Parallel Robust H_∞ Control for Weakly Coupled Bilinear Systems with Parameter Uncertainties Using Successive Galerkin Approximation

Young-Joong Kim and Myo-Taeg Lim

Abstract: This paper presents a new algorithm for the closed-loop H_∞ composite control of weakly coupled bilinear systems with time-varying parameter uncertainties and exogenous disturbance using the successive Galerkin approximation (SGA). By using weak coupling theory, the robust H_∞ control can be obtained from two reduced-order robust H_∞ control problems in parallel. The H_∞ control theory guarantees robust closed-loop performance but the resulting problem is difficult to solve for uncertain bilinear systems. In order to overcome the difficulties inherent in the H_∞ control problem, two H_∞ control laws are constructed in terms of the approximated solution to two independent Hamilton-Jacobi-Isaac equations using the SGA method. One of the purposes of this paper is to design a closed-loop parallel robust H_∞ control law for the weakly coupled bilinear systems with parameter uncertainties using the SGA method. The other is to reduce the computational complexity when the SGA method is applied to the high order systems.

Keywords: Bilinear system, H_∞ control, parallel processing, parameter uncertainty, successive Galerkin approximation, weak coupling.

1. INTRODUCTION

The major importance of bilinear systems indeed lies in their applications to the real world systems as demonstrated in some economic processes, ecology processes, socioeconomic processes and numerous biological processes, such as the population dynamics of biological species, water balance and temperature regulation in the human body, control of carbon dioxide in the lungs, blood pressure, immune system, cardiac regulator, etc. [1,2]. These bilinear systems are linear in control and linear in state but not jointly linear in state and control. It is important to understand the real properties of the system or to guarantee the global stability or improve the performance by applying the various control techniques to the bilinear system rather than its linearized system since the linearization of the bilinear system loses its nature property [2-6].

Many real physical systems are naturally weakly coupled such as power systems, communication satellites, helicopters, chemical reactors, electrical networks, flexible space structures, and mechanical systems in modal coordinates. The weakly coupled linear systems were introduced to the control audience by Kokotovic [7]. Since then many theoretical aspects for weakly coupled linear systems have been studied. These results lead to a reduction in the size of the required computation and allow parallel processing. Specifically, the optimal control is obtained in the form of a feedback law, with the feedback gains calculated from two independent reduced-order optimal control problems [8,9]. By using these results, the optimal control problems for weakly coupled bilinear systems have been studied [5,6].

Recently, robust control is issued and developed by many researchers for linear systems [10-12]. But in the class of bilinear and nonlinear systems, because conditions for the solvability of the robust H_∞ control design problem are hard, still there are a lot of problems to be developed. For bilinear and nonlinear systems with parameter uncertainties, the H_∞ optimal control problem can be reduced to the solution of the Hamilton-Jacobi-Isaac (HJI) equation, which is a nonlinear partial differential equation (PDE) [13]. The solution of a nonlinear PDE is extremely difficult to solve and so researchers have looked for methods of obtaining its approximate solution. In particular, the practical method named successive Galerkin approximation (SGA) to improve a stabilizing feedback control was proposed in [14,15]. The problem of a stabilizing H_∞ control can be reduced to solving a first-order, linear PDE known as the Generalized-Hamilton-Jacobi-Isaac (GHJI) equation [16]. An interesting fact is that when the process is

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Young-Joong Kim and Myo-Taeg Lim are with the School of Electrical Engineering, Korea University, 1, 5-Ka, Anam-Dong, Sungbuk-Ku, Seoul 136-701, Korea (e-mails: {kyjoong, mlim}@korea.ac.kr).

iterated, the solution to the GHJI equation converges uniformly to the solution of the HJI equation which solves the H_∞ optimal control problem [16]. Recently, [14] shows how to find a uniform approximation such that the approximate controls are still stable on a specified set using SGA. However, the SGA method has the difficulty that the complexity of computations increases according to the order of a system or a state variable. Specifically, for using the SGA method, we need N basis functions and must compute n-tuple integrals, where n is order of the system. Moreover, the number of those computations increases according to $O(N^3)$. Therefore, we deal with two reduced-order HJI equations in this paper. The robust H_∞ control law is designed from the solutions of two independent reduced-order HJI equations using the SGA method.

Then, n_1- and n_2-tuple integrals are computed in parallel, and the number of computations is greatly decreased, where $n = n_1 + n_2$. In this paper, a duel successive algorithm (Algorithm 1) is proposed as a heuristic formulation, and it is the modification addressed in the successive approximation reported in [3,4,16]. Since the GHJI equations are the partial differential equations, we hardly solve them. Therefore, we propose the alternative method (Algorithm 2) using Galerkin's approximation. In Algorithm 2, only linear equations remain to be solved.

This paper is summarized as follows. In Section 2, weakly coupled bilinear systems with time-varying parameter uncertainties and exogenous disturbance are studied. In Section 3, we define two independent GHJI equations. The solutions of two GHJI equations are obtained using the SGA method, and then a robust H_∞ control law is designed. In addition, we present new algorithms for the closed-loop parallel H_∞ control of weakly coupled bilinear systems with parameter uncertainties using the SGA method. In Section 4, the proposed algorithm is demonstrated on a real physical bilinear model of a paper making machine. Finally, Section 5 gives our conclusion.

2. ROBUST H_∞ CONTROL FOR WEAKLY COUPLED BILINEAR SYSTEMS WITH PARAMETER UNCERTAINTIES

The weakly coupled bilinear system with time-varying parameter uncertainties and exogenous disturbance under consideration is represented by

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\left\{\begin{bmatrix}A_1 & \varepsilon A_2\\ \varepsilon A_3 & A_4\end{bmatrix}+\begin{bmatrix}\Delta A_1 & \varepsilon\Delta A_2\\ \varepsilon\Delta A_3 & \Delta A_4\end{bmatrix}\right\}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}H_1 & \varepsilon H_2\\ \varepsilon H_3 & H_4\end{bmatrix}\begin{bmatrix}\omega_1\\ \omega_2\end{bmatrix}+\begin{bmatrix}B_1 & \varepsilon B_2\\ \varepsilon B_3 & B_4\end{bmatrix}\begin{bmatrix}u_1\\ u_2\end{bmatrix}+\left\{\begin{bmatrix}x_1\\ x_2\end{bmatrix}\begin{bmatrix}M_a & \varepsilon M_b\\ \varepsilon M_c & M_d\end{bmatrix}\right\}\begin{bmatrix}u_1\\ u_2\end{bmatrix}, \quad (1)$$

$$z=\begin{bmatrix}Cx\\ Du\end{bmatrix} \quad (2)$$

with an initial condition:

$$\begin{bmatrix}x_1(t_0)\\ x_2(t_0)\end{bmatrix}=\begin{bmatrix}x_1^0\\ x_2^0\end{bmatrix}$$

and

$$\left\{\begin{bmatrix}x_1\\ x_2\end{bmatrix}\begin{bmatrix}M_a & \varepsilon M_b\\ \varepsilon M_c & M_d\end{bmatrix}\right\}=\sum_{i=1}^{n_1}x_{1i}\begin{bmatrix}M_{ai} & \varepsilon M_{bi}\\ \varepsilon M_{ci} & 0\end{bmatrix}+\sum_{j=n_1+1}^{n_1+n_2}x_{2(j-n_1)}\begin{bmatrix}0 & \varepsilon M_{bj}\\ \varepsilon M_{cj} & \varepsilon M_{dj}\end{bmatrix}, \quad (3)$$

where $x_1\in\Re^{n_1}$, $x_2\in\Re^{n_2}$, $u_1\in\Re^{m_1}$, $u_2\in\Re^{m_2}$, $M_{ai}\in\Re^{n_1\times m_1}$, $M_{bi}\in\Re^{n_1\times m_2}$, $M_{ci}\in\Re^{n_2\times m_1}$, $M_{di}\in\Re^{n_2\times m_2}$. $x=[x_1^T \ \ x_2^T]^T$ is a state variable, $u=[u_1^T \ \ u_2^T]^T$ is a control input, $z\in\Re^p$ is a controlled output, and $\omega=\begin{bmatrix}\omega_1^T & \omega_2^T\end{bmatrix}^T\in\Re^p$ is an exogenous disturbance. A_i, B_i, H_i, C, D are constant matrices of appropriate dimensions, and ε is a small positive coupling parameter. In addition, ΔA_i represents the uncertainty in the system and satisfies the following assumption.

Assumption 1:

$$\begin{bmatrix}\Delta A_1 & \varepsilon\Delta A_2\\ \varepsilon\Delta A_3 & \Delta A_4\end{bmatrix}=\begin{bmatrix}E_1 & \varepsilon E_2\\ \varepsilon E_3 & E_4\end{bmatrix}Q(t)\begin{bmatrix}F_1 & \varepsilon F_2\\ \varepsilon F_3 & F_4\end{bmatrix}, \quad (4)$$

where E_i and F_i are known real constant matrices with appropriate dimensions and $Q(t)$ is an unknown matrix function with Lebesgue measurable elements such that $Q(t)^TQ(t)\le I$. □

A quadratic cost functional associated with (1)-(2) to be minimized has the following form:

$$J=\frac{1}{2}\int_0^\infty\left(z^Tz-\gamma^2\omega^T\omega\right)dt, \quad (5)$$

where γ is a positive design parameter.

For computational simplification, denote the following notations:

$$\tilde{B}(x)=\begin{bmatrix}B_1 & \varepsilon B_2\\ \varepsilon B_3 & B_4\end{bmatrix}+\left\{\begin{bmatrix}x_1\\ x_2\end{bmatrix}\begin{bmatrix}M_a & \varepsilon M_b\\ \varepsilon M_c & M_d\end{bmatrix}\right\}=\begin{bmatrix}\tilde{B}_1(x_1) & \varepsilon\tilde{B}_2(x)\\ \varepsilon\tilde{B}_3(x) & \tilde{B}_4(x_2)\end{bmatrix} \quad (6)$$

and without loss of generality, we assume that $C^TC=\begin{bmatrix}C_1 & \varepsilon C_2\\ \varepsilon C_2^T & C_3\end{bmatrix}$ and $D^TD=I$.

Since the reduced order technique cannot be applied to HJI equations directly, it is first applied to Riccati equations and then decoupled HJI equations are derived from reduced order Riccati equations.

With the help of [11,13], we can derive the following state dependent Riccati equation for the weakly coupled bilinear system (1)-(2) with respect to the performance criterion (5).

$$PA+A^TP-P\left\{\tilde{B}(x)\tilde{B}(x)^T-\gamma^{-2}HH^T-\sigma EE^T\right\}P$$
$$+C^TC+\frac{1}{\sigma}FF+\delta I=0, \qquad (7)$$

where $\sigma > 0$ is a design parameter and δ is a sufficiently small positive constant. Moreover, H_∞ control law is given by

$$u^*=-\tilde{B}(x)^TPx, \qquad (8)$$

and the disturbance is given by

$$\omega^*=\gamma^{-2}H^TPx, \qquad (9)$$

where P can be partitioned as

$$P=\begin{bmatrix} P_1 & \varepsilon P_2 \\ \varepsilon P_2^T & P_3 \end{bmatrix}. \qquad (10)$$

Setting $\varepsilon^2 = 0$, we can get the following $O(\varepsilon^2)$ approximations:

$$S(x)=\tilde{B}(x)\tilde{B}(x)^T-\gamma^{-2}HH^T-\sigma EE^T$$
$$=\begin{bmatrix} S_1(x_1) & \varepsilon S_2(x) \\ \varepsilon S_2^T(x) & S_3(x_2) \end{bmatrix}, \qquad (11)$$

$$T=CC^T+\frac{1}{\sigma}F^TF=\begin{bmatrix} T_1 & \varepsilon T_2 \\ \varepsilon T_2^T & T_3 \end{bmatrix}. \qquad (12)$$

Partitioning the state dependent Riccati equation (7) according to (10)-(12), and setting $\varepsilon^2 = 0$, we get an $O(\varepsilon^2)$ approximation of (7) in terms of two reduced-order, decoupled Riccati equations:

$$P_1A_1+A_1^TP_1-P_1S_1(x_1)P_1+T_1+\delta I=0, \qquad (13)$$
$$P_3A_4+A_4^TP_3-P_3S_3(x_2)P_3+T_3+\delta I=0 \qquad (14)$$

and non-symmetric Riccati equation with no input and no disturbance:

$$\{A_1-S_1(x_1)P_1\}^TP_2+P_2\{A_4-S_3(x_2)P_3\}$$
$$+P_1A_2+A_3^TP_3-P_1S_2(x)P_3+T_2=0. \qquad (15)$$

A detailed description of reduce-order scheme can be found in [6]. Since (13)-(14) are state dependent Riccati equations, they have no analytical solution.

Focusing the nonlinear H_∞ control in this paper, we deal with HJI equations rather than Riccati equations. HJI equations corresponding to (13) and (14) are given by

$$\frac{\partial J_1}{\partial x_1}^T A_1x_1+\frac{1}{2}x_1^T(T_1+\delta I)x_1-\frac{1}{2}\frac{\partial J_1}{\partial x_1}^T S_1(x_1)\frac{\partial J_1}{\partial x_1}=0, \qquad (16)$$

$$\frac{\partial J_2}{\partial x_2}^T A_4x_2+\frac{1}{2}x_2^T(T_3+\delta I)x_2-\frac{1}{2}\frac{\partial J_2}{\partial x_2}^T S_3(x_2)\frac{\partial J_2}{\partial x_2}=0, \qquad (17)$$

where $\partial J_1/\partial x_1 = P_1x_1$ and $\partial J_2/\partial x_2 = P_3x_2$. Moreover denoting $\partial J_3/\partial x_1 = P_2x_2$ and $\{\partial J_3/\partial x_2\}^T = x_1^TP_2$, we obtain the following equation equivalent to (15) after substitutions:

$$x_1^T\{A_1-S_1(x_1)P_1\}^T\frac{\partial J_3}{\partial x_1}+\frac{\partial J_3}{\partial x_2}^T\{A_4-S_3(x_2)P_3\}x_2$$
$$+x_1^T\left\{P_1A_2+A_3^TP_3-P_1S_2(x)P_3+T_2\right\}x_2=0. \qquad (18)$$

Unfortunately, they still have no analytical solution. However, we can obtain approximate solutions of (16) and (17) using successive Galerkin approximation. If the solutions of (16)-(17) are found, then the solution of (18) can be easily found using Galerkin approximation.

2. DESIGN OF H_∞ CONTROL LAW FOR WEAKLY COUPLED BILINEAR SYSTEMS WITH PARAMETER UNCERTAINTIES USING SGA

In order to design the H_∞ control law u^*, we present the scheme to find the solutions for (16)-(17) using the SGA method.

Assumption 2: Ω_1 and Ω_2 are compact sets of $\Re^{n_1}$ and $\Re^{n_2}$, respectively. The state x_1 and x_2 are bounded on Ω_1 and Ω_2, respectively. □

Under Assumption 2, the successive approximation, which is the duel iteration in policy space to solve HJI equations is proposed as follows.

Algorithm 1: Duel Successive Approximation

Let an initial control law $u_1^{(0)}:\Re^{m_1}\times\Omega_1\rightarrow\Re$, be stabilizing for the system $\dot{x}_1=A_1x_1+\tilde{B}(x_1)u_1(x_1)$ with no uncertainty and no disturbance (i.e., $\Delta A_1=0$, $\omega_1^{(0,0)}=0$).

Obtain $J_1^{(1,0)}$ from

$$\frac{\partial J_1^{(1,0)}}{\partial x_1}^T\left\{A_1x_1+\tilde{B}_1(x_1)u_1^{(0)}\right\}+\frac{1}{2}x_1^TC_1x_1$$
$$+\frac{1}{2}u_1^{(0)T}u_1^{(0)}=0. \qquad (19)$$

While $\left\| J_1^{(i,j)} - J_1^{(i-1,j)} \right\| > \alpha$

Set $j = 0$ and $\omega_1^{(i,0)} = 0$

While $\left\| J_1^{(i,j)} - J_1^{(i,j-1)} \right\| > \alpha$

Obtain $J_1^{(i,j)}$ from the GHJI equation defined as:

$$\frac{\partial J_1^{(i,j)}}{\partial x_1}^T A_1 x_1 + \frac{1}{2}\frac{\partial J_1^{(i,j-1)}}{\partial x_1}^T S_1(x_1)\frac{\partial J_1^{(i,j-1)}}{\partial x_1} + \frac{1}{2}x_1^T(T_1 + \delta I)x_1 - \frac{\partial J_1^{(i,j)}}{\partial x_1}^T S_1(x_1)\frac{\partial J_1^{(i,j-1)}}{\partial x_1} = 0. \tag{20}$$

Update the disturbance:

$$\omega_1^{(i,j+1)} = \gamma^{-2} H_1^T \frac{\partial J_1^{(i,j)}}{\partial x_1}. \tag{21}$$

Set $j = j + 1$.

End j loop.

Update the control law:

$$u_1^{(i,j+1)} = -\tilde{B}_1(x_1)^T \frac{\partial J_1^{(i,j)}}{\partial x_1}. \tag{22}$$

Set $i = i + 1$.

End i loop. □

In Algorithm 1, α is an arbitrary small positive design parameter.

Since the GHJI equation (20) is a linear partial differential equation, it is still difficult to solve. In this paper, we seek an approximate solution of this equation using Galerkin's projection method. A detailed description of the SGA method can be found in [14,17,18].

Given an initial control $u_1^{(0)}$, we can compute an approximation to its cost $J_{1N_1}^{(0,0)} = \mathbf{c}_{1N_1}^{(0,0)T}\Phi_{1N_1}$ where $\mathbf{c}_{1N_1}^{(0,0)}$ is the solution of Galerkin approximation of (19), i.e.,

$$a_1^{(0,0)}\mathbf{c}_{1N_1}^{(0,0)} + b_1^{(0,0)} = 0, \tag{23}$$

where

$$a_1^{(0,0)} = \left\langle \nabla_1\Phi_{1N_1} A_1 x_1, \Phi_{1N_1} \right\rangle_{\Omega_1} + \left\langle \nabla_1\Phi_{1N_1}\tilde{B}_1(x_1)u_1^{(0)}, \Phi_{1N_1} \right\rangle_{\Omega_1},$$

$$b_1^{(0,0)} = \frac{1}{2}\left\langle x_1^T C_1 x_1, \Phi_{1N_1} \right\rangle_{\Omega_1} + \frac{1}{2}\left\langle u_1^{(0)T}u_1^{(0)}, \Phi_{1N_1} \right\rangle_{\Omega_1}.$$

In the above equations, Φ_{1N_1} denotes the vector of basis functions and $\nabla\Phi_{1N_1}$ denotes the Jacobian matrix of Φ_{1N_1}.

After duel iterative steps, we can obtain the approximation to its cost $J_{1N_1}^{(i,j)} = \mathbf{c}_{1N_1}^{(i,j)T}\Phi_{1N_1}$ where $\mathbf{c}_{1N_1}^{(i,j)}$ is the solution of Galerkin approximation of (20), i.e.,

$$a_1^{(i,j)}\mathbf{c}_{1N_1}^{(i,j)} + b_1^{(i,j)} = 0, \tag{24}$$

where

$$a_1^{(i,j)} = \left\langle \nabla_1\Phi_{1N_1} A_1 x_1, \Phi_{1N_1} \right\rangle_{\Omega_1} - \left\langle \nabla_1\Phi_{1N_1} S_1(x_1)\nabla\Phi_{1N_1}^T \mathbf{c}_{1N_1}^{(i-1,j-1)}, \Phi_{1N_1} \right\rangle_{\Omega_1},$$

$$b_1^{(i,j)} = \frac{1}{2}\left\langle x_1^T(T_1 + \delta I)x_1, \Phi_{1N_1} \right\rangle_{\Omega_1} + \frac{1}{2}\left\langle \mathbf{c}_{1N_1}^{(i-1,j-1)T}\nabla_1\Phi_{1N_1} S_1(x_1)\nabla_1\Phi_{1N_1}^T \mathbf{c}_{1N_1}^{(i-1,j-1)}, \Phi_{1N_1} \right\rangle_{\Omega_1}.$$

Moreover, we can obtain the updated disturbance that is based on the approximated solution, $J_{1N_1}^{(i,j)}$:

$$\omega_{1N_1}^{(i,j+1)} = \gamma^{-2} H_1^T \frac{\partial J_{1N_1}^{(i,j)}}{\partial x_1} = \gamma^{-2} H_1^T \nabla_1\Phi_{1N_1}^T \mathbf{c}_{1N_1}^{(i,j)}, \tag{25}$$

and the updated control law:

$$u_{1N_1}^{(i,j+1)} = -\tilde{B}_1(x_1)^T \frac{\partial J_{1N_1}^{(i,j)}}{\partial x_1} = -\tilde{B}_1(x_1)^T \nabla_1\Phi_{1N_1}^T \mathbf{c}_{1N_1}^{(i,j)}. \tag{26}$$

Similarly, given an initial control $u_2^{(0,0)}$, we can compute an approximation to its cost $J_{2N_2}^{(i,j)} = \mathbf{c}_{2N_2}^{(i,j)T}\Phi_{2N_2}$. The following theorem shows the existence of an unique solution of SGA.

Theorem 1: Suppose that $\{\phi_k\}_1^N$ is linearly independent and $\partial\phi_k / \partial x \neq 0$, then there exists a unique solution, $\mathbf{c}_N$.

Proof: Suppose that $\{\phi_k\}_1^N$ is linearly independent then Φ_N is linearly independent. Suppose $\partial\phi_k / \partial x \neq 0$, such that $\nabla\Phi_N \neq 0$, then linearly independent Φ_N implies that $\nabla\Phi_N$ is linearly independent. This implies that $\left\langle \nabla\Phi_N Ax, \Phi_N \right\rangle_\Omega - \left\langle \nabla\Phi_N S(x)\nabla\Phi_N^T \mathbf{c}_N, \Phi_N \right\rangle_\Omega$ is invertible. This implies that $a_1^{(i,j)}$ is invertible in (24) for every i and j. Therefore, there exists a unique solution to a linear equation (24). □

From the solutions of Galerkin approximations of (16) and (17), P_1 and P_3 can be determined. Then, we can obtain the approximate solution of (18).

Defining $J_{3N_3}^{(i,j)} = \mathbf{c}_{3N_3}^{(i,j)T}\Phi_{3N_3}$, we can denote that $\partial J_3/\partial x_1 = \nabla_1 \mathbf{c}_{3N_3}^{(i,j)T}\Phi_{3N_3}$ and $\partial J_3/\partial x_2 = \nabla_2 \mathbf{c}_{3N_3}^{(i,j)T}\Phi_{3N_3}$. Using these notations, we can derive the Galerkin approximation of (18) as follows:

$$a_3\mathbf{c}_{3N_3} + b_3 = 0, \tag{27}$$

where

$$a_3 = \left\langle \nabla_1\Phi_{3N_3}\{A_1 - S_1(x_1)P_1\}x_1, \Phi_{3N_3}\right\rangle_{\Omega_3} + \left\langle \nabla_2\Phi_{3N_3}\{A_4 - S_3(x_2)P_3\}x_2, \Phi_{3N_3}\right\rangle_{\Omega_3},$$

$$b_3 = \left\langle x_1^T\left\{P_1A_2 + A_3^TP_3 - P_1S_2(x)P_3 + T_2\right\}x_2, \Phi_{3N_3}\right\rangle_{\Omega_3}.$$

In this case, $\Omega_3 = \Omega_1 \cup \Omega_2$ and P_3 can be determined without an iterative step.

Hence, we propose a new algorithm which designed an H_∞ control law with two independent reduced-order HJB equations (16), (17), and (18) using the SGA method for weakly coupled bilinear systems with time-varying parameter uncertainties and exogenous disturbance.

Algorithm 2: Duel Successive Galerkin Approximation

Let an initial control law $u_1^{(0)}: \Re^{m_1} \times \Omega_1 \to \Re$, be stabilizing for the system $\dot{x}_1 = A_1x_1 + \tilde{B}(x_1)u_1(x_1)$ with no uncertainty and no disturbance (i.e., $\Delta A_1 = 0$, $\omega_1^{(0,0)} = 0$).

Initial step: Compute

$$a_1^{(0,0)} = \left\langle \nabla_1\Phi_{1N_1}A_1x_1, \Phi_{1N_1}\right\rangle_{\Omega_1} + \left\langle \nabla_1\Phi_{1N_1}\tilde{B}_1(x_1)u_1^{(0)}, \Phi_{1N_1}\right\rangle_{\Omega_1},$$

$$b_1^{(0,0)} = \frac{1}{2}\left\langle x_1^TC_1x_1, \Phi_{1N_1}\right\rangle_{\Omega_1} + \frac{1}{2}\left\langle u_1^{(0)T}u_1^{(0)}, \Phi_{1N_1}\right\rangle_{\Omega_1},$$

and

$$a_2^{(0,0)} = \left\langle \nabla_2\Phi_{2N_2}A_4x_2, \Phi_{2N_2}\right\rangle_{\Omega_2} + \left\langle \nabla_2\Phi_{2N_2}\tilde{B}_3(x_2)u_2^{(0)}, \Phi_{2N_2}\right\rangle_{\Omega_2},$$

$$b_2^{(0,0)} = \frac{1}{2}\left\langle x_2^TC_3x_2, \Phi_{2N_2}\right\rangle_{\Omega_2} + \frac{1}{2}\left\langle u_2^{(0)T}u_2^{(0)}, \Phi_{2N_2}\right\rangle_{\Omega_2}.$$

Find $\mathbf{c}_{1N_1}^{(0,0)}$ and $\mathbf{c}_{2N_2}^{(0,0)}$ satisfying the following linear equations:

$$a_1^{(0,0)}\mathbf{c}_{1N_1}^{(0,0)} + b_1^{(0,0)} = 0,$$
$$a_2^{(0,0)}\mathbf{c}_{2N_2}^{(0,0)} + b_2^{(0,0)} = 0.$$

Routine for P_1:

While $\left\|\mathbf{c}_{1N_1}^{(i,j)} - \mathbf{c}_{1N_1}^{(i-1,j)}\right\| > \alpha$

Set $j = 0$ and $\omega_1^{(i,0)} = 0$.

While $\left\|\mathbf{c}_{1N_1}^{(i,j)} - \mathbf{c}_{1N_1}^{(i,j-1)}\right\| > \alpha$

Compute

$$a_1^{(i,j)} = \left\langle \nabla_1\Phi_{1N_1}A_1x_1, \Phi_{1N_1}\right\rangle_{\Omega_1} - \left\langle \nabla_1\Phi_{1N_1}S_1(x_1)\nabla_1\Phi_{1N_1}^T\mathbf{c}_{1N_1}^{(i,j-1)}, \Phi_{1N_1}\right\rangle_{\Omega_1},$$

$$b_1^{(i,j)} = \frac{1}{2}\left\langle x_1^T(T_1 + \delta I)x_1, \Phi_{1N_1}\right\rangle_{\Omega_1} + \frac{1}{2}\left\langle \mathbf{c}_{1N_1}^{(i,j-1)T}\nabla_1\Phi_{1N_1}S_1(x_1)\nabla_1\Phi_{1N_1}^T\mathbf{c}_{1N_1}^{(i,j-1)}, \Phi_{1N_1}\right\rangle_{\Omega_1}.$$

Find $\mathbf{c}_{1N_1}^{(i,j)}$ satisfying the following linear equations:

$$a_1^{(i,j)}\mathbf{c}_{1N_1}^{(i,j)} + b_1^{(i,j)} = 0.$$

Update the disturbance:

$$\omega_{1N_1}^{(i,j+1)} = \gamma^{-2}H_1^T\nabla_1\Phi_{1N_1}^T\mathbf{c}_{1N_1}^{(i,j)}.$$

Set $j = j + 1$.

End j loop.

Update the control law:

$$u_{1N_1}^{(i,j+1)} = -\tilde{B}_1(x_1)^T\nabla_1\Phi_{1N_1}^T\mathbf{c}_{1N_1}^{(i,j)}.$$

Set $i = i + 1$.

End i loop.

Determine P_1.

Routine for P_3:

While $\left\|\mathbf{c}_{2N_2}^{(i,j)} - \mathbf{c}_{2N_2}^{(i-1,j)}\right\| > \alpha$

Set $j = 0$ and $\omega_2^{(i,0)} = 0$.

While $\left\|\mathbf{c}_{2N_2}^{(i,j)} - \mathbf{c}_{2N_2}^{(i,j-1)}\right\| > \alpha$

Compute

$$a_2^{(i,j)} = \left\langle \nabla_2\Phi_{2N_2}A_4x_2, \Phi_{2N_2}\right\rangle_{\Omega_2}$$

$$-\left\langle \nabla_2 \Phi_{2N_2} S_3(x_2) \nabla_2 \Phi_{2N_2}^T \mathbf{c}_{2N_2}^{(i,j-1)}, \Phi_{2N_2} \right\rangle_{\Omega_2},$$

$$b_2^{(i,j)} = \frac{1}{2}\left\langle x_2^T (T_3 + \delta I) x_2, \Phi_{2N_2} \right\rangle_{\Omega_2} + \frac{1}{2}\left\langle \mathbf{c}_{2N_2}^{(i,j-1)T} \nabla_2 \Phi_{2N_2} S_3(x_2) \nabla_2 \Phi_{2N_2}^T \mathbf{c}_{2N_2}^{(i,j-1)} \Phi_{2N_2} \right\rangle_{\Omega_2}.$$

Find $\mathbf{c}_{2N_2}^{(i,j)}$ satisfying the following linear equations:

$$a_2^{(i,j)} \mathbf{c}_{2N_2}^{(i,j)} + b_2^{(i,j)} = 0.$$

Update the disturbance:

$$\omega_{2N_2}^{(i,j+1)} = \gamma^{-2} H_4^T \nabla_2 \Phi_{2N_2}^T \mathbf{c}_{2N_2}^{(i,j)}.$$

Set $j = j+1$.
End j loop.
Update the control law:

$$u_{2N_2}^{(i,j+1)} = -\tilde{B}_4(x_2)^T \nabla_2 \Phi_{2N_2}^T \mathbf{c}_{2N_2}^{(i,j)}.$$

Set $i = i+1$.
End i loop.
Determine P_3.

Routine for P_2:
Compute

$$a_3 = \left\langle \nabla_1 \Phi_{3N_3} \{A_1 - S_1(x_1) P_1\} x_1, \Phi_{3N_3} \right\rangle_{\Omega_3} + \left\langle \nabla_2 \Phi_{3N_3} \{A_4 - S_3(x_2) P_3\} x_2, \Phi_{3N_3} \right\rangle_{\Omega_3},$$

$$b_3 = \left\langle x_1^T \{P_1 A_2 + A_3^T P_3 - P_1 S_2(x) P_3 + T_2\} x_2, \Phi_{3N_3} \right\rangle_{\Omega_3}.$$

Find $\mathbf{c}_{3N_3}$ satisfying the following linear equations:

$$a_3 \mathbf{c}_{3N_3} + b_3 = 0.$$

Determine P_2.

Final step:
The approximate parallel H_∞ control law is given by

$$u_{pN} = -\tilde{B}^T \begin{bmatrix} P_1 & \varepsilon P_2 \\ \varepsilon P_2^T & P_3 \end{bmatrix} x. \tag{28}$$

□

The following theorem shows that the approximate parallel H_∞ control law, u_{pN}, designed by the proposed algorithm converges to the H_∞ optimal control law, u^*.

Theorem 2: For any small positive constant β, we can choose N for a sufficiently large i to satisfy that:

$$\left\| u^* - u_{pN}^{(i)} \right\| < \beta. \tag{29}$$

Proof: It was proved that u^* converges to u_N pointwise on Ω for finite N in [14], where u_N is a control law designed using the SGA. It implies that for a sufficiently large i, we can choose N satisfying $\left\| u_p - u_{pN}^{(i)} \right\| < \tilde{\beta}$, where u_p is the parallel H_∞ control law obtained by the reduced order scheme for weakly coupled bilinear systems and $\tilde{\beta}$ is a small positive constant. With the help of weakly coupling theory, $u_p = u^* + O(\varepsilon^2)$. This implies that for any small positive constant β, we can choose N for a sufficiently large i satisfying (29). □

4. CASE STUDY: A PAPER MAKING MACHINE

In order to demonstrate the efficiency of the proposed method for the parallel H_∞ control for weakly coupled bilinear systems with time-varying parameter uncertainties and exogenous disturbance using Algorithm 2, we have run a fourth-order real example, a paper making machine control problem reported in [19].

The problem matrices have the following values:

$$A = \begin{bmatrix} -1.93 & 0 & 0 & 0 \\ 0.394 & -0.426 & 0 & 0 \\ 0 & 0 & -0.63 & 0 \\ 0.095 & -0.103 & 0.413 & -0.426 \end{bmatrix},$$

$$B = \begin{bmatrix} 1.274 & 1.274 \\ 0 & 0 \\ 1.34 & -0.65 \\ 0 & 0 \end{bmatrix}, \quad M_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0.755 & 0.366 \\ 0 & 0 \end{bmatrix},$$

$$M_2 = M_4 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad M_3 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -0.718 & -0.718 \\ 0 & 0 \end{bmatrix},$$

$$C^T C = \begin{bmatrix} 1 & 0 & 0.13 & 0 \\ 0 & 1 & 0 & 0.09 \\ 0.13 & 0 & 0.1 & 0 \\ 0 & 0.09 & 0 & 0.2 \end{bmatrix}, \quad H = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Initial states are chosen as $x(t_0) = [3.7\ 3.2\ 4\ 2.8]^T$, time-varying parameter uncertainties are chosen as $1.2\sin(0.5\pi t)I$, and exogenous disturbance is chosen as $[0.4\sin(\pi t)\text{-}0.7\cos(\pi t)\quad 0.8\cos(\pi t)\text{-}0.6\sin(\pi t)]^T$. The simulation results are presented in Figs. 1-5, where the dashed lines are the trajectories that are obtained from the full-order SGA method which is presented in [16],

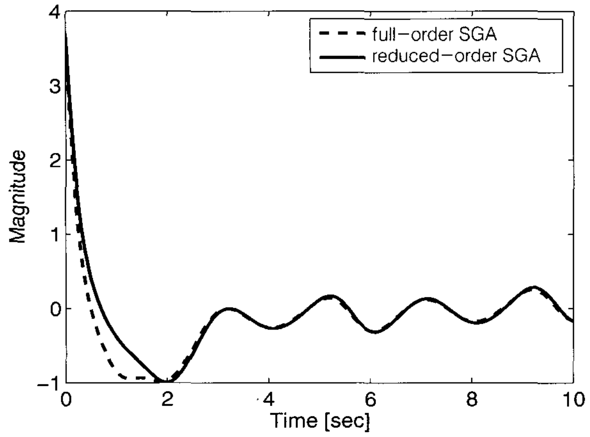

Fig. 1. Trajectories of x_1.

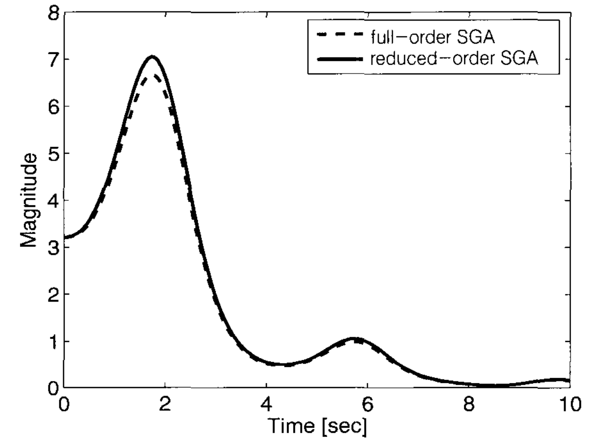

Fig. 2. Trajectories of x_2.

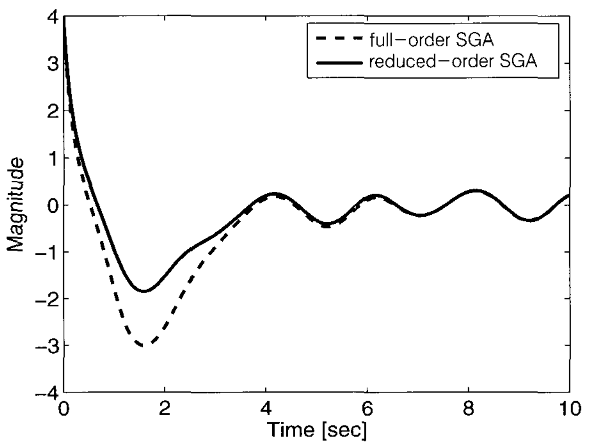

Fig. 3. Trajectories of x_3.

and the solid lines are the trajectories that are obtained from the proposed Algorithm 2. Fig. 5 indicates that the performance criterion trajectory of the proposed Algorithm 2 is better than that of the full-order SGA method, because errors of the full-order SGA method are bigger than those of the proposed algorithm. In the full-order SGA method, eight-dimensional basis are used and four-tuple integrals of $8 \times (1 + 8 + 64) = 584$

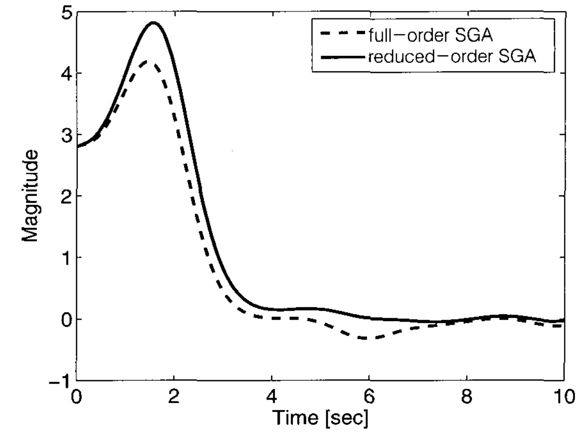

Fig. 4. Trajectories of x_4.

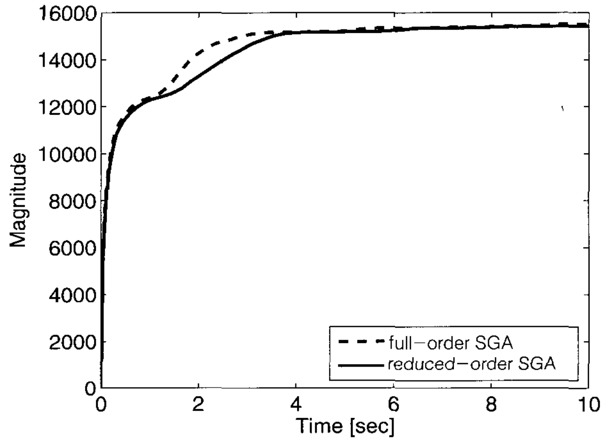

Fig. 5. Trajectories of performance criteria.

times are performed. But, in the proposed algorithm, we can use only three-dimensional basis and compute two-tuple integrals of $3 \times (1 + 3 + 9) = 39$ times for each reduced-order problem in parallel, and compute four-tuple integrals of $8 \times (1 + 8) = 72$ times based on eight-dimensional basis for the problem according to (18). Therefore, the computational complexity is greatly reduced.

5. CONCLUSIONS

We have presented the closed-loop H_∞ control scheme for weakly coupled bilinear systems with time-varying parameter uncertainties and exogenous disturbance and developed a new algorithm using the duel successive Galerkin approximation for the scheme. The difficulty of the SGA method is a computational complexity, but in the proposed algorithm, it can be greatly reduced. The presented simulation results for a fourth-order real example, a paper making machine control problem, show that the performance trajectories of the proposed algorithm are superior to those of the full order SGA method. It should be noted that the proposed algorithm is more effective than the full order SGA method.

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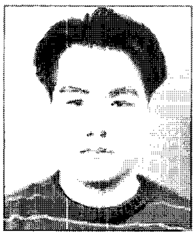

Young-Joong Kim received the B.S., M.S., and Ph.D. degrees in Electrical Engineering from Korea University, Seoul, in 1999, 2001, and 2006, respectively. Since 2006, he has been a Postdoctoral Fellow in the School of Electrical Engineering at Korea University. His research interests include optimal control, robust control and visual control of autonomous mobile robots. He is a member of KIEE.

Myo-Taeg Lim received the B.S. and M.S. degrees in Electrical Engineering from Korea University, Seoul, in 1985 and 1987, respectively. In addition, he received the M.S. and Ph.D. degrees in Electrical Engineering from Rutgers University, U.S.A., in 1990 and 1994, respectively. Since 1996, he has been a Professor in the School of Electrical Engineering at Korea University. His research interests include robust control, multivariable system theory, and computer-aided control systems design. He is a Member of KIEE.