Kangweon-Kyungki Math. Jour. **13** (2005), No. 2, pp. 217–221

PRIME BI-IDEALS OF GROUPOIDS

S. K. Lee

Abstract. Kehayopulu and Tsingelis [2] studied prime ideals of groupoids. Also the author studied prime left (right) ideals of groupoids. In this paper, we give some results on prime bi-ideals of groupoids.

1. Introduction

Kehayopulu and Tsingelis([2]) studied prime ideals of groupoids. Also the author([3]) studied prime left (right) ideals of groupoids. In this paper, we give some results on prime bi-ideals of groupoids.

2. Main results

G will denote a groupoid. A non-empty subset L of G is a *left ideal* if $GL \subseteq L$, a non-empty subset R of G is a *right ideal* if $RG \subseteq R$, and a non-empty subset I of G is an *ideal* if it is a left and right ideal of G. A non-empty subset Q of G is a *quasi-ideal* if $QG \cap GQ \subseteq Q$. A non-empty subset B of G is a *bi-ideal* if $BGB \subseteq B$([2-4]).

It is clear that every ideal is an one-sided (left or right) ideal, every one-sided ideal is a quasi-ideal and every quasi-ideal is a bi-ideal. The converses are not necessarily true.

A bi-ideal B of a groupoid G is *prime* if for $x, y \in G$, $xGy \subseteq B$ implies $x \in B$ or $y \in B$. A bi-ideal B of a groupoid G is *semi-prime* if $xGx \subseteq B$ implies $x \in B$. A nonempty subset I of a groupoid G is a *prime ideal* if I is an ideal of G such that for any ideals A, B of G, $AB \subseteq I$ implies $A \subseteq I$ or $B \subseteq I$([3, 4]).

Received July 15, 2005.
2000 Mathematics Subject Classification: 06F05, 20N02.
Key words and phrases: left(right, bi-, quasi) ideal, ideal, prime, semiprime.

Notation. $(x)_l = x \cup Gx$ (resp. $(x)_r = x \cup xG$) is the principal left (resp. right) ideal generated by x.

Proposition 1. *A bi-ideal B of a groupoid G is prime if and only if for a right ideal R and a left ideal L of G $RL \subseteq B$ implies $R \subseteq B$ or $L \subseteq B$.*

Proof. Suppose that $RL \subseteq B$ for a right ideal R, a left ideal L of G and $R \nsubseteq B$. Then there exists $x \in R \setminus B$. Let $y \in L$. Then $xGy \subseteq RGL \subseteq RL \subseteq B$. Since B is a prime bi-ideal and $x \notin B$, we have $y \in B$. Thus $L \subseteq B$.

Conversely, suppose that for a right ideal R and a left ideal L of G $RL \subseteq B$ implies $R \subseteq B$ or $L \subseteq B$. If $xGy \subseteq B$ for $x, y \in G$, then $(xG)(Gy) \subseteq xGy \subseteq B$. Since xG is a right ideal and Gy is a left ideal of G, by hypothesis, $xG \subseteq B$ or $Gy \subseteq B$.

If $xG \subseteq B, x^2 \in xG \subseteq B$. Thus

$$(x)_r(x)_l = (x \cup xG)(x \cup Gx) = x^2 \cup xGx \cup xG^2x \subseteq x^2 \cup xG \subseteq B.$$

Since $(x)_r$ is a right ideal and $(x)_l$ is a left ideal of G containing x, $(x)_r \subseteq B$ or $(x)_l \subseteq B$ by hypothesis. Hence $x \in B$.

If $Gy \subseteq B$, $y \in B$ by the similar method.

Therefore B is prime. □

Proposition 2. *If a bi-ideal B of a groupoid G is prime, B is a left or right ideal of G.*

Proof. Since BG is a right ideal of G and GB a left ideal of G such that $(BG)(GB) \subseteq BGB \subseteq B$, we get $BG \subseteq B$ or $GB \subseteq B$ by Proposition 1. Hence B is a left ideal or a right ideal of G. □

Notation. Now let $L_B, {}_BR, I_L$ and ${}_RI$ for a bi-ideal B of a groupoid G as follows;

$$L_B := \{x \in B \mid Gx \subseteq B\} \quad \text{and} \quad {}_BR := \{x \in B \mid xG \subseteq B\}$$

$$I_L := \{y \in L_B \mid yG \subseteq L_B\} \quad \text{and} \quad {}_RI := \{y \in {}_BR \mid Gy \subseteq {}_BR\}$$

Then we have the following results.

Proposition 3. *Let B be a prime bi-ideal of a groupoid G. Then L_B* (resp. *${}_BR$) is a left* (resp. *right*) *ideal of G contained in B if L_B* (resp. *${}_BR$) is non-empty.*

Proof. Let $x \in L_B \neq \emptyset$ and $z \in G$. Then $zx \in Gx \subseteq B$. Since $Gzx \subseteq G^2x \subseteq Gx \subseteq B$, we have $zx \in L_B$. Thus $GL_B \subseteq L_B$, so L_B is a left ideal of G.

Also we can prove R_B is a right ideal of G contained in B by the similar method. □

Proposition 4. *Let B be a bi-ideal of a groupoid G. Then I_L* (resp. *${}_RI$) is the largest ideal of G contained in B if I_L* (resp. *${}_RI$)is non-empty. Furthermore I_L coincides with ${}_RI$.*

Proof. Let $x \in I_L$. Then $xG \subseteq L_B$. Thus for any $z \in G$, $xz \in xG \subseteq L_B$. Hence $xzG \subseteq xG^2 \subseteq xG \subseteq L_B$. Therefore $xz \in I_L$, and so $I_LG \subseteq I_L$. So I_L is a right ideal of G. Since $I_L \subseteq L_B \subseteq B$, we have $x \in L_B$ and $x \in B$. Thus $Gx \subseteq B$ and $Gzx \subseteq G^2x \subseteq Gx \subseteq B$ for any $z \in G$. Hence $zx \in L_B$. Since L_B is a left ideal of G by Proposition 3 and $xG \subseteq L_B$, we get $zxG \subseteq GL_B \subseteq L_B$. Therefore $zx \in I_L$, and so $GI_L \subseteq I_L$. So I_L is a left ideal of G. It follows that I_L is an ideal of G contained in B.

Let I be any ideal of G contained in B. Then $GI \subseteq I \subseteq B$, and so $I \subseteq L_B$. Since $IG \subseteq I \subseteq L_B$, we get $I \subseteq I_L$. Therefore I_L is the largest ideal of G contained in B.

Also we can prove that ${}_RI$ is the largest ideal of S contained in B by the similar method.

Furthermore, since I_L and ${}_RR$ are the largest ideals of S contained in B, I_L coincides with ${}_RI$. □

Notation. We denote I_B as $I_B :\equiv I_L = {}_RI$ by Proposition 4.

Proposition 5. *If B is a prime bi-ideal of a groupoid G, then I_B is a prime ideal of G contained in B.*

Proof. We note that for a bi-ideal B, I_B is an ideal of S by Proposition 4. Suppose that $XY \subseteq I_B$ for any ideals X, Y of G. Since X is a right ideal, Y is a left ideal of G and $I_B \subseteq L_B \subseteq B$, we get $X \subseteq B$ or

$Y \subseteq B$ by Proposition 1. Since I_B is the largest ideal contained in B, we get $X \subseteq I_B$ or $Y \subseteq I_B$. It follows that I_B is a prime ideal of G. $\square$

COROLLARY 1. *If B be a semi-prime bi-ideal of a groupoid G, then I_B is a semi-prime ideal of S if I_B is non-empty.*

PROPOSITION 6. *If a bi-ideal B of a groupoid G is semi-prime, then for any left* (resp. *right*) *ideal L* (resp. R)*of G, $L^2 \subseteq B$ implies $L \subseteq B$ (resp. $R^2 \subseteq B$ implies $R \subseteq B$).*

Proof. Suppose that $L^2 \subseteq B$ for a left ideal L of G. If $L \nsubseteq B$, then there exists $x \in L \setminus B$. Since $xGx \subseteq LGL \subseteq L^2 \subseteq B$ and B is semi-prime, we get $x \in B$. It is impossible. Hence $L \subseteq B$.

We can prove that if $R^2 \subseteq B$, then $R \subseteq B$ by the similar method. $\square$

PROPOSITION 7. *If a bi-ideal B of a groupoid G is semi-prime, then B is a quasi-ideal of G.*

Proof. Let $y \in BG \cap GB$. Then $yGy \subseteq (BG)G(GB) \subseteq BGB \subseteq B$. Since B is semi-prime, we have $y \in B$. Thus $BG \cap GB \subseteq B$. Therefore B is a quasi-ideal. $\square$

We note that a groupoid G is *regular* if for any $x \in G$ there exists a in G such that $x = xax$.

PROPOSITION 8. *A groupoid G is regular if and only if every bi-ideal of G is semi-prime.*

Proof. Let G be regular and B a bi-ideal of G. Suppose that $xGx \subseteq B$ for $x \in G$. Since G is regular, there exists $a \in G$ such that $x = xax \in xGx \subseteq B$. Hence B is semi-prime.

Conversely, suppose that every bi-ideal of G is semi-prime. Let $B = aGa$ for $a \in G$. Then $BSB = (aGa)G(aGa) \subseteq aGa = B$, and so B is a bi-ideal of G. Thus aGa is semi-prime for all $a \in G$. Since $aGa \subseteq aGa = B$, we get $a \in aGa = B$. Hence for all $a \in G$, there exists $x \in G$ such that $a = axa$. Therefore G is regular. $\square$

From Proposition 7 and 8, we have the following corollary.

Corollary 2. *In regular groupoids, the set of bi-ideals coincides with the set of quasi-ideals.*

Example. Let $G := \{a, b, c, d, f\}$ be the set with Cayley table([1]):

$\cdot$	a	b	c	d	f
a	a	a	a	a	a
b	a	a	a	b	c
c	a	b	c	a	a
d	a	a	a	d	f
f	a	d	f	a	a

G is a groupoid. The left ideals of G are $\{a\}, \{a, b, d\}, \{a, c, f\}$ and G. The right ideals of G are $\{a\}, \{a, b, c\}, \{a, d, f\}$ and G. The bi-ideals of G are $\{a\}, \{a, b\}, \{a, c\}, \{a, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, f\}$, $\{a, d, f\}$ and G.

We note that $\{a, b\}, \{a, c\}, \{a, d\}$ are not one-sided ideals of G and $\{a, b, c\}$, $\{a, b, d\}, \{a, c, f\}, \{a, d, f\}$ are one-sided ideals of G. In this groupoid, the set of quasi-ideal coincides with the set of bi-ideals.

Furthermore $\{a, b\}$, $\{a, c\}$, $\{a, d\}$ are not prime. Indeed $cGd \subseteq \{a, b\}$, but c and d are not elements of $\{a, b\}$. $bGf \subseteq \{a, c\}$, but b and f are not elements of $\{a, c\}$. $fGb \subseteq \{a, d\}$, but f and b are not elements of $\{a, d\}$.

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Department of Mathematics Education and RINS
Gyeongsang National University
Jinju 660-701, Korea
E-mail: `sklee@gsnu.ac.kr`