner product of dual vectors $\hat{a}$ and $\hat{b}$ is defined by

$$\langle \hat{a}, \hat{b} \rangle = \langle a, b \rangle + \xi(\langle a, b^* \rangle + \langle a^*, b \rangle)$$

with $\hat{a} = a + \xi a^*$ and $\hat{b} = b + \xi b^*$, where the Lorentzian inner product of the vectors $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3) \in \mathbb{R}_1^3$ is

$$\langle a, b \rangle = a_1 b_1 + a_2 b_2 - a_3 b_3.$$

A dual vector $\hat{a}$ is said to be timelike if $\langle \hat{a}, \hat{a} \rangle < 0$, spacelike if $\langle \hat{a}, \hat{a} \rangle > 0$ or $\hat{a} = 0$ and lightlike (or null) if $\langle \hat{a}, \hat{a} \rangle = 0$ and $\hat{a} \neq 0$, where $\langle , \rangle$ is a Lorentzian inner product with signature $(+, +, -)$. The norm of a dual vector $\hat{a}$ is defined to be

$$\|\hat{a}\| = \|a\| + \xi \frac{\langle a, a^* \rangle}{\|a\|}.$$

We denote the set of all dual Lorentzian vectors by $\mathbb{D}_1^3$. Then we have the following definition.

The hyperbolic and Lorentzian unit spheres are

$$H_0^2 := \{\hat{a} = a + \xi a^* \in \mathbb{D}_1^3 \,\big|\, \langle \hat{a}, \hat{a} \rangle = -1; \; a, a^* \in \mathbb{R}_1^3\}$$

and

$$S_1^2 = \{\hat{a} = a + \xi a^* \in \mathbb{D}_1^3 \,\big|\, \langle \hat{a}, \hat{a} \rangle = 1; \; a, a^* \in \mathbb{R}_1^3\},$$

respectively [6, 10].

The dual Lorentzian cross-product of $\hat{a}$ and $\hat{b}$ is defined as

$$\hat{a} \times \hat{b} = a \times b + \xi(a \times b^* + a^* \times b)$$

with the Lorentzian cross-product a and b

$$a \times b = (a_1, a_2, a_3) \times (b_1, b_2, b_3) = (a_3 b_2 - a_2 b_3, \; a_1 b_3 - a_3 b_1, \; a_1 b_2 - a_2 b_1)$$

(see [1], [2]).

3. A ruled surface

Given two points p and q in the a rigid body $\mathcal{M}$ we define the line $\ell(t)$ joining them by the equation

(3.1) $$\ell(t) = p + t(q - p), \qquad t \in \mathbb{R}.$$

The trajectory traced by $\ell(t)$, denoted $\mathcal{L}(\psi, t)$, is a ruled surface in the fixed reference frame $\mathcal{F}$ defined by the equation

(3.2) $$\mathcal{L}(\psi, t) = P(\psi) + t(Q(\psi) - P(\psi)) = P(\psi) + t X(\psi),$$

where $P(\psi)$ and $Q(\psi)$ are the trajectories of p and q, and ψ is the motion parameter. $P(\psi)$ is a general non-null Lorentzian space curve, called the directrix of the ruled surface, and $X(\psi) = Q(\psi) - P(\psi)$ is a curve on the hyperbolic unit sphere H_0^2 of radius $r = \|p - q\|$ called the spherical indicatrix. $X(\psi)$ is unique since any other pair of points p' and

q' chosen on $\ell(t)$ results in the same spherical indicatrix, differing only in the radius r of the hyperbolic unit sphere. In contrast, the directrix is not unique since any curve on $\mathcal{L}(\psi, t)$ of the form

$$(3.3) \qquad C(\psi) = P(\psi) + \mu(\psi)X(\psi),$$

where $\mu(\psi)$ is a smooth function, may be used as its directrix.

Now we consider three unit Lorentzian vectors T, G, and X which form a trihedron defining respectively the central normal, asymptotic normal, and the direction along generator L_X of the ruled surface $\mathcal{L}(\psi, t)$, where the vectors T and G are spacelike and the vector X is timelike. Thus

$$\begin{aligned}
\langle T, T \rangle &= \langle G, G \rangle = 1, \quad \langle X, X \rangle = -1 \quad \text{and} \\
\langle T, G \rangle &= \langle T, X \rangle = \langle G, X \rangle = 0.
\end{aligned}$$

The central normal T of the ruled surface $\mathcal{L}(\psi, t)$ is given by

$$(3.4) \qquad T = \frac{dX/d(\psi)}{\|dX/d(\psi)\|}.$$

The set of striction points on $\mathcal{L}(\psi, t)$ is its striction curve, $C(\psi)$. It is defined in terms of the directrix $P(\psi)$ by relation

$$(3.5) \qquad C(\psi) = P(\psi) - \frac{\langle \frac{dP}{d\psi}, \frac{dX}{d\psi} \rangle}{\langle \frac{dX}{d\psi}, \frac{dX}{d\psi} \rangle} X(\psi)$$

[7]. The trihedron T, G, X with its origin located on the striction curve of $\mathcal{L}(\psi, t)$ is the natural trihedron of the ruled surface.

4. The scalar Lorentzian formulation

The differential geometry of surfaces uses the angular variation of a natural reference frame on the surface measured relative to itself to characterize its local properties. The equations which result are the structure equations of the surface and are a generalization of the Frenet equations of a non-null Lorentzian curve. General surfaces do not have the characteristic striction curve that a ruled surface has. The presence of this curve allows the consideration of the angular and positional variation of the natural trihedron T, G, X as it follows $C(\psi)$. The resulting equations completely characterize the local properties of the ruled surface in a simple form.

The shape of the ruled surface $\mathcal{L}(\psi, t)$ is independent of the parameter ψ chosen to identify the sequence of lines along it therefore we choose

a standard parametrization. It convenient to use the arc-length of the spherical indicatrix $X(\psi)$ as this standard parameter. The arc-length parameter s is defined by the equation

$$(4.1) \qquad s(\psi) = \int_0^{\psi} \left\| \frac{dX}{d\psi} \right\| d\psi.$$

Here $\delta = \left\| \frac{dX}{d\psi} \right\|$ is called the speed of $X(\psi)$. If $\delta \neq 0$ then equation (4.1) can be inverted to yield $\psi(s)$ allowing the definition of $X(\psi(s)) = X(s)$. $X(s)$ has unit speed, that is its tangent vector is of unit magnitude.

The angular variation of the frame T, G, X is obtained by computing $\frac{dT}{ds}$ and $\frac{dG}{ds}$ in terms of T, G, X. Since $\langle T, X \rangle = 0$, differentiating this expression with respect to s we get $\langle \frac{dT}{ds}, X \rangle = -1$. Defining

$$\gamma = \langle \frac{dT}{ds}, G \rangle$$

as the function γ we obtain the geodesic Frenet equations of $X(\psi)$:

$$(4.2) \qquad \begin{array}{rcl} \frac{dX}{ds} & = & T \\ \frac{dT}{ds} & = & X + \gamma G \\ \frac{dG}{ds} & = & -\gamma T. \end{array}$$

The function γ is the geodesic curvature of $X(\psi)$. These equations may be viewed as a set of linear differential equations in the components of the trihedron T, G, X. If the geodesic curvature γ is specified these equations can be solved for $X(\psi)$. Thus γ completely characterizes the spherical indicatrix of $\mathcal{L}(\psi, t)$ [3].

The formula for γ is obtained in terms of $X(\psi)$ and its derivatives with respect to ψ is:

$$(4.3) \qquad \gamma = \frac{1}{\left\| \frac{dX}{d\psi} \right\|^3} \langle \frac{d^2 X}{d\psi^2}, \frac{dX}{d\psi} \times X \rangle.$$

The positional variation of the trihedron T, G, X is given by $\frac{dC}{ds}$; note ψ is replaced by s, the arc-length of $X(\psi)$. $\frac{dC}{ds}$ expanded in terms of the frame T, G, X is

$$(4.4) \qquad \frac{dC}{ds} = -\langle \frac{dC}{ds}, X \rangle X + \langle \frac{dC}{ds}, T \rangle T + \langle \frac{dC}{ds}, G \rangle G.$$

Using (3.5) and the chain rule differentiation we compute

$$(4.5) \qquad \frac{dC}{ds} = -\Gamma X + \Delta G,$$

where

(4.6)
$$\Delta = \frac{1}{\delta}\langle\frac{dP}{d\psi}, G\rangle = \frac{1}{\left\|\frac{dX}{d\psi}\right\|^2}\langle\frac{dP}{d\psi}, \frac{dX}{d\psi} \times X\rangle,$$

and

(4.7)
$$\Gamma = \frac{1}{\delta}[\langle\frac{dP}{d\psi}, X\rangle - \frac{d\mu}{d\psi}].$$

If the Lorentzian vectors G and X are known from the geodesic Frenet equations then, given Δ and Γ, (4.5) is a set of linear differential equations which can be solved for $C(s)$. Thus the three functions γ, Δ, and Γ completely define the ruled surface $\mathcal{L}(\psi, t)$. The functions γ, Δ, and Γ are called the curvature functions of the ruled surface $\mathcal{L}(\psi, t)$.

5. The central normal surface

As the trihedron T, G, X moves along the striction curve of $\mathcal{L}(\psi, t)$ the two Lorentzian vectors T and G generate ruled surfaces associated with $\mathcal{L}(\psi, t)$. Of primary importance is the ruled surface generated by T called the central normal surface of $\mathcal{L}(\psi, t)$. This surface, denoted $L_T(s, t)$, is defined by the equation

(5.1)
$$L_T(s, t) = C(s) + tT(s).$$

The unit surface normal U_T of $L_T(s, t)$ is obtained as

$$U_T(s, t) = \frac{\frac{dL}{ds} \times \frac{dL}{dt}}{\left\|\frac{dL}{ds} \times \frac{dL}{dt}\right\|}$$

(5.2)
$$= \frac{[\frac{dC}{ds} + t\frac{dT}{ds}] \times T}{\{\left|\langle(\frac{dC}{ds} + t\frac{dT}{ds}), (\frac{dC}{ds} + t\frac{dT}{ds})\rangle + \langle\frac{dC}{ds}, T\rangle^2\right|\}^{1/2}}.$$

The asymptotic normal B and central normal N directions of $L_T(s, t)$ are given by

(5.3)
$$B = -\frac{1}{\varkappa}\frac{dT}{ds} \times N,$$
$$N = \frac{1}{\varkappa}\frac{dT}{ds},$$

where $\varkappa = \left\|\frac{dT}{ds}\right\|$ [4]. Thus the natural trihedron of $L_T(s, t)$ is the reference frame T, N, B.

The striction curve of $L_T(s, t)$ is obtained as

(5.4)
$$C_T(s) = C(s) - P(s)T(s),$$

where

$$(5.5) \qquad P(s) = \frac{\langle \frac{dC}{ds}, \frac{dT}{ds} \rangle}{\left\| \frac{dT}{ds} \right\|^2} = \frac{\Gamma + \Delta\gamma}{\gamma^2 - 1}.$$

The frame T, N, B happens to be the Frenet reference frame of the spherical indicatrix $X(\psi)$ considered as a general non-null Lorentzian space curve. Thus, we get the Frenet equations

$$(5.6) \qquad \begin{aligned} \frac{dT}{ds} &= \varkappa N, \\ \frac{dN}{ds} &= -\varkappa T + \tau B, \\ \frac{dB}{ds} &= \tau N, \end{aligned}$$

where $\varkappa$ and τ are the curvature and torsion of $X(\psi)$. These equations describe the angular variation of the T, N, B frame of the central normal surface, $L_T(s,t)$.

The positional variation of the trihedron T, N, B as it moves along the striction curve $C_T(s)$ is given by:

$$(5.7) \qquad \frac{dC_T(s)}{ds} = \mathcal{T}T - \mathcal{K}B,$$

where

$$(5.8) \qquad \mathcal{K} = \langle \frac{dC_T(s)}{ds}, B \rangle = \frac{\Delta + \Gamma\gamma}{\varkappa},$$

and

$$(5.9) \qquad \mathcal{T} = \langle \frac{dC_T(s)}{ds}, T \rangle = -\frac{dP}{ds}.$$

The functions $\varkappa$, τ, $\mathcal{K}$ and $\mathcal{T}$ characterize the central normal surface $L_T(s,t)$ in the same way that γ, Δ, and Γ characterize $\mathcal{L}(\psi,t)$. The extra function $\varkappa$ appears because the parametrization of $L_T(s,t)$ was not normalized by the arc-length of its spherical indicatrix $T(s)$ as was done for $\mathcal{L}(\psi,t)$.

The relative orientation of the two reference frames T, G, X and T, N, B is given by the relations

$$(5.10) \qquad \begin{aligned} N &= \tfrac{1}{\varkappa}X + \tfrac{\gamma}{\varkappa}G, \\ B &= \tfrac{1}{\varkappa}G + \tfrac{\gamma}{\varkappa}X. \end{aligned}$$

The first of these is obtained by equating $\frac{dT}{ds}$ in equations (4.2) and (5.6) and the second by computing the Lorentzian cross-product of the resulting expression with T. Equation (5.10) shows that in the plane

normal to T the frame G, X is rotated relative to the frame N, B by the hyperbolic angle

$$(5.11) \qquad\qquad \rho = \operatorname{arcsh}(\frac{1}{\varkappa}).$$

This hyperbolic angle is the Lorentzian spherical radius of curvature of $X(\psi)$ and is measured from the Lorentzian vector B to X.

The plane spanned by the vectors T, N is the osculating plane of $X(\psi)$. In the osculating plane lies the circle which best approximates $X(\psi)$, called the osculating circle. It is defined by the vector equation

$$S(\psi) = \operatorname{sh}\rho \sin\psi T - \operatorname{sh}\rho \cos\psi N + \operatorname{ch}\rho B.$$

The radius of the osculating circle is $R = \operatorname{sh}\rho$.

The functions γ, $\varkappa$ and τ are not independent. The relations between them are obtained from equations (5.10). Since N and B are unit spacelike and timelike vectors, respectively, we have

$$(5.12) \qquad\qquad \varkappa = \sqrt{\gamma^2 - 1}.$$

Taking into consideration hyperbolic angle ρ between the frames G, X and N, B and the equation $\frac{dB}{ds} = \tau N$ we obtain

$$(5.13) \qquad\qquad \tau = \frac{d\rho}{ds}.$$

Since it is possible to obtain the curvature functions γ, Δ, and Γ directly from the functions $\varkappa$, $\mathcal{K}$, ρ and P, using equations (5.5), (5.8) and (5.12), this latter set of functions also completely characterizes the ruled surface $\mathcal{L}(\psi, t)$.

6. The shape of a ruled surface

The results derived thus far provide a kinematic interpretation of the shape of the surface $\mathcal{L}(\psi, t)$ in terms of the motion of a line, L_X, directed along $X(\psi)$ and passing through the striction curve $C(\psi)$. At a particular instant $\psi = \psi_0$ the velocity of L_X is given by its instantaneous rotation about the central tangent vector G together with its instantaneous translation along G. The ratio of these two quantities is the function Δ, defined in (4.6), known as the distribution parameter of $\mathcal{L}(\psi, t)$.

The curvature of the path of L_X is most easily described by considering the velocity of the line L_T along the central normal T of $\mathcal{L}(\psi, t)$. L_T rotates about and translates along its central tangent B with angular velocity $\varkappa$ and linear velocity $\mathcal{K}$, refer to (5.6) and (5.8). The ratio of

these values is the distribution parameter $\Delta_T = \frac{\mathcal{K}}{\varkappa}$ of $L_T(s,t)$. The line along the central tangent of $L_T(s,t)$ is L_B. L_T is perpendicular to and intersects both L_X and L_B. P, given (5.5), is the distance from L_B to L_X; and ρ, defined by (5.11), is the hyperbolic angle from L_B to L_X measured about L_T.

The line L_B (the central tangent of the central normal surface) is fixed to the second order as L_X moves about it with pitch $\Delta_T = \frac{\mathcal{K}}{\varkappa}$, holding P and ρ constant. Thus locally the ruled surface $\mathcal{L}(\psi,t)$ is traced during a screw displacement of pitch Δ_T about the axis L_B, by the line L_X located at a distance P and hyperbolic angle ρ relative to L_B. L_B is called the Disteli axis of the ruled surface, it is analogous to the center of curvature of a non-null Lorentzian curve.

7. The dual Lorentzian vector formulation

In this section we present the differential geometry of ruled surfaces in terms of three dimensional dual Lorentzian vector calculus. The result is a set of dual functions which characterize the ruled surface, (see [9]).

Dual Lorentzian vector calculus allows the Plücker vectors X and $P \times X$ of a line L (X is the direction of L and P is any point on L) to be assembled into a single dual timelike vector $\hat{X} = X + \xi P \times X$. The symbol ξ is called the dual unit. Operations with ξ are the same as with real numbers except that $\xi^2 = 0$. All the operations of vector algebra are available for the manipulation of dual Lorentzian vectors. This fact reduces computations representing lines in the Lorentzian space to simple vector operations.

A ruled surface $\mathcal{L}(\psi,t) = P(\psi) + tX(\psi)$ is written as the dual Lorentzian vector function $\hat{X}(\psi)$ given by

$$(7.1) \qquad \hat{X}(\psi) = X(\psi) + \xi P(\psi) \times X(\psi).$$

Since the spherical image $X(\psi)$ is a unit timelike vector the dual vector $\hat{X}(\psi)$ also has unit magnitude as is seen from the computation:

$$(7.2) \qquad \begin{aligned} \langle \hat{X}(\psi), \hat{X}(\psi) \rangle &= \langle X + \xi P \times X, X + \xi P \times X \rangle \\ &= \langle X, X \rangle + \xi(\langle X, P \times X \rangle \\ &\quad + \xi^2 \langle P \times X, P \times X \rangle) \\ &= \langle X, X \rangle = -1. \end{aligned}$$

Thus the ruled surface becomes a dual non-null Lorentzian curve on a dual hyperbolic unit sphere H_0^2.

The dual arc-length of the ruled surface $\hat{X}(\psi)$ is defined as

$$(7.3) \qquad \hat{s}(\psi) = \int_0^\psi \left\| \frac{d\hat{X}(\psi)}{d\psi} \right\| d\psi.$$

The integrand of (7.3) is the dual velocity, $\hat{\delta}$, of $\hat{X}(\psi)$ and is given by

$$(7.4) \qquad \hat{\delta} = \left\| \frac{d\hat{X}(\psi)}{d\psi} \right\| = \left\| \frac{dX}{d\psi} \right\| \left(1 + \xi \frac{\langle \frac{dX}{d\psi}, P \times X \rangle}{\left\| \frac{dX}{d\psi} \right\|^2} \right),$$

which in view of equation (4.6) is

$$(7.5) \qquad \hat{\delta} = \delta(1 + \xi\Delta).$$

As long as $\delta \neq 0$, the dual function $\hat{s}(\psi)$ can be inverted to yield the real function of a dual parameter $\psi(\hat{s})$ which we use to reparameterize the dual non-null Lorentzian curve. To compute derivatives with respect to $\hat{s}$ we note that by definition $\hat{s}(\psi(\hat{s})) = \hat{s}$; differentiation of this expression with respect to $\hat{s}$ yields

$$(7.6) \qquad \frac{d\hat{s}}{d\psi} \cdot \frac{d\psi}{d\hat{s}} = 1,$$

from which we obtain

$$\frac{d\psi}{d\hat{s}} = \frac{1}{\hat{\delta}}$$

and

$$(7.7) \qquad \frac{d}{d\hat{s}} = \frac{1}{1 + \xi\Delta} \frac{d}{ds}.$$

We reparameterize $\hat{X}(\psi)$ to obtain $\hat{X}(\psi(\hat{s}))$, such that

$$(7.8) \qquad \hat{T} = \frac{d\hat{X}}{d\hat{s}} = \frac{d\hat{X}}{d\psi} \cdot \frac{d\psi}{d\hat{s}} = \frac{1}{\hat{\delta}} \frac{d\hat{X}}{d\psi}.$$

Thus as in the case of spherical curves the dual arc-length parameter normalizes the representation of the ruled surface $\hat{X}(\psi)$ such that its dual tangent vector $\hat{T}$ has unit magnitude.

Using (7.7) we derive the dual Lorentzian form of the geodesic Frenet equations in exactly the same way as the geodesic Frenet equations (4.2):

$$(7.9) \qquad \begin{array}{rcl} \frac{d\hat{X}}{d\hat{s}} & = & \hat{T}, \\ \frac{d\hat{T}}{d\hat{s}} & = & \hat{X} + \hat{\gamma}\hat{G}, \\ \frac{d\hat{G}}{d\hat{s}} & = & -\hat{\gamma}\hat{T}. \end{array}$$

The dual geodesic curvature $\hat{\gamma}$ is defined as

$$(7.10) \qquad \hat{\gamma} = \langle \frac{d\hat{T}}{d\hat{s}}, \hat{G} \rangle = \frac{1}{\hat{\delta}^3} \langle \frac{d^2\hat{X}}{d\psi^2}, \frac{d\hat{X}}{d\psi} \times \hat{X} \rangle.$$

As in the case of spherical curves we can derive the dual Lorentzian form of the usual Frenet equations of $\hat{X}(\psi)$:

$$(7.11) \qquad \begin{aligned} \frac{d\hat{T}}{d\hat{s}} &= \hat{\varkappa}\hat{N}, \\ \frac{d\hat{N}}{d\hat{s}} &= -\hat{\varkappa}\hat{T} + \hat{\tau}\hat{B}, \\ \frac{d\hat{B}}{d\hat{s}} &= \hat{\tau}\hat{N}. \end{aligned}$$

The trihedrons $\hat{T}, \hat{G}, \hat{X}$ and $\hat{T}, \hat{N}, \hat{B}$ are related by the equations

$$(7.12) \qquad \begin{aligned} \hat{N} &= \tfrac{1}{\hat{\varkappa}}\hat{X} + \tfrac{\hat{\gamma}}{\hat{\varkappa}}\hat{G}, \\[1mm] \hat{B} &= \tfrac{\hat{\gamma}}{\hat{\varkappa}}\hat{X} + \tfrac{1}{\hat{\varkappa}}\hat{G}. \end{aligned}$$

The dual hyperbolic angle $\hat{\rho}$ between the frames $\hat{G}, \hat{X}$ and $\hat{N}, \hat{B}$ is defined by the relation

$$(7.13) \qquad \hat{\rho} = \text{arcsh}(\frac{1}{\hat{\varkappa}}).$$

From (7.12) and (7.13) we also obtain the relations

$$(7.14) \qquad \begin{aligned} \hat{\varkappa} &= \sqrt{\hat{\gamma}^2 - 1}, \\ \hat{\tau} &= \frac{d\hat{\rho}}{d\hat{s}}. \end{aligned}$$

8. Uniting the two formulations

It is easy to see the dual Lorentzian vector calculus is a convenient tool for the analysis of ruled surfaces in the form of dual Lorentzian spherical curves. The derivations consistently yield formulas which are identical to those obtained in the differential geometry of non-null Lorentzian curves on a hyperbolic unit sphere. In a way this tool is too efficient, it suppresses important geometric concepts such as the striction curve and Disteli axis as well as the distribution parameter and the other curvature functions. We now open up the dual Lorentzian vector formalism to expose the relation between the reference frames $\hat{T}, \hat{G}, \hat{X}$ and $\hat{T}, \hat{N}, \hat{B}$ and the natural trihedron, central normal surface, and Disteli axis of $\mathcal{L}(\psi, t)$; and to obtain the dual curvature functions $\hat{\gamma}, \hat{\varkappa}$ and $\hat{\tau}$ in terms of the functions γ, Δ, Γ and $\varkappa, \mathcal{K}, \rho$, and P.

The dual Lorentzian vector $\hat{T}$ obtained in (7.8) represents a line which we denote as L_T, we will soon see that it is the central normal of $\hat{X}(\psi)$. The Plücker vectors of L_T provide information about a point C on L_T via the relation

$$(8.1) \qquad \hat{T} = T + \xi C \times T.$$

To determine C we compute $\frac{d\hat{X}}{d\hat{s}}$ directly and obtain

$$(8.2) \qquad \frac{d\hat{X}}{d\hat{s}} = T + \xi \Big(P - \frac{\langle \frac{dP}{d\psi}, T \rangle}{\langle \frac{dX}{d\psi}, \frac{dX}{d\psi} \rangle} X \Big) \times T.$$

Thus C is the striction point of $\hat{X}(\psi)$, see equation (3.5); and we see that $\hat{T}$ is the central normal of $\mathcal{L}(\psi, t)$.

As one might expect the dual Lorentzian vector $\hat{G} = \hat{T} \times \hat{X}$ defines the line corresponding to the central tangent of $\mathcal{L}(\psi, t)$

$$(8.3) \qquad \begin{aligned} \hat{G} &= (T + \xi C \times T)(X + \xi C \times X) \\ &= T \times X + \xi[C \times (T \times X)]. \end{aligned}$$

Thus the trihedron of dual Lorentzian vectors $\hat{T}$, $\hat{G}$, $\hat{X}$, is the natural trihedron of the ruled surface $\mathcal{L}(\psi, t)$. The dual geodesic Frenet equations (7.11) define the differential motion of this reference frame.

The geodesic curvature $\hat{\gamma}$ characterizes the motion of the $\hat{T}$, $\hat{G}$, $\hat{X}$ frame. We determine $\hat{\gamma}$ by expanding $\frac{d\hat{T}}{d\hat{s}}$ to obtain

$$(8.4) \qquad \frac{d\hat{T}}{d\hat{s}} = \frac{1}{1 + \xi\Delta} \Big[\frac{dT}{ds} + \xi(\frac{dC}{ds} \times T + C \times \frac{dT}{ds}) \Big].$$

This expression is further expanded using equations (4.2) and (4.5) to yield

$$(8.5) \quad \frac{d\hat{T}}{d\hat{s}} = \frac{1}{1 + \xi\Delta} (X + \gamma G + \xi((-\Gamma X + \Delta G) \times T + C \times (X + \gamma G))).$$

Computing the Lorentzian cross-product and collecting terms we obtain

$$(8.6) \quad \frac{d\hat{T}}{d\hat{s}} = \frac{1}{1 + \xi\Delta} [(\gamma + \xi\Gamma)(G + \xi C \times G) + (1 - \xi\Delta)(X + \xi C \times X)].$$

Comparing (7.9) and (8.6) we see that $\hat{\gamma}$ is defined in terms of γ, Δ and Γ as:

$$(8.7) \qquad \hat{\gamma} = \frac{\gamma + \xi\Gamma}{1 + \xi\Delta} = \gamma + \xi(\Gamma - \gamma\Delta).$$

Now consider the dual Lorentzian unit vector $\hat{N}$ defined in (7.11). Expanding $\frac{d\hat{T}}{d\hat{s}}$ (which equals $\hat{\varkappa}\hat{N}$), this time in the $\hat{T}$, $\hat{N}$, $\hat{B}$ frame, we obtain

$$(8.8) \qquad \frac{d\hat{T}}{d\hat{s}} = \frac{1}{1+\xi\Delta}[\varkappa N + \xi\varkappa(C - \frac{\langle\frac{dC}{ds}, N\rangle}{\varkappa}T) \times N + \xi\langle\frac{dC}{ds}, B\rangle N].$$

This expression may be rewritten in terms of the striction curve $C_T(s)$ of the central normal surface $L_T(s,t)$ to yield

$$(8.9) \qquad \frac{d\hat{T}}{d\hat{s}} = \frac{1}{1+\xi\Delta}[(\varkappa + \xi\mathcal{K})(N + \xi C_T \times N)].$$

Thus $\hat{N}$ is the line through the striction point C_T of $L_T(s,t)$ in the direction of its central normal N. Comparing (8.9) and (7.11) we see that the dual curvature $\hat{\varkappa}$ is given by

$$(8.10) \qquad \hat{\varkappa} = \frac{1}{1+\xi\Delta}(\varkappa + \xi\mathcal{K}) = \varkappa + \xi(\mathcal{K} - \varkappa\Delta).$$

The dual Lorentzian vector $\hat{B} = \hat{T} \times \hat{N}$ is the central tangent of $L_T(s,t)$, that is $\hat{B}$ is the Disteli axis of the ruled surface $\mathcal{L}(\psi, t)$. The dual Lorentzian Frenet frame $\hat{T}$, $\hat{N}$, $\hat{B}$ is the natural trihedron of the central normal surface $\hat{T}(s)$.

We now examine the dual torsion $\hat{\tau}$ of $\hat{X}(\psi)$. It is defined by

$$(8.11) \qquad \hat{\tau} = -\langle\frac{d\hat{N}}{d\hat{s}}, \hat{B}\rangle.$$

Differentiating (8.9) to obtain $\frac{d\hat{N}}{d\hat{s}}$ and using (5.6) and (5.7) to simplify the resulting expression, we obtain

$$(8.12) \qquad \hat{\tau} = -\tau + \xi(\tau\Delta - \mathcal{T}).$$

Consider now the coefficients $\frac{1}{\hat{\varkappa}}$ and $\frac{\hat{\gamma}}{\hat{\varkappa}}$ in equations (7.12). Using the formulas (8.7) and (8.10) for $\hat{\varkappa}$ and $\hat{\gamma}$, we obtain

$$(8.13) \qquad \begin{array}{rcl} \frac{1}{\hat{\varkappa}} & = & \frac{1}{\varkappa} + \xi\frac{\varkappa\Delta - \gamma\mathcal{K}}{\varkappa^2}, \\[2mm] \frac{\hat{\gamma}}{\hat{\varkappa}} & = & \frac{\gamma}{\varkappa} + \xi\frac{\varkappa\Gamma - \gamma\mathcal{K}}{\varkappa^2}. \end{array}$$

P, which is defined by equation (5.5), is the normal distance along the central tangent of $\mathcal{L}(\psi, t)$ measured from the Disteli axis L_B to the line L_X. The dual number $\hat{\rho} = \rho + \xi P$ is the dual hyperbolic angle of L_X

measured relative to L_B. The sh and ch functions of a dual hyperbolic angle are defined to be

$$
\text{(8.14)} \qquad
\begin{aligned}
\text{sh}\hat{\rho} &= \text{sh}\rho + \xi P \text{ch}\rho \\
\text{ch}\hat{\rho} &= \text{ch}\rho + \xi P \text{sh}\rho.
\end{aligned}
$$

Substituting (5.11) which defines ρ into (8.13) and comparing the result to (8.14) we see that

$$
\text{(8.15)} \qquad
\begin{aligned}
\frac{1}{\hat{\varkappa}} &= \text{sh}\hat{\rho}, \\[2mm]
\frac{\hat{\gamma}}{\hat{\varkappa}} &= \text{ch}\hat{\rho}.
\end{aligned}
$$

The dual hyperbolic angle $\hat{\rho} = \rho + \xi P$ is the dual pseudo spherical radius of curvature.

The formulas (8.7), (8.10), (8.12) and (8.13) unite the dual and scalar Lorentzian formulations of the differential geometry of ruled surfaces. It seems clear that no matter how the ruled surface is represented the curvature functions γ, Δ, and Γ, or the equivalent set $\varkappa$, $\mathcal{K}$, ρ, and P, contain the fundamental geometric information describing the shape of the surface.

EXAMPLE 1. The ruled surface traced by a line fixed in a rigid body undergoing a screw displacement of constant pitch is fundamental to the curvature theory of ruled surfaces. It is the analog for a ruled surface of the osculating circle of a curve [4]. Now we shall examine its properties in some detail.

This surface is generated by the line L_S carried by a the helix of the 1st kind of radius a and pitch b in $\mathbb{R}^3_1$ with respect to Lorentzian inner product with signature $(+, +, -)$. Align the axis of the helix of the 1st kind with the timelike z-coordinate axis so the equation of the helix of the 1st kind is

$$
P(\psi) = (a\cos\psi, a\sin\psi, b\psi).
$$

The direction, $S(\psi)$, of L_S lies in the plane normal to the radius vector $r = (\cos\psi, \sin\psi, 0)$ and at a hyperbolic angle θ from the timelike z-axis direction z, that is,

$$
\begin{aligned}
S(\psi) &= -\text{sh}\theta r^{\perp} + \text{ch}\theta z \\
&= -\text{sh}\theta(-\sin\psi, \cos\psi, 0) + \text{ch}\theta(0,0,1) \\
&= (\text{sh}\theta\sin\psi, -\text{sh}\theta\cos\psi, \text{ch}\theta).
\end{aligned}
$$

Thus we get the ruled surface $L_S(\psi, t)$:

$$
\begin{aligned}
L_S(\psi, t) &= P(\psi) + tS(\psi) \\
&= (a\cos\psi, a\sin\psi, b\psi) + t(\text{sh}\theta\sin\psi, -\text{sh}\theta\cos\psi, \text{ch}\theta),
\end{aligned}
$$

where θ, a, and b are constants. Therefore, the dual vector function representing $L_S(\psi, t)$ is given by

$$
\begin{aligned}
\text{(8.16)} \qquad \hat{S}(\psi) &= S(\psi) + \xi P(\psi) \times S(\psi) \\
&= (\text{sh}\theta\sin\psi, -\text{sh}\theta\cos\psi, \text{ch}\theta) \\
&\quad + \xi(-b\psi\text{sh}\theta\cos\psi - a\sin\psi\text{ch}\theta, \\
&\qquad a\cos\psi\text{ch}\theta - b\psi\text{sh}\theta\sin\psi, -a\text{sh}\theta).
\end{aligned}
$$

We already know quite a bit about the ruled surface $\hat{S}(\psi)$ from the way it was constructed. Its central normal surface L_T is the helicoid of the 1st kind swept out by the radius vector $r = (a\cos\psi, a\sin\psi, 0) = T$ given by the equation

$$
L_T(\psi, t) = (0, 0, b\psi) + t(a\cos\psi, a\sin\psi, 0).
$$

The distribution parameter Δ_T of the central normal surface is it, the pitch of the helix of the 1st kind $P(\psi)$. The Disteli axis of $L_S(\psi, t)$ is the timelike z-coordinate axis, and the dual pseudo spherical radius of curvature $\hat{\rho}$ is simply

$$
\text{(8.17)} \qquad\qquad \rho + \xi P = \theta + \xi a.
$$

We now determine the curvature functions Δ, γ, Γ, $\varkappa$, τ, $\mathcal{K}$ and $\mathcal{T}$, from the relations defined above. First we note that since ρ and P are constants we obtain from (5.9) and (5.13) that $\mathcal{T} = 0$ and $\tau = 0$. From equations (5.11) and (5.12) we easily have $\varkappa = \frac{1}{\text{sh}\theta}$ and $\gamma = \frac{\text{ch}\theta}{\text{sh}\theta}$. Since $\Delta_T = \frac{\mathcal{K}}{\varkappa}$ we get $\mathcal{K} = \frac{b}{\text{sh}\theta}$. Now, only Δ and Γ remain to be determined.

The functions Δ and Γ can be obtained directly from equations (5.5) and (5.8) since the functions $\varkappa$, $\mathcal{K}$ and P are known. However we determine them indepently using formulas (7.4) and (7.10) to illustrate computations using the dual vector form of $L_S(\psi, t)$.

To compute $\hat{\delta}$ in (7.4), we first determine

$$
\begin{aligned}
\frac{d\hat{S}(\psi)}{d\psi} &= (\text{sh}\theta\cos\psi, \text{sh}\theta\sin\psi, 0) \\
&\quad + \xi((-b\text{sh}\theta - a\text{ch}\theta)\cos\psi + b\psi\text{sh}\theta\sin\psi, \\
&\qquad (-a\text{ch}\theta - b\text{sh}\theta)\sin\psi - b\psi\text{sh}\theta\cos\psi, 0).
\end{aligned}
$$

Computing the magnitude of $\frac{d\hat{S}(\psi)}{d\psi}$ we get

$$\hat{\delta} = \left\| \frac{d\hat{S}(\psi)}{d\psi} \right\| = \text{sh}\theta \left(1 - \xi \frac{b\text{sh}\theta + a\text{ch}\theta}{\text{sh}\theta} \right),$$

which leads to the result

$$\Delta = -\frac{b\text{sh}\theta + a\text{ch}\theta}{\text{sh}\theta}.$$

To determine the dual geodesic curvature $\hat{\gamma}$ of $\hat{S}(\psi)$, we first determine

$$\frac{d^2\hat{S}(\psi)}{d\psi^2} = (-\text{sh}\theta \sin\psi, \text{sh}\theta \cos\psi, 0)$$
$$+\xi((2b\text{sh}\theta + a\text{ch}\theta)\sin\psi + b\psi\text{sh}\theta \cos\psi,$$
$$(-a\text{ch}\theta - 2b\text{sh}\theta)\cos\psi + b\psi\text{sh}\theta \sin\psi, 0),$$

and then compute

$$\frac{d^2\hat{S}(\psi)}{d\psi^2} \times \frac{d\hat{S}(\psi)}{d\psi} = (0, 0, -\text{sh}^2\theta) + \xi(0, 0, 2a\text{ch}\theta\text{sh}\theta + 3b\text{sh}^2\theta).$$

Using (7.10) we obtain

$$\hat{\gamma} = \frac{\text{ch}\theta}{\text{sh}\theta} + \xi \frac{a}{\text{sh}^2\theta},$$

and from (8.7) we obtain that

$$\Gamma = -a - \frac{b\text{ch}\theta}{\text{sh}\theta}.$$

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