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A Rectangular Fin Optimization Including Comparison Between 1-D and 2-D Analyses

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Both 1-D and 2-D analytic methods are used for a rectangular fin optimization. Optimum heat loss is taken as 98% of the maximum heat loss. Temperature profile using 2-D analytic method and relative error of temperature along the fin length between 1-D and 2-D analytic methods are presented. Increasing rate of the optimum heat loss with the variation of Biot number and decreasing rate of that with the variation of the fin base length are listed. Optimum fin tip length using 2-D analytic method and relative error of that between 1-D and 2-D analytic methods are presented as a function of Biot numbers ratio.

Key Words : Analytical Method, Optimization, Heat Loss, Biot Number

Nomenclature

Bi : Fin top and bottom Biot number, $(h\,l')/k$
Bi_e : Fin tip Biot number, $(h_e l')/k$
h : Fin top and bottom heat transfer coefficient [W/m² ℃]
h_e : Fin tip heat transfer coefficient [W/m² ℃]
k : Thermal conductivity of fin material [W/m ℃]
l' : One half fin base height [m]
L'_b : Fin base length [m]
L_b : Dimensionless fin base length, L'_b/l'
L'_e : Fin tip length [m]
L_e : Dimensionless fin tip length, L'_e/l'
q : Heat loss per unit width [W/m]
Q : Dimensionless heat loss, $q/(k\varphi_i)$
T : Fin temperature [℃]
T_b : Fin base temperature [℃]
T_i : Temperature of inside wall [℃]
T_∞ : Ambient temperature [℃]
x' : Length directional variable [m]
x : Dimensionless length directional variable, x'/l'
y' : Height directional variable [m]
y : Dimensionless height directional variable, y'/l'

Greek symbol

β : Ratio of Biot numbers, Bi_e/Bi
θ : Dimensionless temperature, $(T-T_\infty)/(T_i-T_\infty)$
λ_n : Eigenvalues $(n=1,2,3,\cdots)$
φ_i : Adjusted temperature of inside wall [℃], (T_i-T_∞)

Subscript

1 : One-dimensional analysis
2 : Two-dimensional analysis
b : Fin base
e : Fin tip
i : Inside wall
∞ : Surrounding

Superscript

' : Dimensional quantity
* : Optimum

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1. Introduction

Extended surfaces or fins are designed to increase the rate of heat transfer to a surrounding fluid in many engineering applications such as the cooling of combustion engines, many kinds of heat exchangers, air craft and so on. Optimization procedures of various shapes of fins have been studied. For a given heat dissipation rate, the geometric shape that minimizes the material volume could be found. For this kind of optimization, Hrymak et al.(1985) presented an efficient numerical method to discover the optimal shape for a fin subject to both convective and radiating heat loss. One of the alternative ways is to fix a suitable simple profile, and then to determine the dimensions of fin to yield the maximum heat dissipation for a given fin volume or mass. For example, Ullmann and Kalman (1989) considered a problem of increasing the heat dissipation of annular fins (four different cross-section shapes) at a defined magnitude of mass. Chung et al.(1989) dealt with the optimum design of convective longitudinal fins of a trapezoidal profile. Yeh (1996) determines the optimum dimensions of a one-dimensional longitudinal rectangular fin and a cylindrical pin fin. Kang and Chung (2003) analyze and optimize a design for a rectangular profile annular fin. Casarosa and Franco (2001) investigated the optimum design of single longitudinal fins with constant thickness by means of an accurate mathematical method yielding the solution of constrained minimization (maximization) problems considering different uniform heat transfer coefficients on the fin faces and on the tip. Also Razelos and Satyaprakash (1993) present an analysis of trapezoidal profile longitudinal fins that delineates their thermal performance and an improved solution of the optimal problem. Another alternative way is to fix a fin height and to choose the 98% of the maximum heat loss as the optimum heat loss. For this optimum procedure, Kang (2001) showed the optimum heat loss from a thermally asymmetric rectangular fin using three-dimensional analysis. Recently, Kang and Look (2004) present the optimum heat loss and dimensions for a thermally and geometrically asymmetric trapezoidal fin.

In all these papers, fin base temperature is given as constant for the boundary condition and the effect of base thickness is not considered. In this study, for a straight rectangular fin, inside wall temperature is given and the effect of the fin base thickness is shown. The fin height is fixed and the optimum heat loss is taken as 98% of the maximum heat loss for given conditions. For this optimum criterion (Kang, 2001 ; Kang and Look, 2004), the optimum heat loss and dimensions are analyzed and the relative errors of the optimum values between 1-D and 2-D analyses are presented.

2. Analytical Methods

2.1 1-D Analysis

One-dimensional energy balance equation under steady state for a rectangular fin shown in Fig. 1 is given as a dimensionless form by Eq. (1).

$$\frac{d^2\theta_1}{dx^2}-Bi\cdot\theta_1=0 \tag{1}$$

Two boundary conditions are required to solve the Eq. (1) and these conditions are shown as Eqs. (2) and

$$-\frac{d\theta_1}{dx}\bigg|_{x=L_b}=\frac{1}{L_b}(1-\theta_1|_{x=L_b}) \tag{2}$$

$$\frac{d\theta_1}{dx}\bigg|_{x=L_e}+Bi_e\cdot\theta_1|_{x=L_e}=0 \tag{3}$$

The solution for the dimensionless temperature distribution $\theta_1(x)$ within the rectangular fin is written in Eq. (4).

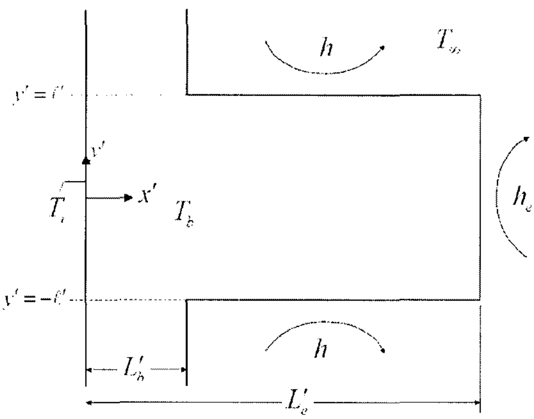

Fig. 1 Geometry of a rectangular fin

$$\theta_1(x)=\frac{f_1(x)+g_1(x)}{A+B} \quad (4)$$

where

$$f_1(x)=\sqrt{Bi}\cdot\cosh\{\sqrt{Bi}\cdot(L_e-x)\} \quad (5)$$

$$g_1(x)=Bi_e\cdot\sinh\{\sqrt{Bi}\cdot(L_e-x)\} \quad (6)$$

$$A=(L_b\cdot Bi_e+1)\cdot\sqrt{Bi}\cdot\cosh\{\sqrt{Bi}\cdot(L_e-L_b)\} \quad (7)$$

$$B=(L_b\cdot Bi+Bi_e)\cdot\sinh\{\sqrt{Bi}\cdot(L_e-L_b)\} \quad (8)$$

By applying Eq. (4) to Fourier's law, the heat loss per unit width conducted into the fin through the fin base is calculated by

$$q_1=k\cdot A(x')\frac{dT}{dx'}\bigg|_{x'=L_b'} \quad (9)$$

Then dimensionless heat loss is given by

$$Q_1=\frac{q_1}{k\cdot\varphi_i}=\frac{C+D}{A+B} \quad (10)$$

where

$$C=Bi\cdot\sinh\{\sqrt{Bi}\cdot(L_e-L_b)\} \quad (11)$$

$$D=Bi_e\cdot\sqrt{Bi}\cdot\cosh\{\sqrt{Bi}\cdot(L_e-L_b)\} \quad (12)$$

2.2 2-D Analysis

Dimensionless two-dimensional governing differential equation under steady state for a rectangular fin shown in Fig. 1 is

$$\frac{\partial^2\theta_2}{\partial x^2}+\frac{\partial^2\theta_2}{\partial y^2}=0 \quad (13)$$

Four boundary conditions are required to solve the Eq. (13). These boundary conditions are shown as Eqs. (14) ~ (17).

$$-\frac{\partial\theta_2}{\partial x}\bigg|_{x=L_b}=\frac{1}{L_b}(1-\theta_2|_{x=L_b}) \quad (14)$$

$$\frac{\partial\theta_2}{\partial y}\bigg|_{y=0}=0 \quad (15)$$

$$\frac{\partial\theta_2}{\partial x}\bigg|_{x=L_e}+Bi_e\cdot\theta_2|_{x=L_e}=0 \quad (16)$$

$$\frac{\partial\theta_2}{\partial y}\bigg|_{y=1}+Bi\cdot\theta_2|_{y=1}=0 \quad (17)$$

The solution for the temperature distribution $\theta_2(x,y)$ within the rectangular fin obtained using separation of variables method with Eqs. (13) ~ (16) is

$$\theta_2(x,y)=\sum_{n=1}^{\infty}\frac{A_n\cdot f_2(x)\cdot\cos(\lambda_n y)}{B_n+C_n} \quad (18)$$

where

$$A_n=\frac{4\sin(\lambda_n)}{2\lambda_n+\sin(2\lambda_n)} \quad (19)$$

$$B_n=\cosh(\lambda_n L_b)-L_b\cdot\lambda_n\cdot\sinh(\lambda_n L_b) \quad (20)$$

$$C_n=f_n\cdot\{\sinh(\lambda_n L_b)-L_b\cdot\lambda_n\cdot\cosh(\lambda_n L_b)\} \quad (21)$$

$$f_n=-\frac{\lambda_n\cdot\tanh(\lambda_n L_e)+Bi_e}{\lambda_n+Bi_e\cdot\tanh(\lambda_n L_e)} \quad (22)$$

$$f_2(x)=\cosh(\lambda_n x)+f_n\cdot\sinh(\lambda_n x) \quad (23)$$

The eigenvalues λ can be obtained from Eq. (24), which comes from Eq. (17).

$$\lambda_n\cdot\tan(\lambda_n)=Bi \quad (24)$$

The heat loss per unit width conducted into the fin through the fin base using 2-D analysis is calculated by

$$q_2=\int_{-l'}^{l'}-k\frac{\partial T}{\partial x'}\bigg|_{x'=L_b'}dy' \quad (25)$$

Dimensionless heat loss is written as

$$Q_2=\frac{q_2}{k\cdot\varphi_i}=-\sum_{n=1}^{\infty}\frac{8D_n\cdot\sin^2(\lambda_n)}{B_n+C_n} \quad (26)$$

where

$$D_n=\sinh(\lambda_n L_b)+f_n\cdot\cosh(\lambda_n L_b) \quad (27)$$

3. Results and Discussions

Figure 2 represents the variation of the temperature along the fin center-line (base to tip) for $L_b=1$ and $L_2=4$ in the case of $Bi=0.01$, 0.05 and 0.1 using 2-D analytic method. It is shown that the temperature at the fin base decreases and the temperature along the fin center-line decreases more rapidly as fin top and bottom Biot number increases.

Relative error of the temperature along the fin length between 1-D and 2-D analyses under the same condition as given in Fig. 2 is shown in Fig. 3. Relative error increases rapidly first and then increases slowly as x increases. It shows that the

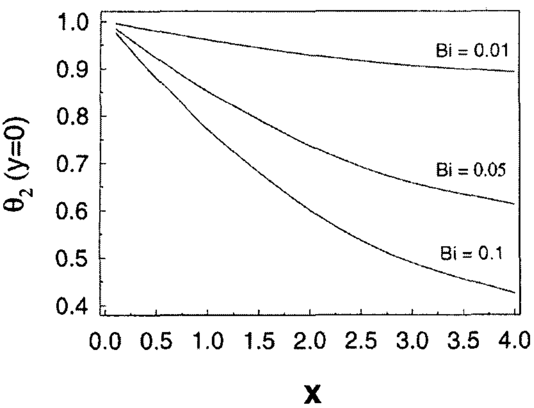

Fig. 2 Temperature profile along the fin length ($\beta=1$, $L_b=0.1$, $L_e=4$)

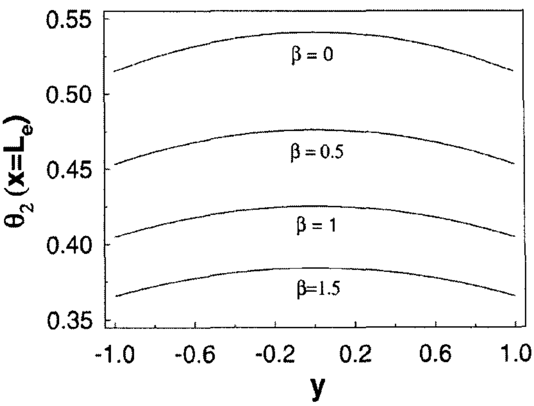

Fig. 4 Temperature profile along the fin height ($Bi=0.1$, $L_b=0.1$, $L_e=4$)

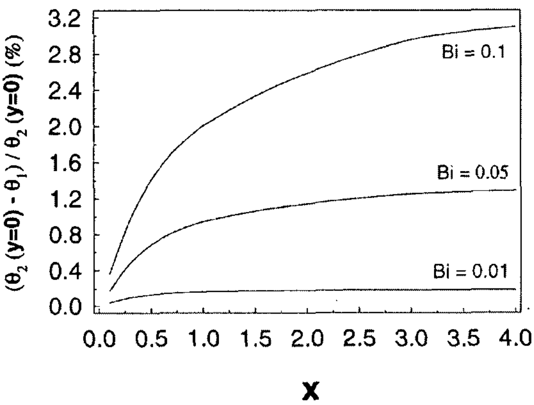

Fig. 3 Relative error of temperature between 1-D and 2-D analyses ($\beta=1$, $L_b=0.1$, $L_e=4$)

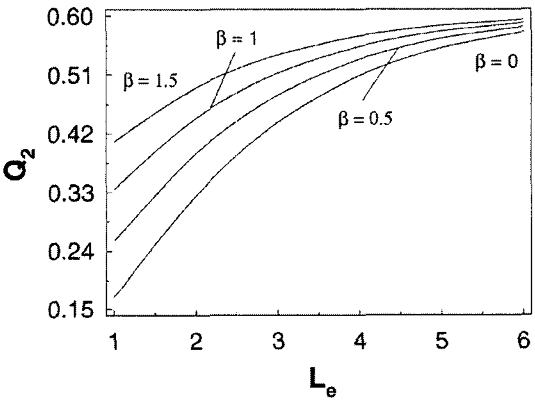

Fig. 5 Heat loss versus the fin tip length ($Bi=0.1$, $L_b=0.1$)

relative error reaches to 3.1% for $Bi=0.1$ while that is within 0.2% in the case of $Bi=0.01$ at the fin tip.

Figure 4 presents the variation of the temperature along the fin height at the fin tip for several values of β. As expected, the temperature at the center is the highest and it decreases along the top and bottom directions symmetrically. From this figure, it can be guessed that the temperature at the fin tip decreases rapidly first and then decreases slowly as β increases in this relatively short fin case.

Figure 5 represents the heat loss versus the fin tip length for several values of β. Heat loss increases rapidly first and then increases slowly as the fin tip length increases. It also shows that heat loss increases as β increases at the same fin tip length and the effect of β on the heat loss becomes smaller as the fin tip length increases. This figure implies that there is no effect of β on the optimum heat loss if the fin tip length for the optimum heat loss is long enough to eliminate the effect of fin tip heat transfer coefficient.

Table 1 lists the relative error of the optimum heat loss between 1-D and 2-D analyses for all β and $L_b=0.1$. The condition 'for all β' means that β has no effect on the optimum heat loss. It can be known that the relative error increases from 0.02% to 1.45% as Bi increases from 0.001 to 0.1 and the optimum heat loss calculated using 1-D analysis is greater than that calculated using 2-D analysis.

Table 1 Relative error of the optimum heat loss between 1-D and 2-D analyses (all β, L_b=0.1)

Bi	0.001	0.01	0.1
$(Q_2^*-Q_1^*)/Q_2^*(\%)$	−0.02%	−0.16%	−1.45%

Table 2 Increasing rate of the optimum heat loss (all β, L_b=0.1)

Bi	I.R. of Q_2^* (%)
0.01 → 0.03	71.44
0.03 → 0.05	28.09
0.05 → 0.1	39.19
0.1 → 0.15	20.85

Increasing rate of the optimum heat loss obtained by using 2-D analysis for all β and L_b=0.1 is listed in Table 2. As already mentioned in Table 1, β has no effect on the optimum heat loss. It is noted that increasing rate of the optimum heat loss decreases as Bi increases. It means that the optimum heat loss increases rapidly first and then increases slowly with the increase of Bi.

Figure 6 shows the optimum fin tip length as a function of Bi for several values of β in the case of L_b=0.1. For all β, the optimum fin tip length decreases somewhat rapidly first and then decreases slowly as Bi increases. Even though β has no effect on the optimum heat loss, the optimum fin tip length decreases as β increases for the same value of Bi. This phenomenon explains physically that the optimum heat loss is obtained at shorter fin tip length as fin tip heat transfer coefficient increases.

Figure 7 represents the relative error of the optimum fin tip length between 1-D and 2-D analyses as a function of β. For all given five fin top and bottom Biot numbers, the relative error varies from relatively large value to smaller value as β increases from 0 to 1.8. The range of relative error becomes wide as Bi increases, for example, the relative error in the case of Bi=0.03 decreases from 0.49% to 0.40% while that for Bi=0.2 decreases from 3.1% to −4.1% as β increases from 0 to 1.8. For given range of β, the relative error seems to be tolerated until Bi=0.1.

Table 3 lists decreasing rate of the optimum heat loss for all β and Bi=0.1 with the increase of the fin base length. It is noted that the optimum heat loss decreases as the fin base length increases. It is because that the resistance between the inside wall and the fin base increases as the fin base length increases. It also shows that the decreasing rate of the optimum heat loss becomes smaller as the fin base length increases.

The optimum fin tip length for Bi=0.1 is represented as a function of β in Fig. 8. It shows

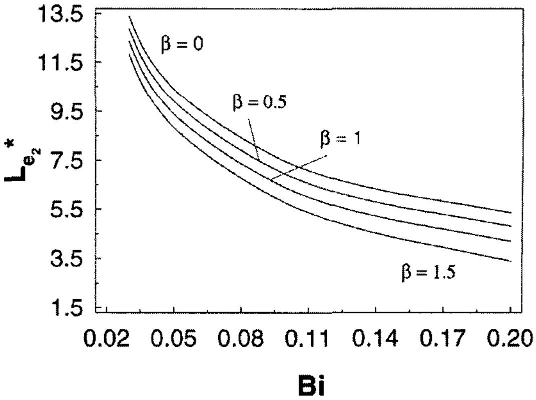

Fig. 6 Optimum fin tip length versus Bi (L_b=0.1)

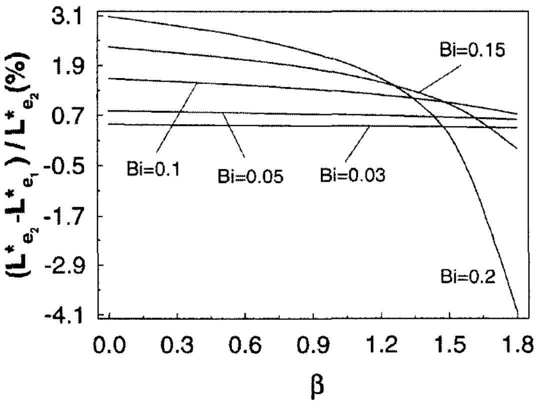

Fig. 7 Relative error of the optimum fin tip length between 1-D and 2-D analyses (L_b=0.1)

Table 3 Decreasing rate of the optimum heat loss (all β, Bi=0.1)

L_b	D.R. of Q_2^* (%)
0.1 → 0.2	2.96
0.2 → 0.3	2.86
0.3 → 0.4	2.78

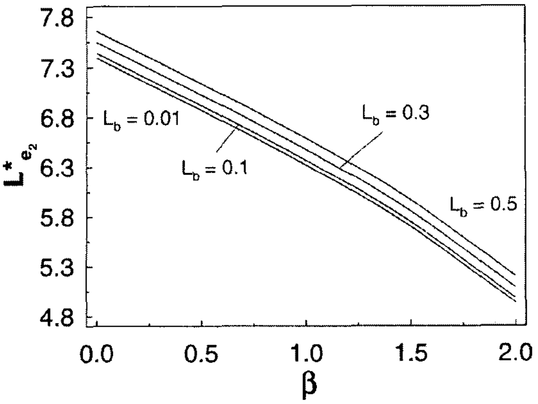

Fig. 8 The optimum fin tip length versus β (Bi=0.1)

that the optimum fin tip length decreases monotonically as β increases. It also indicates that the optimum fin tip length increases as the fin base length increases for a fixed value of β. It must be noted that the actual optimum fin length decreases with the increase of the fin base length since the fin length is the difference between the fin tip length and the fin base length.

4. Conclusions

The following conclusions can be drawn from the results.

(1) Under usual circumstances (i.e. $Bi<0.1$), relative errors of the optimum heat loss and fin tip length between 1-D and 2-D analyses are within 1.6% for given range of other variables.

(2) The optimum heat loss increases while the optimum fin tip length decreases as fin top and bottom Biot number increases.

(3) The actual optimum fin length decreases even though the optimum fin tip length increases with the increase of the fin base length.

(4) The optimum fin tip length decreases remarkably as β decreases even though the optimum heat loss is independent on the variation of β.

References

Casarosa, C. and Franco, A., 2001, "On the Optimum Thermal Design of Individual Longitudinal Fins with Rectangular Profile," *Heat Transfer Engineering*, Vol. 22, No. 1, pp. 51~71.

Chung, B. T. F., Abdalla, M. H. and Liu, F., 1989, "Optimization of Convective Longitudinal Fins of Trapezoidal Profile," *Chem. Eng. Comm.*, Vol. 80, pp. 211~223.

Hrymak, A. N., McRae, G. J. and Westerberg, A. W., 1985, "Combined Analysis and Optimization of Extended Heat Transfer Surfaces," *ASME J. of Heat Transfer*, Vol. 107, No. 3, pp. 527~532.

Kang, H. S. and Chung, B. T. F., 2003, "Two-Dimensional Analytical Solutions for Optimization of Convective Annular Fin of Rectangular Profile," *2003 ASME International Mechanical Engineering Congress & Exposition*, Washington, DC, IMECE2003-41508, Nov. 15-21.

Kang, H. S. and Look, D. C. Jr., 2004, "Optimization of Thermally and Geometrically Asymmetric Trapezoidal Fins," *AIAA J. of Thermophysics and Heat Transfer*, Vol. 18, No. 1, pp. 52~57.

Kang, H. S., 2001, "Optimization of a 3-D Thermally Asymmetric Rectangular Fin," *KSME International Journal*, Vol. 15, No. 11, pp. 1541~1547.

Razelos, P. and Satyaprakash, B. R., 1993, "Analysis and Optimization of Convective Trapezoidal Profile Longitudinal Fins," *ASME J. of Heat Transfer*, Vol. 115, No. 2, pp. 461~463.

Ullmann, A. and Kalman, H., 1989, "Efficiency and Optimized Dimensions of Annular Fins of Different Cross-Section Shapes," *Int. J. of Heat and Mass Transfer*, Vol. 32, pp. 1105~1110.

Yeh, R. H., 1996, "Errors in One-Dimensional Fin Optimization Problem for Convective Heat Transfer," *Int. J. of Heat and Mass Transfer*, Vol. 39, No. 14, pp. 3075~3078.