A Boundary Element Solution Approach for the Conjugate Heat Transfer Problem in Thermally Developing Region of a Thick Walled Pipe

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This paper presents a sole application of boundary element method to the conjugate heat transfer problem of thermally developing laminar flow in a thick walled pipe when the fluid velocities are fully developed. Due to the coupled mechanism of heat conduction in the solid region and heat convection in the fluid region, two separate solutions in the solid and fluid regions are sought to match the solid-fluid interface continuity condition. In this method, the dual reciprocity boundary element method (DRBEM) with the axial direction marching scheme is used to solve the heat convection problem and the conventional boundary element method (BEM) of axisymmetric model is applied to solve the heat conduction problem. An iterative and numerically stable BEM solution algorithm is presented, which uses the coupled interface conditions explicitly instead of uncoupled conditions: Both the local convective heat transfer coefficient at solid-fluid interface and the local mean fluid temperature are initially guessed and updated as the unknown interface thermal conditions in the iterative solution procedure. Two examples imposing uniform temperature and heat flux boundary conditions are tested in thermally developing region and compared with analytic solutions where available. The benchmark test results are shown to be in good agreement with the analytic solutions for both examples with different boundary conditions.

Key Words : Conjugate Heat Transfer, Boundary Element Method, Thermally Developing Region, Thick Walled Pipe, Solid-Fluid Interface

Nomenclature

A, B, m : Parameters for the complete elliptic integral of the 1st kind
b : Heat source-like term
$E(m)$: Complete elliptic integral of the 1st kind of modulus m
f : Interpolating function
F : Matrix of its element f
G : Coefficient matrix involving T^* or u^*
Gz : Graetz number
g : Boundary integrals involving T^* or u^*
H : Coefficient matrix involving q^*
h : Convective heat transfer coefficient or boundary integrals involving q^*
K : Ratio of thermal conductivity of pipe wall to that of the fluid considered
k : Thermal conductivity
L : Number of internal points or length of thermal boundary section
N : Number of boundary elements
Nu : Nusselt number
n : Normal unit vector
p : Pressure
Pe : Peclet number
Pr : Prandtl number
q : Normal derivative of T or u

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r, z : Cylindrical coordinates
T : Temperature
u : General dependent variable
w : Axial flow velocity
x, y, z : Cartesian coordinates

Greek symbols

α : Thermal diffusivity
β : Initially unknown coefficient
Γ : Boundary
δ : Kronecker delta
ε : Relative error
θ : Angle
ζ : Parameter taking the values between 0 and 1
λ : Relaxation factor
μ : Coefficient of viscosity
ϕ : Linear interpolation function
Ω : Solution domain

Subscripts

e : Pipe entrance of heating section
f : Fluid region
i : Source point or inner boundary
j : Collocation point
o : Outer boundary
s : Solid region
w : Wall surface
∞ : Ambient condition

Superscripts

m : Time level
$-$: Specified value
$*$: Fundamental solution
$\hat{}$: Particular solution

1. Introduction

When the thermal boundary condition is known at the outer surface of the thick walled pipe and the thermal conditions at the solid-fluid interface are not known a priori, heat convection in a pipe flow becomes the conjugate heat transfer problem. For the thick walled pipes, thermal boundary conditions imposed at the outer wall surface are usually different from the inner surface conditions due to the axial heat conduction across the walls surrounding the fluid. This effect is known to be especially significant at the thermally developing region in the pipe (Barozzi and Pagliarini, 1985).

Analytic and numerical solution methods for the conjugate heat transfer problem have been studied for many years. Mori et al. (1976 ; 1974) used the analytic and experimental methods to study the conjugate heat transfer problems in a parallel plate channel and in a circular pipe with fully developed velocity profile. Barozzi and Pagliarini (1985) proposed a combining method of the superposition principle with the finite element method (FEM) for the case of fully developed laminar flow in a pipe. For the same subject, Campo and Schuler (1998) applied the finite difference method (FDM) as well. The simultaneously developing flow situations both for the temperature and the velocity profiles were also investigated using the FEM by Pagliarini (1991). Recently the boundary element methods (BEM) combined with the FDM or FVM are applied to the conjugate heat transfer problems of the parallel plate channel and rectangular ducts, where heat conduction in the solid region was solved using the BEM, while heat convection in the fluid region was solved using the FDM or FVM (He et al., 1995 ; Divo et al., 2002 ; Al-Bakhit and Fakheri, 2006). In these studies, they presented iterative solution algorithm, and showed some merits of the BEM solution method.

This paper presents the sole application of BEM to the conjugate heat transfer problem in thermally developing laminar flow of a thick walled pipe when the fluid velocities are fully developed. For the solution of this conjugate heat transfer problem, BEM has the distinct advantage over FDM, FVM or FEM since the boundary meshing is good enough to meet the requirement of temperature and heat flux continuity at the solid-fluid interface. And, in general, BEM's heat flux calculation is likely to be more accurate and convenient than the FDM or FEM's heat flux calculation (Banerjee, 1994). Due to the coupled mechanism of heat conduction in the solid and heat convection in the fluid, two separate solutions in the solid and fluid regions are sought to match the solid-fluid interface continuity condition. In this method, the dual reciprocity boundary element method (DRBEM) with the axial direction marching scheme (Choi, 1999) is used to solve the heat convection problem and the conventional BEM with the axisymmetric model is applied to solve the heat conduction problem. Calculations are performed iteratively until the continuities of temperature and heat flux at the interfacial boundary are satisfied. Initial guess for the unknown interfacial thermal boundary conditions is made to start the iterative calculation procedure.

In this study, an iterative and numerically stable BEM solution algorithm is presented, which uses the coupled interface conditions explicitly instead of uncoupled ones (Hribersek and Kuhn, 2000). Both the local convective heat transfer coefficient at the solid-fluid interface and the local mean fluid temperature are initially guessed and updated as the unknown interfacial thermal conditions in the iterative solution procedure. Numerical solutions by the BEM are obtained for the constant temperature and heat flux imposed-boundary conditions at the outer surface of the thick walled pipe. The solutions in the thermally developing region are compared to previous study results to validate the salient feature of the present approach. The effects of relevant parameters on the wall heat conduction are also investigated and discussed.

2. Governing Equations and Boundary Conditions

The conjugate heat transfer problem to be analyzed is shown in Fig. 1. Fluid flow situation inside of the thick walled pipe is assumed to be fully developed steady laminar and the fluid has constant transport properties of incompressible Newtonian fluid. At the entrance of the finite heating region, the fluid temperature is uniform. The outer surface of pipe section with a finite length is subjected to a uniform heat flux, and both end sections except the heated pipe section are thermally insulated. Due to the effect of the pipe thickness in the thermally developing region, wall temperature variations in both the longitudinal and radial directions are important in this problem. Based on the assumptions mentioned above, the governing equations and related boundary conditions can be expressed as follows :

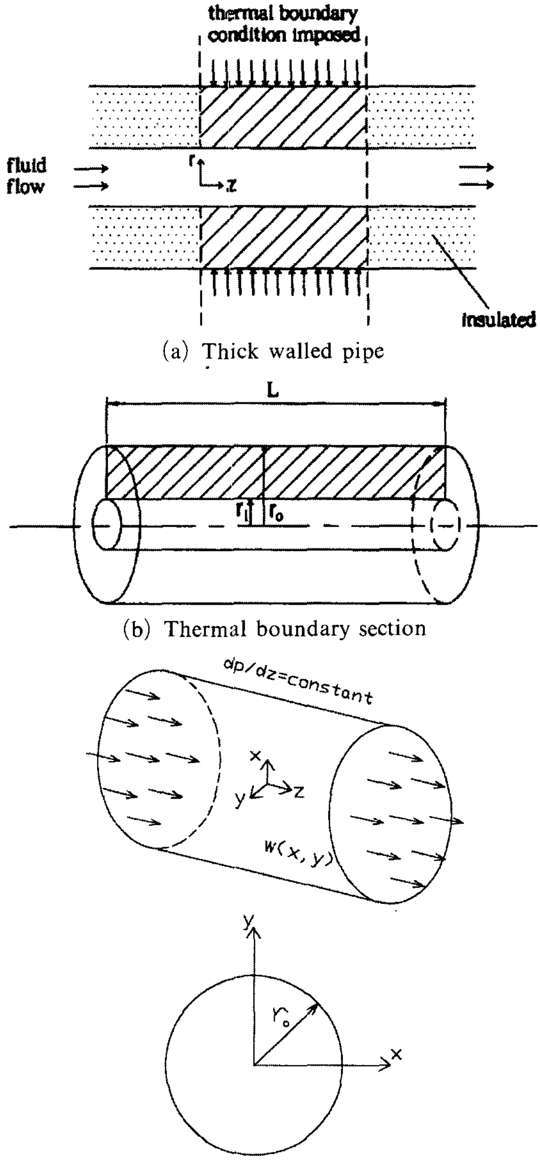


Fig. 1 Schematics of the problem to be analyzed

In the solid region,

$$\nabla^2 T_s = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T_s}{\partial r}\right) + \frac{\partial^2 T_s}{\partial z^2} = 0 \qquad (1a)$$
$$(r, z) \in \Omega_s$$

$$\text{at } z=0,\ \frac{\partial T_s}{\partial z}=0 \tag{1b}$$

$$\text{at } z=L,\ \frac{\partial T_s}{\partial z}=0 \tag{1c}$$

$$\text{at } r=r_o,\ T_s=\bar{T}=T_w$$
$$\text{and/or } \frac{\partial T_s}{\partial n}=\bar{q}=\frac{q_w}{k_s} \tag{1d}$$

In the fluid region,

$$\nabla^2 w=\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}=\frac{1}{\mu}\frac{dP}{dz},\quad (x,y)\in\Omega_f \tag{2a}$$

$$\nabla^2 T_f=\frac{\partial^2 T_f}{\partial x^2}+\frac{\partial^2 T_f}{\partial y^2}=\frac{w}{\alpha}\frac{\partial T_f}{\partial z},\quad (x,y,z)\in\Omega_f \tag{2b}$$

$$\text{at } (x,y)\in\Gamma,\ w=0 \tag{2c}$$

$$\text{at } z=0,\ T=T_e \tag{2d}$$

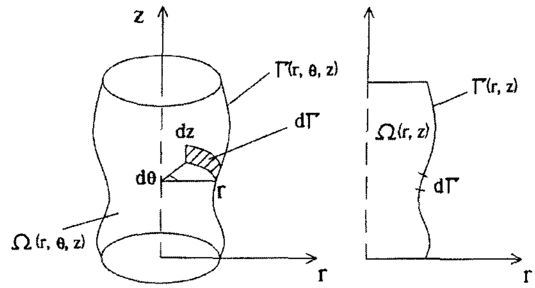
Continuity condition at the solid-fluid interface is given as

$$\text{at } r=r_i,\ T_s=T_f \text{ and } k_s\frac{\partial T_s}{\partial n}=k_f\frac{\partial T_f}{\partial n} \tag{3}$$

where Ω, w, μ, p, T, k, and α represent solution domain, flow velocity in axial direction, coefficient of viscosity, pressure, temperature, thermal conductivity, and thermal diffusivity, respectively. In addition the subscripts s, f, i, o, w, and e denote solid region, fluid region, inner radius of the pipe, outer radius of the pipe, boundary condition given at the wall, and initial condition given at the pipe entrance of heating section respectively. For the coordinate system adopted, z coordinate represents the axial direction and x-y coordinates are attached to the cross-sectional surface while r coordinate is the radial direction from the centerline of the pipe. It should be noted that the different coordinate system is used for solid and fluid region to reflect the mathematical and geometrical characteristics of the problems for the present BEM analysis.

3. Axisymmetric Boundary Element Equation for the Solid Region

Considering a situation of steady-state axisymmetric heat conduction in a homogeneous isotropic medium of domain Ω bounded by surface Γ, as shown in Fig. 2, the boundary integral equation for the three-dimensional problem can be written as



(a) 3-D domain (b) 2-D domain

Fig. 2 Axisymmetric body for the BEM analysis

$$c_i T_i+\int_\Gamma Tq^* d\Gamma=\int_\Gamma qT^* d\Gamma-\int_\Gamma \frac{h}{k}TT^* d\Gamma \tag{4}$$

where T^* is the fundamental solution, q^* is the normal derivative of T^* along the boundary, and c_i is a constant that depends on the geometry at the point i under consideration (Banerjee, 1994). In this equation, it is noted that the convective heat transfer coefficient h is set to zero for the temperature or heat flux boundary conditions. However, if a convective boundary condition is imposed on the boundary, q is set to $(h/k)\,T_\infty$ with ambient temperature T_∞, and h is not equal to zero.

Now for the axisymmetric problem, the differential surface boundary $d\Gamma$ should be replaced by $rd\theta d\Gamma$ where the differential of Γ, $d\Gamma$ turns out to be the differential line boundary:

$$\begin{aligned} &c_i T_i+\int_\Gamma\int_\theta Tq^* r d\theta d\Gamma \\ &=\int_\Gamma\int_\theta qT^* r d\theta d\Gamma-\int_\Gamma\int_\theta \frac{h}{k}TT^* r d\theta d\Gamma \end{aligned} \tag{5}$$

Because T and q are independent of angular direction θ, Eq. (5) can be rewritten as

$$\begin{aligned} &c_i T_i+\int_\Gamma T\int_{\theta=0}^{2\pi} q^* d\theta r d\Gamma \\ &=\int_\Gamma q\int_{\theta=0}^{2\pi} T^* d\theta r d\Gamma-\int_\Gamma \frac{h}{k}T\int_{\theta=0}^{2\pi} T^* d\theta r d\Gamma \end{aligned} \tag{6}$$

And let us define the θ integrated weighting function as

$$\bar{T}^* = \int_{\theta=0}^{2\pi} T^* d\theta \text{ and } \bar{q}^* = \int_{\theta=0}^{2\pi} q^* d\theta \qquad (7)$$

Then an axisymmetric boundary element equation is obtained by discretizing the integrals with boundary elements as follows:

$$c_i T_i + \sum_{j=1}^{N} \int_{\Gamma_j} T\bar{q}^* r d\Gamma = \sum_{j=1}^{N} \int_{\Gamma_j} T\bar{q}^* r d\Gamma - \sum_{j=1}^{N} \frac{1}{k} \int_{\Gamma_j} hT\bar{T}^* r d\Gamma \qquad (8a)$$

where

$$\bar{T}^* = \frac{4E(m)}{(A+B)^{1/2}},\ q^* = \frac{\partial \bar{T}^*}{\partial n}. \qquad (8b)$$

$$m = \frac{2B}{(A+B)},\ A = r_i^2 + r^2 + (z_i - z)^2,\ B = 2r_i r \qquad (8c)$$

In Eqs. (8b) and (8c), $E(m)$ is the complete eliptic integral of the 1st kind of modulus m and the subscript i represents the source point. Detailed expressions of the weighting functions can be found in the references (Ligget and Liu, 1993; Long et al., 1993).

In the numerical solution of the equation, the linear boundary element type is chosen for the present study. Thus T and q in the integrals can be modeled using the linear interpolation functions, and the integrals are to be evaluated by the various numerical quadrature schemes (Choi and Jo, 2003). In this study, the Gaussian quadrature method is suitable for the case $i \neq j$. However, in case where i and j are on the same element, the singularity of the fundamental solution requires a special integration scheme. For this purpose, the composite Gaussian quadrature formula is used, which is accurate and easy to implement.

Assembling the above integrals on each element, then Eq. (8a) can be expressed as

$$\sum_{j=1}^{N} \left(H_{ij} + \frac{1}{k} G_{ij} h_j \right) T_j = \sum_{j=1}^{N} G_{ij} q_j \qquad (9)$$

where

$$H_{ij} = \bar{H}_{ij} + c_i \delta_{ij}$$

$$\bar{H}_{ij} = h_{i(j-1)}^2 + h_{ij}^1 = \int_{\Gamma_{j-1}} \phi_2 \bar{q}^* r d\Gamma + \int_{\Gamma_j} \phi_1 \bar{q}^* r d\Gamma \qquad (10)$$

$$G_{ij} = g_{i(j-1)}^2 + g_{ij}^1 = \int_{\Gamma_{j-1}} \phi_2 \bar{T}^* r d\Gamma + \int_{\Gamma_j} \phi_1 \bar{T}^* r d\Gamma$$

4. Dual Reciprocity Boundary Element Equation for the Fluid Region

For the DRBEM solution of thermally developing flow inside an arbitrarily shaped duct (see Figure 3), Eqs. (2a) and (2b) with specified boundary and inlet conditions can be generalized as the following type of Poisson equation (Choi and Jo, 2003).

Momentum equation:

$$\nabla^2 u(x,y) = b(x,y),\ (x,y) \in \Omega \qquad (11)$$

Energy equation:

$$\nabla^2 u(x,y) = b(x,y,z,u),\ (x,y) \in \Omega \qquad (12)$$

The boundary conditions at the duct surface (i.e., at the solid-fluid interface) are

$$\text{at } (x,y) \in \Gamma_1,\ u(x,y) = \bar{u} \qquad (13)$$

$$\text{at } (x,y) \in \Gamma_2,\ q(x,y) = \frac{\partial u(x,y)}{\partial n} = \bar{q} \qquad (14)$$

and the inlet boundary condition for the energy equation is expressed as

$$\text{at } z=0,\ u(x,y) = u_o \qquad (15)$$

As shown in Fig. 4, $\Gamma_1 + \Gamma_2 = \Gamma$ is the total boundary of solution domain Ω, n denotes the normal to the boundary, the over-bar denotes a specified

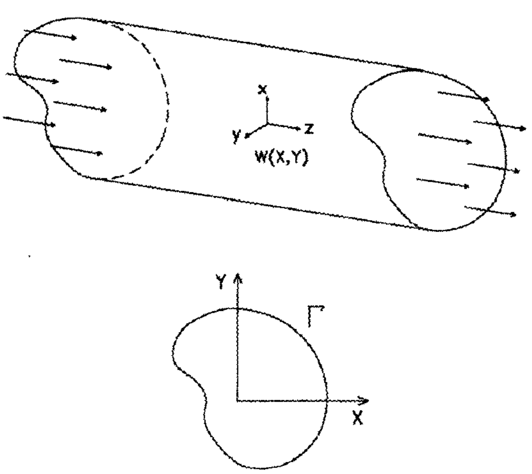


Fig. 3 Fully developed flow inside an arbitrarily shaped duct

value, and u_o is the given inlet temperature condition. Now the variables for the convective heat transfer problem can be written as

Momentum equation :

$$u(x,y)=w \tag{16}$$

$$b(x,y)=\frac{1}{\mu}\frac{dp}{dz}=\text{constant} \tag{17}$$

Energy equation :

$$u(x,y)=T \tag{18}$$

$$b(x,y,z,u)=\frac{1}{a}w(x,y)\frac{\partial T(x,y,z)}{\partial z} \tag{19}$$

Applying the usual boundary element technique to Eq. (11) or Eq. (12) based on the use of the fundamental solution and reciprocity principle (Green's theorem) (Choi and Jo, 2000), a boundary integral equation can be deduced as

$$c_i u_i+\int_\Gamma uq^* d\Gamma-\int_\Gamma qu^* d\Gamma=\int_\Omega bu^* d\Omega \tag{20}$$

The key method of DRBEM is to take the domain integral of Eq. (20) to the boundary and remove the needs of complicated domain discretization. To do this, either the source term $b(x,y)$ or $b(x,y,z,u)$ is expanded as its values at each node j using a set of interpolating functions f_j as

$$b(x,y)\cong\sum_{j=1}^{N+L}\beta_j f_j(x,y)$$
$$\text{or } b(x,y,z,u)\cong\sum_{j=1}^{N+L}\beta_j(z)f_j(x,y) \tag{21}$$

where the β_j is a set of initially unknown coefficients and $N+L$ is the number of boundary nodes plus internal points. The interpolating functions $f_j(x,y)$ and the particular solutions $\hat{u}_j$ are linked as

$$\nabla^2\hat{u}_j=f_j \tag{22}$$

so that the domain integral can be transferred to the boundary.

Substituting Eq. (22) into Eq. (21), and applying integration by parts twice for the domain integral term of Eq. (20) lead to the following dual reciprocity boundary integral equation.

$$c_i u_i+\int_\Gamma uq^* d\Gamma+\int_\Gamma qu^* d\Gamma$$
$$=\sum_{j=1}^{N+L}\beta_j\left(c_i\hat{u}_{ij}+\int_\Gamma \hat{u}_j q^* d\Gamma-\int_\Gamma \hat{q}_j u^* d\Gamma\right) \tag{23}$$

As for the interpolating function f, a radial basis function $f=1+r$ is chosen which was chosen to be generally sufficient to use for nonlinear problems (Patridge et al., 1992). Here, r stands for the distance from a source point i, or DRBEM collocation point j to a field point (x,y).

For the numerical solution of the integral Eq. (23), the boundary is discretized into N elements and the values of $u, q, \hat{u}$ and $\hat{q}$ are modeled in terms of their nodal values and two linear interpolating functions ϕ_1 and ϕ_2. Therefore, Eq. (23) results in the following dual reciprocity boundary element equation.

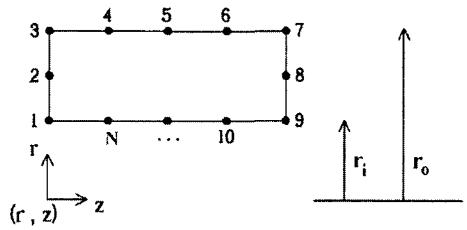
$$\sum_{k=1}^{N}H_{ik}u_k-\sum_{k=1}^{N}G_{ik}q_k$$
$$=\sum_{k=1}^{N+L}\beta_j\left(\sum_{k=1}^{N}H_{ik}\hat{u}_{kj}-\sum_{k=1}^{N}G_{ik}\hat{q}_{kj}\right) \tag{24}$$

where

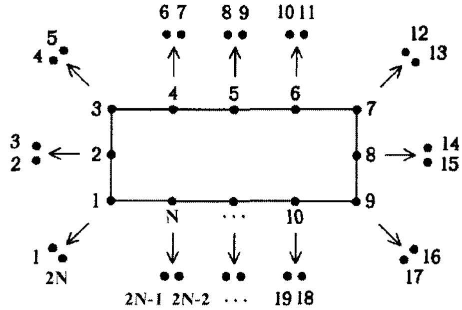
$$H_{ik}=\bar{H}_{ik}+c_i\delta_{ik}$$

$$\bar{H}_{ik}=h^2_{i(k-1)}+h^1_{ik}=\int_{\Gamma_{k-1}}\phi_2 q^* d\Gamma+\int_{\Gamma_k}\phi_1 q^* d\Gamma \tag{25}$$

$$G_{ik}=g^2_{i(k-1)}+g^1_{ik}=\int_{\Gamma_{k-1}}\phi_2 u^* d\Gamma+\int_{\Gamma_k}\phi_1 u^* d\Gamma$$



(a) Single node system



(b) Double nodes system

Fig. 4 Boundary element discretization scheme for corner treatment where δ_{ik} is the Kronecker delta, and H_{ik} and G_{ik} represent the influence coefficient matrices resulting from the above discretization process. They are the functions of geometry and can be readily evaluated with the use of quadrature.

5. Numerical Implementation

For computer implementation of the numerical solution for the present conjugate heat transfer problem, the boundary element equation of axisymmetric model Eq. (9) can now be written in matrix form as

$$\left([H]+\frac{1}{k}[G][C]\right)\{T\}=[G]\{q\} \qquad (26)$$

where, $[C]$ is a diagonal matrix containing the convective heat transfer coefficients. This equation represents a system of N unknowns with N Specified boundary variables, which can be readily solved for the unknowns.

Dual reciprocity boundary element equation Eq. (24) for the fluid region can now be expressed in a matrix form as

$$[H]\{u\}-[G]\{q\}=([H][\hat{U}]-[G][\hat{Q}])\{\beta\} \qquad (27)$$

It is noted that column vector $\{\beta\}$ can be evaluated from Eq. (21) as $\{\beta\}[F]^{-1}\{b\}$ with the chosen interpolating function f_j and the function $b(x,y)$ of governing equation.

To get the temperature T of the final solution, the flow velocity w must be obtained first by solving momentum equation and, in this case, the right hand side of Eq. (27) becomes a known vector. Thus, introducing the boundary conditions into the nodes $\{u\}$ and $\{q\}$, and then after rearranging the terms of each side lead to a set of simultaneous equations. The solutions of flow velocity field can now be readily obtained from these algebraic equations.

For the temperature solution, the source term b is expressed as follows as the fluid temperature T varies from the duct entrance through the exit in the axial direction.

$$\frac{1}{\alpha}w(x,y)\frac{\partial T(x,y,z)}{\partial z}=\frac{1}{\alpha}w(x,y)\dot{T}(x,y,z)=[U_D]\{\dot{T}\} \cong \sum_{j=1}^{N+L}\beta_j(z)f_j(x,y)=[F]\{\beta\} \qquad (28)$$

where $[U_D]$ represents a diagonal matrix with the element of $w(x,y)/\alpha$. Substituting the above source term expression into Eq. (27) with the use of general variable u instead of temperature T, the following equation can be obtained :

$$[C]\{\dot{u}\}+[H]\{u\}=[G]\{q\} \qquad (29)$$

where $[C]=([H][\hat{U}]-[G][\hat{Q}])[F]$.

For simplicity in finding the numerical solutions for the system, the two-level line integration scheme (Jo et al., 1999) is employed in this study. A linear approximation for the variations of T and q within each axial distance step z is adopted in the form,

$$\begin{aligned} u&=(1-\zeta_u)u^m+\zeta_u u^{m+1} \\ q&=(1-\zeta_q)q^m+\zeta_q q^{m+1} \\ \dot{u}&=\frac{1}{\Delta z}(u^{m+1}-u^m) \end{aligned} \qquad (30)$$

where ξ_u and ξ_q are parameters, taking values in the range from 0 through 1, which position the values of u and q, respectively, between the calculation levels m and $m+1$. For the present study a fully implicit scheme is chosen. Thus, substituting Eqs. (30) into Eq. (29) with $\xi_u=1$ and $\xi_q=1$ gives as

$$\left(\frac{[C]}{\Delta z}+[H]\right)u^{m+1}-[G]q^{m+1}=\frac{[C]}{\Delta z}u^m \qquad (31)$$

where the right hand side of Eq. (31) is known at the current calculation level $m+1$ because it involves values which are specified as the initially known values or calculated at the previous calculation level m.

The temperature solution can be obtained by using the axial direction marching scheme that solves the above equation in each location step with moving forward from the duct inlet to the downstream in the axial direction.

6. Corner Treatment

When the boundary with discretized linear elements is smooth, both unique temperature and heat flux (or flux) at the second extreme node of element j-1 which is to be the first extreme node of next consecutive element j have the same values unless the fluxes are prescribed as different.

However, in general, a boundary element discretization and actual physical geometry of the calculation domain can yield a series of nodal points of geometric discontinuity so that the boundary at a geometrically discontinuous nodal point, joining any two contiguous elements of the fixed boundaries, may be prescribed differently (Jo et al., 1999).

For the case where either a unique temperature value or a unique flux value is prescribed at a boundary node regardless of its geometric discontinuity, the use of single nodes is generally sufficient for modeling by two contiguous linear elements. On the other hand, when a discontinuous value of flux at a boundary node is required, double nodes are used at the nodal point in order to allow for such discontinuity (see Figure 4). In the present study, the single node has been used for the temperature boundary condition and the double nodes for the heat flux boundary condition, respectively.

7. Solution Algorithm

For the present conjugate heat transfer problem, two separate solutions for the solid and fluid regions are sought to match the solid-fluid interface continuity condition explicitly because of coupled mechanism of the heat conduction in the solid and the heat convection in the fluid. Therefore the axisymmetric heat conduction problem for the solid region of a thick walled pipe is solved first by assuming thermal boundary condition of the solid-fluid interface, which is an unknown a priori. Then after the heat convection problem in thermally developing laminar flow inside the pipe with fully developed velocity profile is solved at the solid-fluid interface boundary with the boundary conditions obtained from the solution for the solid region. Calculations proceed iteratively until the continuities of temperature and heat flux at the interfacial boundary are satisfied.

Initial guess for the unknown interfacial thermal boundary conditions is needed to start the iterative calculation procedure. In the present study, two solution algorithms for the interface continuity conditions are tested. One is that the temperature or heat flux condition at the interface is guessed and updated iteratively until the interface continuity condition is satisfied. This algorithm works well when the temperature condition is imposed on the outer surface of thick walled pipe. It is conformed in this study and shown also from the previous paper (He et al., 1995). However, when the heat flux condition is imposed, the converged solution cannot be obtained except a few limited cases. The onset of instabilities leading to divergence is due to the direct imposition of temperature at interface, which

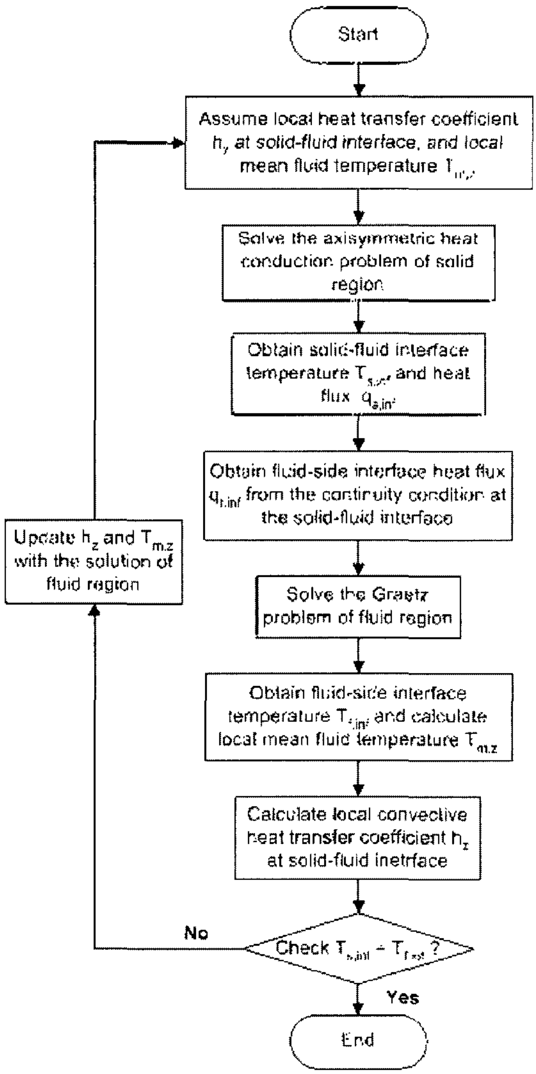


Fig. 5 Solution algorithm for the thermally developing conjugate heat transfer problem

turns out to be an extremely rigid constraint. To avoid such difficulty, the computation should be started with a guessed distribution of temperature almost equal to the exact solution, which will be impractical.

The other is to use the coupling conditions for the interface continuity condition. They are the local convective heat transfer coefficient at the interface and the local mean fluid temperature of the pipe flow. These conditions are initially guessed and updated iteratively as shown in Fig. 5. It is seen in this study that this algorithm works well for both the temperature and heat flux boundary conditions at the outer surface of thick walled pipe. Thus, the heat transfer coefficient is considered to be an effective interface condition for transmitting thermal information through an interface during the iteration process.

8. Results and Discussion

To illustrate the validity of the present BEM technique for solving the conjugate heat transfer problem, some numerical tests were performed for the thermally developing laminar flow through a thick walled pipe with following conditions (see Fig. 1). Fluid is assumed to be water in the present study.

$$r_i = 0.02\ \text{m}$$

$$r_o = 0.32\ \text{m}$$

$$L = 0.4\ \text{m}$$

$$\frac{1}{\mu}\frac{dp}{dz} = -36.75\,(\text{m}\cdot\text{s})^{-1}$$

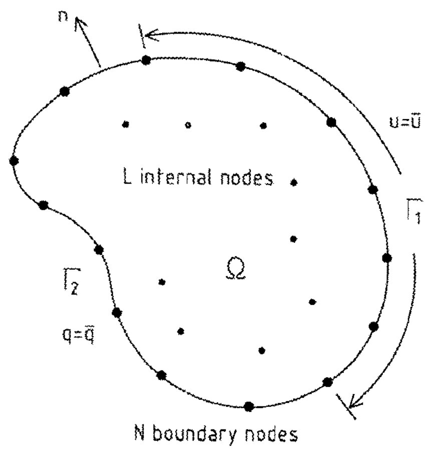
$$\alpha = 0.147\times10^{-6}\,\text{m}^2/\text{s}$$

$$k_f = 0.613\ \text{W/mK}$$

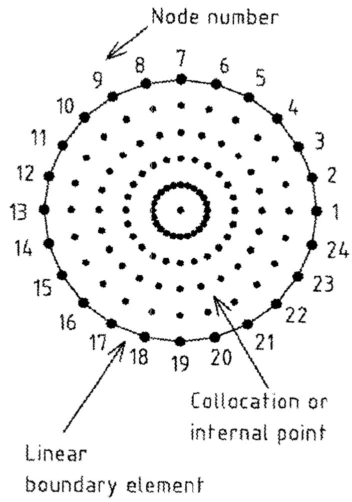
$$\text{Pr} = 5.83$$

Thus flow conditions will have Peclet number $Pe=500$ and Graetz number $Gz=50$, where the dimensionless numbers are defined as $Pe=2w_m r_i/\alpha$, $Gz=4w_m r_i^2/L\alpha$. For the conjugate heat transfer conditions, ratio of thermal conductivity of the pipe wall to that of the fluid $(K=k_s/k_f)$ considered in the present test calculations for comparison with Mori's study (Mori et al., 1974) ranges from $K=1$ to 5000 for the case where a constant heat flux at the outer surface of the pipe is specified, and from $K=0.1$ to 100 for the case where a constant temperature at the outer surface of the pipe is specified.

Now, for the present calculations, the solution domain of fluid region is discretized with 24 boundary nodes and 217 internal DRBEM collocation nodes which are distributed with 10 subintervals from a center node of the pipe to each boundary nodes, as shown in Fig. 6. The calculation step in the axial direction is chosen as $\Delta z=$ 0.002 m. Thus the solid-fluid interfacial boundary of the solution domain region is also discretized



(a) Boundary Γ and domain Ω



(b) Boundary element nodes and internal points

Fig. 6 Geometric definitions for the DRBEM analysis with $\Delta z=0.002$ m, which results in 200 boundary nodes at the interface side and 200 nodes at the opposite side as well. Each insulated sides are discretized with 6 elements respectively. Since the velocity and temperature solutions are to be given at the boundary nodes and internal points only, Reynolds number and Nusselt number are numerically obtained from the mean flow velocity w_m and the mixed mean temperature $T_{m,z}$ at each cross-section as

$$w_m=\frac{2}{r_i^2}\int_{r=0}^{r=r_i} wrdr$$

$$T_{m,z}=\frac{2}{r_i^2 w_m}\int_{r=0}^{r=r_i} wTrdr$$

$$Re=\frac{w_m D}{\nu}=\frac{w_m 2r_i}{\nu}$$

$$Nu_z=\frac{hD}{k_f}=2r_i\frac{-(\partial T/\partial n)_{r=r_i}}{T_w-T_{m,z}}$$

And for the present iterative scheme, relative error for the convergence criteria of the solution algorithm is calculated as the following equation.

$$\varepsilon=|1.0-\frac{T_{s,inf}}{T_{f,inf}}| \tag{32}$$

Here the subscript inf means the interface condition. To ensure convergence, local convective heat transfer coefficient at the interface and the local mean fluid temperature of the pipe flow is updated with under-relaxation factor λ.

$$(h_z)_{update}=\lambda(h_z)_{new}+(1-\lambda)(h_z)_{old} \tag{33}$$

$$(T_{m,z})_{update}=\lambda(T_{m,z})_{new}+(1-\lambda)(T_{m,z})_{old} \tag{34}$$

Then, in this study, values of $\varepsilon=0.1\times10^{-3}$, $\lambda=0.8$ are taken.

Two cases of boundary condition given by either uniform temperature or uniform heat flux are tested for the present study, where inlet fluid temperature is chosen as $T_e=30$℃. Numerical solutions are verified by the comparison with the only available analytic solution for this kind of problem, where a power series form with the first five terms is employed to assume the temperature distribution at the solid-fluid interface (Mori et al., 1976 ; 1974). Figs. 7 and 8 respectively represent the variations of the local Nusselt number

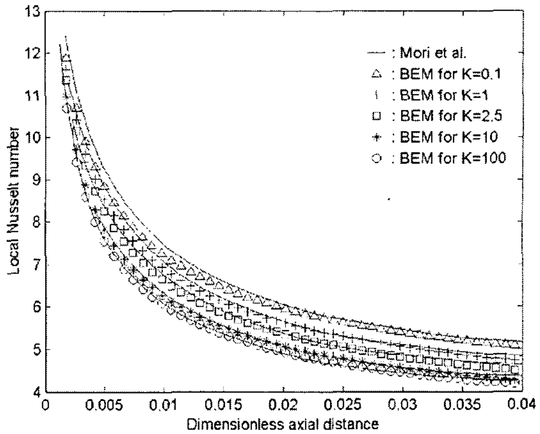


Fig. 7 Comparison for the variations of local Nusselt number when constant temperature is imposed on the outer surface of the pipe

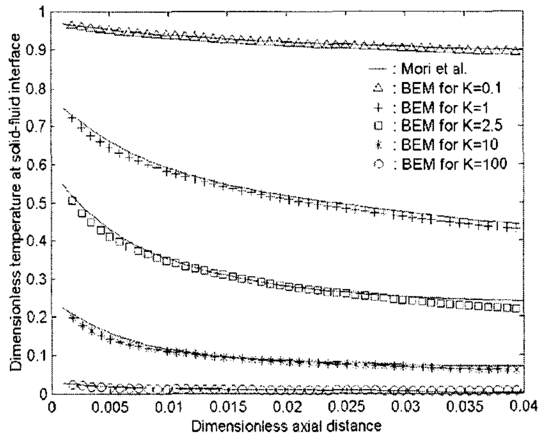


Fig. 8 Comparison for the variations of interface temperature when constant temperature is imposed on the outer surface of the pipe

Nu_z and the dimensionless interface temperature $(T-T_w)/(T_e-T_w)$ along the dimensionless axial distance z/r_iPe when the uniform temperature $T_w=100$℃ is specified on the outer surface of the pipe. Results for the five cases of K value show that the present BEM solution is in good agreement over the whole dimensionless axial distance with the Mori's analytic solution (Mori et al., 1976).

For the case of uniform heat flux boundary condition, where $(\partial T/\partial n)_w=10.0$ ℃/m is specified on the outer surface of the pipe, the calculation results of BEM solution are depicted in

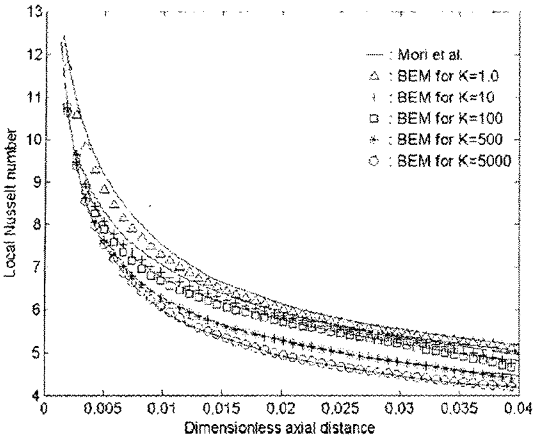


Fig. 9 Comparison for the variations of local Nusselt number when constant heat flux is imposed on the outer surface of the pipe

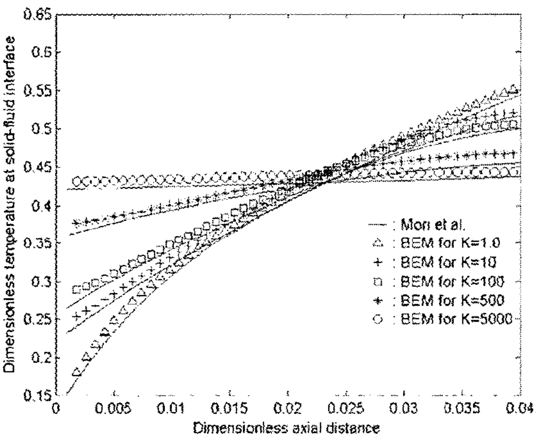


Fig. 10 Comparison for the variations of interface temperature when constant heat flux is imposed on the outer surface of the pipe

Figs. 9 and 10 which respectively shows the numerically calculated Nusselt number variations and the variations of the interface temperature $(T-T_e)/Kr_o(\partial T/\partial n)_w$ along the dimensionless axial distance. From the figures it can be also found that the computed values are in good agreement with Mori's analytic values (Mori et al., 1974).

As the results, the present BEM solutions for the conjugate heat transfer problem generally are in good agreement with the Mori's analytic solutions for the thermally developing region of thick walled pipe in which the flow is laminar and fully developed. When we note that the analytic solution also involves some approximations as mentioned above, minor discrepancies between the BEM and Mori's solutions can be regarded as the acceptable. Therefore the present analysis method is considered to be valid for the present problem with any kind of thermal boundary conditions.

9. Conclusions

A boundary element method (BEM) was applied for the solution of the conjugate heat transfer problem of a thick walled pipe with thermally developing laminar internal flow of which the velocity profile is fully developed. The dual reciprocity BEM was applied to the solution of fluid region inside pipe while the axisymmetric BEM was used to solve the solid region of thick walled pipe. In this study, an iterative and numerically stable BEM solution algorithm was presented which uses the coupled interface conditions explicitly instead of the uncoupled ones. Both the local convective heat transfer coefficient at solid-fluid interface and the local mean fluid temperature are guessed initially and updated as the unknown interface thermal conditions for the iterative solution procedure in the calculations. Two example cases of imposing either uniform temperature or uniform heat flux boundary conditions were tested and compared with available analytic solutions. Test results are shown to be in good agreement with the analytic solutions for both examples with different boundary conditions. As a final remark, it should be noted that the BEM solution method for the thermally developing conjugate heat transfer problem presented in this paper is a general one that can be extended to the arbitrary duct geometry and boundary conditions without any modification.

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