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LOCAL CONVERGENCE OF NEWTON'S METHOD FOR PERTURBED GENERALIZED EQUATIONS

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ABSTRACT. A local convergence analysis of Newton's method for perturbed generalized equations is provided in a Banach space setting. Using center Lipschitzian conditions which are actually needed instead of Lipschitzian hypotheses on the Fréchet-derivative of the operator involved and more precise estimates under less computational cost we provide a finer convergence analysis of Newton's method than before [5]–[7].

1. INTRODUCTION

In this study we are concerned with the problem of approximating a solution of the equation

$$o \in f(x) + g(x) + F(x), \tag{1}$$

where X, Y are Banach spaces, $f\colon X \to Y$ is a Fréchet-differentiable operator, $g\colon X \to Y$ is a continuous operator, and $F \rightrightarrows Y$ is a closed set-valued mapping.

Equation (1) is the perturbed problem for

$$o \in f(x) + F(x), \tag{2}$$

where g in (1) is the perturbed operator.

Many problems, e.g. in engineering and economics can be viewed as special cases of equation (1) [2]–[11].

The most popular method for generating a sequence approximating a solution of equation (1) is undoubtedly Newton's method in the form

$$o \in f(x_n) + g(x_n) + F'(x_n)(x_{n+1} - x_n) + F(x_{n+1}), \quad (n \geq 0) \tag{3}$$

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where $F'(x)$ denotes the Fréchet-derivative of operator F [9], and x_0 is an initial guess in some neighborhood of the solution denoted by x^*. A local as well as semilocal convergence analysis for method (3) involving nonlinear equations has been given in [2], [3] and the references there.

In the case of generalized equations of the form (1) Geoffory and Pietrus provided a local convergence analysis for method (3) in [7]. Here we noticed that some of their hypotheses are not really needed in the proof. Therefore, we managed under weaker hypotheses and less computational cost to provide a finer convergence analysis including more precise estimates on the distances involved.

A survey on results involving generalized equations can be found in [1]–[11] and the references there.

2. Local Convergence Analysis of Method (3)

In order for us to introduce our results we also first need to introduce some terminology and a fixed point theorem already used in [6].

As in [2], [7] we denote by $A(x,y)$ the approximation of $f(x)+g(x)+F(x)$. That is we set

$$A(x,y) = f(y) + f'(y)(x-y) + g(y) + F(x) \quad \text{for all } x,y \in X. \tag{4}$$

It is convenient for us to define operator $Q_n\colon X \to Y$ by

$$\begin{aligned} Q_n(x) &= f(x^*) + f'(x^*)(x-x^*) + g(x^*) - f(x_n) \\ &\quad - f'(x_n)(x-x_n) - g(x_n) \quad (n \geq 0), \end{aligned} \tag{5}$$

and set-valued map $T_n\colon X \rightrightarrows Y$ by

$$T_n(x) = A(\cdot, x^*)^{-1}[Q_n(x)]. \tag{6}$$

Note that $x_1 \in X$ is a fixed point of T_0 if and only if the following implication holds true:

$$\begin{aligned} x_1 \in T_0(x_1) &\Leftrightarrow Q_0(x_1) \in A(x_1, x^*) \\ &\Leftrightarrow o \in f(x_0) + g(x_0) + f'(x_0)(x_1-x_0) + F(x_1). \end{aligned} \tag{7}$$

That is x_1 satisfies (3). In general if x_n plays the role of x_0, method (3) is used to show x_{n+1} is a fixed point of T_n etc. This way we generate a sequence $\{x_n\}$ satisfying (3).

We will make the assumptions:

(A_1) Operator $f\colon X \to Y$ is Fréchet-differentiable and its derivative is L-Lipschitz continuous and L_0-center-Lipschitz continuous in a neighborhood U of x^*. That is

$$\|F'(x) - F'(y)\| \leq L\|x - y\| \quad \text{for all } x, y \in U, \tag{8}$$

and

$$\|F'(x) - F'(x^*)\| \leq L_0\|x - x^*\| \quad \text{for all } x \in U. \tag{9}$$

(A_2) Operator $g\colon X \to Y$ is K_0-center-Lipschitz in a neighborhood U of x^*.

(A_3) The set-valued mapping $A(\cdot, x^*)^{-1}\colon Y \rightrightarrows X$ is M-pseudo-Lipschitz at 0 for x^*, i.e. there exist neighborhoods U of x^* and V of 0 such that

$$e\big(A(\cdot, x^*)^{-1}(y) \cap U, A(\cdot, x^*)(z)\big) \leq M\|y - z\| \tag{10}$$

for all M such that

$$\alpha_0 = M\left(\frac{L}{2} + K_0\right) < 1, \tag{11}$$

where,

$$e(A, B) = \sup_{x \in A} \operatorname{dist}(x, B) \tag{12}$$

denotes the excess e from a set B to the set A. The importance of introducing such a type of continuity due to Aubin has been explained in detail in [1], [5], [6], [11].

From now on we denote for $x \in X$, $r > 0$

$$U(x, r) = \{v \in X \mid \|x - v\| \leq r\}. \tag{13}$$

We need the following generalization of a fixed point theorem by Ioffe–Tikhomirov [6], [8]:

Lemma 1. *Let (X, ρ) be a Banach space. Let T be a map from X into the closed subsets of X, let $q_0 \in X$ and let $r > 0$ and $\lambda \in [0, 1)$ be such that:*

$$\operatorname{dist}(q_0, T(q_0)) \leq r(1 - \lambda), \tag{14}$$

and

$$e\big(T(x_1) \cap U(q_0, r), T(x_2)\big) \leq \lambda\rho(x_1, x_2) \quad \textit{for all } x_1, x_2 \in U(q_0, r). \tag{15}$$

Then, T has a fixed point in $U(q_0, r)$. Moreover if T is single-valued, then x is the unique fixed point of T in $U(q_0, r)$.

We can show the main local convergence result of Newton's method (3):

Theorem 2. *Under assumptions* (A_1)–(A_3) *and for any* $c \in (\alpha_0, 1)$ *there exists* $\delta > 0$ *such that for any initial guess* $x_0 \in U(x^*, \delta)$ *there exists a sequence* $\{x_n\}$ *generated by Newton's method (3) such that*

$$\|x_{n+1} - x^*\| \leq c\|x_n - x^*\|^2 \quad (n \geq 0). \tag{16}$$

To prove Theorem 2 we need the auxiliary result:

Proposition 3. *Under the hypotheses of Theorem 2 there exist* $\delta > 0$ *such that for all* $x_0 \in U(x^*, \delta)$ $(x_0 \neq x^*)$, *the map* T_0 *has a fixed point* x_1 *in* $U(x^*, \delta)$.

Proof. By (A_3) there exist positive constants a and b such that

$$\begin{aligned} &e\big(A(\cdot, x^*)^{-1}(y) \cap U(x^*, a), A(\cdot, x^*)^{-1}(z)\big) \\ &\leq M\|y - z\| \text{ for all } y, z \in U(0, b). \end{aligned} \tag{17}$$

Choose $\delta > 0$ to be fixed and

$$\delta \in (0, \delta_0), \tag{18}$$

where,

$$\delta_0 = \min\left\{\frac{a}{c}, \frac{b}{2(L + 2K_0)}\right\}. \tag{19}$$

Let $q_0 = x^*$. We will show conditions (14) and (15) of Lemma 1 hold true.

Let $x_0 \neq x^*$, $x_0 \in U(x^*, \delta)$. Using (5), (A_2), (8) and (9) we get

$$\begin{aligned} \|Q_0(x^*)\| &= \|f(x^*) - f(x_0) - f'(x_0)(x^* - x_0) + g(x^*) - g(x_0)\| \\ &\leq \frac{L}{2}\|x^* - x_0\|^2 + K_0\|x^* - x_0\|. \end{aligned} \tag{20}$$

For δ sufficiently small and (18)

$$\|Q_0(x^*)\| \leq \left(\frac{L}{2} + K_0\right)\|x^* - x_0\| \leq b. \tag{21}$$

In view of (17) we have:

$$e\big(A(\cdot, x^*)^{-1}(0) \cap U(x^*, a), A(\cdot, x^*)^{-1}(Q_0(x^*))\big) \leq M\|Q_0(x^*)\|, \tag{22}$$

and

$$\operatorname{dist}(x^*, T_0(x^*)) \leq M\left(\frac{L}{2} + K_0\right)\|x^* - x_0\|. \tag{23}$$

By the choice of c there exists $\lambda \in (0,1)$ such that $c(1-\lambda) \geq M\left(\frac{L}{2} + K_0\right)$, and hence

$$\operatorname{dist}(x^*, T_0(x^*)) \leq c(1 - \lambda)\|x^* - x_0\|. \tag{24}$$

Let $q_0 = x^*$, $r = r_0 = c\|x^* - x_0\|$. It follows (14) holds.

We shall show (15) also holds true.

In view of $\delta \leq \frac{a}{c}$, we get $r_0 \leq a$. Let $x \in U(x^*, \delta)$. We can obtain using (8), (9), (A_2), and the choice of δ:

$$\begin{aligned}\|Q_0(x)\| &\leq \|f(x^*) - f(x) - f'(x^*)(x - x^*)\| \\ &\quad + \|f(x) - f(x_0) - f'(x_0)(x - x_0)\| + \|g(x^*) - g(x_0)\| \\ &\leq \frac{L_0}{2}\|x^* - x_0\|^2 + \frac{L}{2}\|x - x_0\|^2 + K_0\|x^* - x_0\| \\ &\leq 4\delta\left(\frac{\overline{L}}{2} + K_0\right) \leq b, \qquad (25)\end{aligned}$$

where,

$$\overline{L} = \frac{L + L_0}{2}. \qquad (26)$$

Moreover, for $x^1, x^2 \in U(x^*, r_0)$, we get

$$\begin{aligned}e\big(T_0(x^1) \cap U(x^*, r_0), T_0(x^2)\big) &\leq e\big(T_0(x^1) \cap U(x^*, \delta), T_0(x^2)\big) \\ &\leq M\|Q_0(x^1) - Q_0(x^2)\| \\ &\leq M\|F'(x^*)(x^1 - x^2) - F'(x_0)(x^1 - x^2)\| \\ &\leq ML_0\|x^* - x_0\|\,\|x^1 - x^2\| \\ &\leq ML_0\delta\|x^1 - x^2\|. \qquad (27)\end{aligned}$$

We can assume that without loss of generality

$$\delta < \frac{\lambda}{ML_0} = \delta_1, \qquad (28)$$

which implies (15). Therefore all conditions of Lemma 1 hold true. Hence, we deduce the existence of a fixed point $x_1 \in U(x^*, r_0)$ for the map T_0.

That completes the proof of Proposition 3. □

Proof of Theorem 2. In view of $x_1 \in U(x^*, r_0)$ we get

$$\|x_1 - x^*\| \leq r_0 = c\|x_0 - x^*\|. \qquad (29)$$

Using induction for $q_0 = x^*$, $r_k = c\|x_k - x^*\|^2$, following the proof of Proposition 3 for the map T_k we conclude the existence of a fixed point x_{k+1} for T_k in $U(x^*, r_k)$. That is

$$\|x_{k+1} - x^*\| \leq c\|x_k - x^*\|^2. \qquad (30)$$

That completes the induction and the proof of the theorem. □

Remark 4. In general

$$L_0 \leq L \tag{31}$$

and

$$K_0 \leq K \tag{32}$$

holds and $\frac{L}{L_0}$, $\frac{K}{K_0}$ can be arbitrarily large [2], [3], where K is the Lipschitz constant of operator g in some neighborhood V of x^*, a hypothesis used in [7] corresponding to our Assumption (A_2). If equality holds in both (31) and (32) then our results reduce to the corresponding ones in [7]. Otherwise our results constitute an improvement since they allow: a larger δ, which implies a wider choice of initial guesses x_0; a smaller choice of c which improves the ratio of the quadratic convergence of Newton's method (3) given by (16).

These observations/improvements are important in computational mathematics [2], [3], [6], [7], [8], [11].

REFERENCES

1. Aubin, J.-P.: Lipschitz behavior of solutions to convex minimization problems. *Math. Oper. Res.* **9** (1984), 87–111.
2. Argyros, I. K.: A unifying local-semilocal convergence analysis and applications for two-point Newton-like methods in Banach space. *J. Math. Anal. Appl.* **298** (2004), 374–397.
3. Argyros, I. K.: *Newton Methods.* Nova Science Publ. Inc., New York, 2005.
4. Argyros, I. K.: On the secant method for solving nonsmooth equations. *J. Math. Anal. Appl.* (to appear, 2006).
5. Dontchev, A. L. & Hager, W. W.: An inverse function theorem for set-valued maps. *Proc. Amer. Math. Soc.* **121** (1994), 481–484.
6. Dontchev, A. L.: Local convergence of the Newton method for generalized equations, *C.R. Acad. Sci. Paris*, t. 332, Série I (1996), 327–331, Mathematical Analysis.
7. Geoffory, M. H. & Pietrus, A.: An iterative method for perturbed generalized equations, *Comptes rendus de l'academie Bulgare des Sciences, Mathematiques, Methodes de Calculs numeriques*, Tolme **57** (2004), no. 11, 7–12.
8. Ioffe, A. D. & Tikhomirov, V. M.: *Theory of Extremal Problems*, North Holland, Amsterdam, 1979.
9. Kantorovich, L. V. & Akilov, G. P.: *Functional Analysis in Normed Spaces*, Pergamon Press, Oxford, 1982.

10. Rockafellar, R. T.: Lipschitz properties of multifunctions, *Nonlinear Analysis* **9** (1985), 867–885.
11. Xiao, B. & Harker, P. T.: A nonsmooth Newton method for variational inequalities I: Theory, *Math. Programming* **65** (1994), 151–194.

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