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PROXIMAL AND DISTAL HOMOMORPHISMS OF FLOWS

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Abstract. In this paper we study some characterizations of proximal, distal and almost one to one homomorphisms of flows. In particular we show that if the almost one to one proximal extension of a minimal flow is weakly almost periodic, then it is minimal.

1. Introduction

A flow (T, X) is a topological action $(t, x) \mapsto tx$ of the discrete group T on the compact Hausdorff space X. The enveloping semigroup $E(X)$ of the flow is a kind of compactification of the acting group and is itself a flow. The flow is *minimal* if every orbit is dense. A flow (T, X) is *weakly almost periodic* iff each element of $E(X)$ is continuous [3].

If (T, X) and (T, Y) are flows, a *homomorphism* is a continuous equivariant map $\phi : X \to Y$, $\phi(tx) = t\phi(x)$ $(t \in T,\ x \in X)$. We say that a homomorphism $\phi : X \to Y$ is *proximal* (*distal*) if whenever $x_1, x_2 \in \phi^{-1}(y)$ then x_1 and x_2 are proximal (distal). A homomorphism $\phi : X \to Y$ is *almost one to one* if there exists a point $y_0 \in Y$ such that $\phi^{-1}(y_0)$ is a singleton. We say that X is a *proximal*, *distal*, and *almost one to one extension* of Y provided that there exists a proximal, distal, and almost one to one homomorphism of (T, X) onto (T, Y), respectively.

In this paper we investigate some characterizations of proximal, distal and almost one to one homomorphisms of flows.

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2. Some characterizations of proximal, distal and almost one to one homomorphisms

THEOREM 2.1. *[4] A distal extension of a distal flow is distal.*

THEOREM 2.2. *[4] (1) A proximal extension of a minimal flow contains a unique minimal set.*

(2) A distal extension of a minimal flow is a disjoint union of minimal sets.

In [5], Song proved the following theorem :

THEOREM 2.3. *(1) A proximal extension of a proximal flow is proximal.*

(2) A distal extension of a pointwise almost periodic flow is pointwise almost periodic.

(3) An almost one to one extension of a proximal minimal flow is proximal.

The proof of the next corollary follows from Theorem 2.2.1 and Theorem 2.3.3.

COROLLARY 2.4. *An almost one to one extension of a proximal minimal flow contains a unique proximal minimal set.*

THEOREM 2.5. *If the proximal extension of a minimal flow is pointwise almost periodic, then it is minimal.*

Proof. Let (T, X) be pointwise almost periodic. Then $\{\overline{Tx} \mid x \in X\}$ is a partition of X consisting of minimal sets (see Proposition 2.6 in [2]). By Theorem 2.2.1 $\overline{Tx} = N$ for all $x \in X$, where N is a unique minimal set in X. Thus $X = N$.

COROLLARY 2.6. *If the proximal extension of a minimal flow is distal, then it is minimal.*

Proof. Note that if (T, X) is distal, then it is pointwise almost periodic (see Corollary 5.5 in [2]).

THEOREM 2.7. *Suppose* $\phi : (T, X) \to (T, Y)$ *is an almost one to one proximal homomorphism and* (T, Y) *is minimal. Then :*

(1) X *contains a unique minimal set* N.

(2) If $\phi^{-1}(\{y_0\}) = \{x_0\}$, *then* $x_0 \in N$.

(3) For every $x \in X$, *there exists an element* $p \in E(X)$ *such that* $px \in N$.

Proof. (1) By Theorem 2.2.1, it is obvious.

(2) Since $\phi(N) = Y$, it follows that $x_0 \in N$.

(3) Let $\phi^{-1}(\{y_0\}) = \{x_0\}$. Suppose $x \in X$. If $x \in N$, then we are done since $ex = x$, where e is the identity of T. If $x \in X - N$,then we let $\phi(x) = y$. By the minimality of Y, we choose $r \in E(Y)$ such that $y = ry_0$. Then There exists an element $q \in E(X)$ such that $\psi(q) = r$, where $\psi : E(X) \to E(Y)$ is the unique epimorphism induced by ϕ. So we have $\phi(x) = \psi(q)\phi(x_0) = \phi(qx_0)$. Hence x and qx_0 are proximal. Thus there exists a minimal right ideal I in $E(X)$ such that $px = p(qx_0)$ for all $p \in I$. Since $pq(x_0) \in N$, we have $px \in N$.

COROLLARY 2.8. *If the almost one to one proximal extension of a minimal flow is weakly almost periodic, then it is minimal.*

Proof. Note that if (T, X) is weakly almost periodic, then there exists the only minimal ideal I in $E(X)$ such that I is a group.

For a discrete group T, the Stone-Cěch compactification βT of T is a compact Hausdorff space which contains T as a dense subset and has the following universal property. Every map ϕ of T into a compact Hausdorff space X can be extended to a unique continuous map $\psi : \beta T \to X$. Note that βT is a universal point transitive flow for T. For a point transitive flow (X, x_0) and $p \in \beta T$ we shall write $px_0 = \phi(p)$, where $\phi : (\beta T, e) \to (X, x_0)$.

Let us fix from now on a minimal ideal M in βT. We denote by J its set of idempotents and we choose a distinguished idempotent $u \in J$. Denote by G the group uM.

Given a minimal flow X, we choose a point $x_0 \in uX = \{ux \mid x \in X\}$. Under the canonical map $(\beta T, e) \to (X, x_0)$, M is mapped onto X and u onto x_0. Thus (M, u) is a universal minimal pointed flow.

Let (X, x_0) be a pointed minimal flow. We define the Ellis group of (X, x_0) to be $G(X, x_0) = \{\alpha \in G \mid \alpha x_0 = x_0\}$. Clearly $G(X, x_0)$ is a subgroup of G.

LEMMA 2.9. *If X is minimal and v is an idempotent in some minimal ideal in βT, then every pair of points in vX is distal.*

THEOREM 2.10. *[4] If $\phi : (X, x_0) \to (Y, y_0)$ is a homomorphism of pointed minimal flows, then $G(X, x_0) \subset G(Y, y_0)$.*

THEOREM 2.11. *Let $\phi : (X, x_0) \to (Y, y_0)$ be a homomorphism of pointed minimal flows, and let $py_0 = qy_0$ for some $p, q \in M$. Then*
(1) $up^{-1}q \in G(Y, y_0)$.
(2) If ϕ is proximal, then $up^{-1}q \in G(X, x_0)$.

Proof. (1) This follows from the fact that $up^{-1}qy_0 = up^{-1}py_0 = uy_0 = y_0$.

(2) If $py_0 = qy_0$ for some $p, q \in M$, we have $\phi(up^{-1}qx_0) = up^{-1}qy_0 = up^{-1}py_0 = \phi(ux_0) = \phi(x_0)$. Hence $up^{-1}qx_0$ and x_0 are proximal. On the other hand by Lemma 2.9 $up^{-1}qx_0$ and $x_0 = ux_0$ are distal. Hence $up^{-1}qx_0 = x_0$, i.e. $up^{-1}q \in G(X, x_0)$.

THEOREM 2.12. *Let $\phi : (X, x_0) \to (Y, y_0)$ be a homomorphism of pointed minimal flows. Then the following conditions are pairwise equivalent.*
(a) ϕ is proximal.
(b) $G(X, x_0) = G(Y, y_0)$.
(c) $\phi^{-1}(y) \subset Jx$ for any $x \in \phi^{-1}(y)$.

Proof. We prove (c) $\Rightarrow$ (a). The other statements were proved by Glasner in [4]. Let $x_1, x_2 \in \phi^{-1}(y)$. By hypothesis, $x_2 \in Jx_1$. Hence there exists an idempoent $v \in J$ such that $x_2 = vx_1$. Since $vx_2 = v(vx_1)$, it follows that x_1 and x_2 are proximal. This means that ϕ is proximal.

THEOREM 2.13. *[4] Let $\phi : (X, x_0) \to (Y, y_0)$ be a homomorphism of pointed minimal flows. Then the following statements are equivalent.*
(a) ϕ is distal.
(b) For every $y \in Y$ and $p \in M$ such that $py_0 = y$, we have $\phi^{-1}(y) = pG(Y, y_0)x_0$.

THEOREM 2.14. *Let (X, x_0) and (Y, y_0) be pointed minimal flows and let (Y, y_0) be distal. Then: $G(X, x_0) \subset G(Y, y_0)$ iff there exists a homomorphism $\phi : (X, x_0) \to (Y, y_0)$.*

Proof. Suppose $G(X, x_0) \subset G(Y, y_0)$. Define $\phi : (X, x_0) \to (Y, y_0)$ by $\phi(px_0) = py_0$ for each $p \in M$. If $x = px_0 = qx_0$ for $p, q \in M$, we have from Theorem 2.11.1 that $up^{-1}q \in G(X, x_0)$. By assumption, it is clear that $up^{-1}qy_0 = y_0$ and hence $pp^{-1}qy_0 = vqy_0 = py_0$ where $v = pp^{-1} \in J$. Then qy_0 and py_0 are proximal. Since Y is distal, it follows that $qy_0 = py_0$. This means that ϕ is well defined. Since the maps $p \mapsto px_0$ and $p \mapsto py_0$ are continuous, it follows that ϕ is continuous. The converse follows from Theorem 2.10.

THEOREM 2.15. *[4] Let $\phi : (X, x_0) \to (Y, y_0)$ and $\psi : (Z, z_0) \to (Y, y_0)$ be two distal homomorphisms of minimal flows. There exists a homomorphism $\theta : (Z, z_0) \to (X, x_0)$ iff $G(Z, z_0) \subset G(X, x_0)$.*

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