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SUPERCONVERGENCE OF CRANK-NICOLSON MIXED FINITE ELEMENT SOLUTION OF PARABOLIC PROBLEMS

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Abstract. In this paper we extend the mixed finite element method and its L_2-error estimate for postprocessed solutions by using Crank-Nicolson time-discretization method.

Global $O(h^2+(\Delta t)^2)$-superconvergence for the lowest order Raviart-Thomas element $(Q_0 - Q_{1,0} \times Q_{0,1})$ are obtained. Numerical examples are presented to confirm superconvergence phenomena.

1. Introduction

We show a practical discretization technique for the parabolic equations based on the mixed finite element method in a finite element space and study how we could get the global superconvergence for the mixed approximate solutions in the rectangular Raviart-Thomas elements of order 0. There are several time-discretization methods such as Backward Euler method, Crank-Nicolson method, and Runge-Kutta method [3]. We here use Crank-Nicolson method and prove optimal order of convergence. As a result, $O(h^2+(\Delta t)^2)$ - superconvergence for Raviart-Thomas element $Q_0 - Q_{1,0} \times Q_{0,1}$ in regular mesh (not necessarily uniform) is derived.

The paper is organized as follows. The Raviart-Thomas space is introduced in §2. In §3, we devote to descretize the parabolic problem by the Crank-Nicolson mixed finite element method. In §4, we derive the main theory for superconvergence. In §5, Numerical results are given to support the theoretical results.

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2. The Raviart-Thomas Elements

Raviart and Thomas [7] introduced a family of mixed finite elements that satisfy the Ladyzhenskaya-Babuska-Brezzi condition. Their elements are defined as follows:
Let K be an ordinary rectangle or triangle and j a non-negative integer. Set

$$RT_j(K) = V(j,K) \times H(j,K),\ j \geq 0. \tag{1}$$

If K is rectangle, set $V(j,K) = Q_{j,j}(K) \equiv Q_j(K),\ H(j,K) = Q_{j+1,j}(K) \times Q_{j,j+1}(K)$. Then the finite element spaces $V_h \times H_h$ of index j are defined by

$$V_h = \{v \in L_2(\Omega) : v|_K \in V(j,K), \forall K \in \mathrm{T}_h\}, \tag{2}$$

$$H_h = \{\mathbf{p} \in H(\mathrm{div};\Omega) : \mathbf{p}|_K \in H(j,K), \forall K \in \mathrm{T}_h\}, \tag{3}$$

where $H(\mathrm{div};\Omega) = \{\mathbf{p} = (p_1,p_2) : p_i \in L_2(\Omega), i = 1,2, \text{ and } \mathrm{div}\mathbf{p} \in L_2(\Omega)\}$ and $Q_{m,n} = span\{x^i y^j : 0 \leq i \leq m, 0 \leq j \leq n\}$.

If K is triangle, set $V(j,K) = P_j(K),\ H(j,K) = P_j(K)^2 \times \mathbf{x}\hat{P}_j(K)$, where $\hat{P}_j(K)$ is the set of homogeneous polynomials of degree j in the variable $\mathbf{x} = (x,y)$.

The local Raviart-Thomas projection

$$j_h : H(\mathrm{div};K) \to H(j,K),\ \forall K \in \mathcal{T}_h \tag{4}$$

satisfies the following properties [8, 16, 17]:

$$(\mathrm{div}(\mathbf{p} - j_h\mathbf{p}), v) = 0,\ \forall v \in V_h, \tag{5}$$

$$\|j_h\mathbf{p} - \mathbf{p}\|_{0,K} \leq Ch^r\|\mathbf{p}\|_{r,K},\ 1 \leq r \leq j+1, \tag{6}$$

$$\mathrm{div} j_h = i_h \mathrm{div}, \tag{7}$$

where i_h is the local L_2-projection: $L_2(K) \to V(j,K)$. Furthermore, we have [8]

$$(\mathrm{div}\mathbf{q}, u - i_h u) = 0,\ \forall \mathbf{q} \in H_h, \tag{8}$$

$$\|i_h u - u\|_{0,K} \leq Ch^r\|u\|_{r,K},\ 0 \leq r \leq j+1. \tag{9}$$

We choose the lowest order rectangular Raviart-Thomas Element, $Q_0 - Q_{1,0} \times Q_{0,1}$, which is described by

$$\begin{cases} V_h = \{v \in L_2(\Omega) : v|_K \in Q_0(K), \forall K \in \mathrm{T}_h\}, \\ H_h^0 = \{\mathbf{q} \in H_0(\mathrm{div};\Omega) : \mathbf{q}|_K \in Q_{1,0} \times Q_{0,1}, \forall K \in \mathrm{T}_h\}, \end{cases} \tag{10}$$

where $H_0(\text{div};\Omega) = \{\mathbf{q} \in L_2(\Omega)^2 : \text{div}\mathbf{q} \in L_2(\Omega),\ \mathbf{q}\cdot\mathbf{n} = 0 \text{ on } \partial\Omega\} \subset H(\text{div};\Omega)$.
The local L_2-projection operator and the local Raviart-Thomas operator are defined on $Q_0 - Q_{1,0} \times Q_{0,1}$ element by

$$\begin{cases} i_h u \in Q_0, \\ \int_K (u - i_h u) = 0 \end{cases}, \quad \begin{cases} j_h \in Q_{1,0} \times Q_{0,1}, \\ \int_{s_i} (\mathbf{p} - j_h \mathbf{p}) \cdot \mathbf{n} ds = 0,\ i = 1,2,3,4 \end{cases},$$

where $\mathbf{n}$ is the outward unit normal vector to ∂K and s_i is the side of each rectangle elements.

3. Crank-Nicolson Mixed Finite Element Approximation

Consider the mixed approximation for the parabolic equation with Neumann boundary condition.

$$\begin{aligned} &u_t - \text{div}(a(\mathbf{x})\nabla u(\mathbf{x},t)) + b(\mathbf{x})u(\mathbf{x},t) = f(\mathbf{x},t) \text{ in } \Omega \times [0,T), \\ &a(\mathbf{x})\nabla u \cdot \mathbf{n} = 0 \text{ on } \partial\Omega \times [0,T),\ u(\cdot,0) = g(\mathbf{x}) \text{ in } \Omega \times \{0\}, \end{aligned} \tag{11}$$

where Ω is a bounded convex domain in the plane and $\partial\Omega$ is the boundary of Ω. For simplicity of presentation, we assume that $a(\mathbf{x}) = 1,\ b(\mathbf{x}) = 0$.

A mixed formulation for (11) is obtained by introducing a flux variable:

$$\mathbf{p} = \nabla u, \tag{12}$$

which is of more interest in many applications in science and engineering. The problem (11) is equivalent to seeking $(u, \mathbf{p})$ such that

$$\begin{aligned} &\nabla u - \mathbf{p} = 0 \text{ in } \Omega \times [0,T),\ u_t - \text{div}\mathbf{p} = f \text{ in } \Omega \times [0,T), \\ &\mathbf{p}\cdot\mathbf{n} = 0 \text{ on } \partial\Omega \times [0,T),\ u(\cdot,0) = g(\cdot) \text{ in } \Omega \times \{0\}. \end{aligned} \tag{13}$$

Let $V = L_2(\Omega)$ and $\mathcal{H} = H_0(\text{div};\Omega) = \{\mathbf{q} \in L_2(\Omega)^2 : \text{div}\mathbf{q} \in L_2(\Omega),\ \mathbf{q}\cdot\mathbf{n} = 0 \text{ on } \partial\Omega\} \subset H(\text{div};\Omega)$. Using integration by parts, we arrive at the following mixed variational form for (13):

Find $(u, \mathbf{p}) \in V \times \mathcal{H}$ such that

$$(\frac{\partial u}{\partial t}, v) - (\text{div}\mathbf{p}, v) = (f, v),\ \ \forall v \in V,\ \forall t \in [0,T), \tag{14}$$

$$(\mathbf{p},\mathbf{q}) + (u, \text{div}\mathbf{q}) = 0,\ \ \forall \mathbf{q} \in \mathcal{H},\ \forall t \in [0,T), \tag{15}$$

$$u(\cdot,0) = g. \tag{16}$$

Note that the Raviart-Thomas finite element space $V_h \times H_h^0 \subset V \times \mathcal{H}$ satisfies $\text{div}H_h^0 \subset V_h$ and the Ladyzhenskaya-Babuska-Brezzi condition.

Let $\Delta t = \frac{T}{N}$ be the time step and u_h^n be the approximation of $u(t)$ at $t = t_n = n\Delta t$ in V_h. Applying the Crank-Nicolson scheme to time derivative $\frac{\partial u}{\partial t}$ around the point $t_{n-\frac{1}{2}} = (n-\frac{1}{2})\Delta t$, we obtain the following fully discrete formulation:

For each $1 \le n \le N$,

$$(17)\quad (\tfrac{u_h^n - u_h^{n-1}}{\Delta t},\ v) - (\mathrm{div}(\tfrac{\mathbf{p_h}^n + \mathbf{p_h}^{n-1}}{2}),\ v) = (\tfrac{f(t_n)+f(t_{n-1})}{2},\ v),\ \forall v \in V_h,$$

$$(18)\quad (\tfrac{\mathbf{p_h}^n + \mathbf{p_h}^{n-1}}{2},\ \mathbf{q}) + (\tfrac{u_h^n + u_h^{n-1}}{2},\ \mathrm{div}\mathbf{q}) = 0,\ \forall \mathbf{q} \in H_h^0,$$

$$(19)\quad (u_h^0,\ v) = (i_h g,\ v),\ \forall v \in V_h,\ (\mathbf{p_h}^0,\ \mathbf{q}) + (i_h g,\ \mathrm{div}\mathbf{q}) = 0,\ \forall \mathbf{q} \in H_h^0.$$

Let $\varepsilon^n = u_h^n - u^n$ and $\eta^n = \mathbf{p_h}^n - \mathbf{p}^n$. Using (17)-(19), we obtain the error equations as follows.

$$(20)\quad (\frac{\varepsilon^n - \varepsilon^{n-1}}{\Delta t}, v) - (\mathrm{div}(\frac{\eta^n + \eta^{n-1}}{2}), v)$$

$$= (\frac{u^n - u^{n-1}}{\Delta t} - \frac{\partial u^{n-\frac{1}{2}}}{\partial t}, v) - (\frac{u_t^n + u_t^{n-1}}{2} - \frac{\partial u^{n-\frac{1}{2}}}{\partial t}, v),\ \forall v \in V_h,$$

$$(21)\quad (\frac{\eta^n + \eta^{n-1}}{2}, \mathbf{q}) + (\frac{\varepsilon^n + \varepsilon^{n-1}}{2}, \mathrm{div}\mathbf{q}) = 0,\ \forall \mathbf{q} \in H_h^0 \text{ for } n = 1, 2, ..., N.$$

Here, $\frac{u^n - u^{n-1}}{\Delta t} - \frac{\partial u^{n-\frac{1}{2}}}{\partial t}$ is the truncation error associated with the Crank-Nicolson method to the time derivative.

4. Global Superconvergence

In the following discussion, we assume that (x_K, y_K) is the center of K and s_i $(i = 1, 2, 3, 4)$ is its side. s_1 and s_3 are parallel to y-direction and s_2 and s_4 are parallel to x-direction. C denotes a positive constant independent to h, not necessarily the same at each occurrence. $\|\cdot\|_m$ denote the norm $\|\cdot\|_{m,2,\Omega}$, in particular, $\|\cdot\| = \|\cdot\|_0$.

THEOREM 4.1. *If* $\mathbf{p} \in [H^2(\Omega)]^2$, *then*

$$(\mathbf{p} - j_h\mathbf{p}, \mathbf{q}) \ \le\ Ch^2\|\mathbf{p}\|_2\|\mathbf{q}\|.$$

Proof. See J.Pan [11]. □

Lemma 4.1. *For each n, we have*

$$\|\frac{u^n+u^{n-1}}{2}-u^{n-\frac{1}{2}}\|^2 \;\le\; C(\Delta t)^3\int_{t_{n-1}}^{t_n}\|\frac{\partial^2 u}{\partial t^2}\|^2dt,$$
$$\|\frac{u^n-u^{n-1}}{\Delta t}-u_t^{n-\frac{1}{2}}\|^2 \;\le\; C(\Delta t)^3\int_{t_{n-1}}^{t_n}\|\frac{\partial^3 u}{\partial t^3}\|^2dt,$$

where C is a positive constant.

Proof. Use the Taylor theorem with the integral remainder and Hölder inequality. □

Theorem 4.2. *For Q_0-$Q_{1,0}\times Q_{0,1}$Element, there exists a positive constant C such that*

$$\|u_h^n-i_hu^n\|+\|\mathbf{p_h}^n-j_h\mathbf{p}^n\|$$
$$\le\; C(\;(\Delta t)^2(\int_0^{t_n}\|\frac{\partial^3 u}{\partial t^3}\|^2dt)^{\frac{1}{2}}+h^2(\|\mathbf{p}(\cdot,0)\|_2+(\sum_{j=1}^{n}\|\mathbf{p}^{j-\frac{1}{2}}\|_2^2)^{\frac{1}{2}})\;).$$

Proof. Let $\theta^n=u_h^n-i_hu^n,\;\xi^n=\mathbf{p_h}^n-j_h\mathbf{p}^n$. (5) and (8) yield

$$(\frac{\theta^n-\theta^{n-1}}{\Delta t},v)-(\mathrm{div}\frac{\xi^n+\xi^{n-1}}{2},v) \tag{22}$$
$$=(\frac{u^n-u^{n-1}}{\Delta t}-u_t^{n-\frac{1}{2}},v)-(\frac{u_t^n+u_t^{n-1}}{2}-u_t^{n-\frac{1}{2}},v),\;\forall v\in V_h,$$

$$(\frac{\xi^n+\xi^{n-1}}{2},\mathbf{q})+(\frac{\theta^n+\theta^{n-1}}{2},\mathrm{div}\mathbf{q})=(\mathbf{p}^{n-\frac{1}{2}}-j_h\mathbf{p}^{n-\frac{1}{2}},\mathbf{q}),\;\forall\mathbf{q}\in H_h^0. \tag{23}$$

Putting $\bar\theta^n=\frac{\theta^n+\theta^{n-1}}{2}$, $\bar\xi^n=\frac{\xi^n+\xi^{n-1}}{2}$ and taking $v=\bar\theta^n$, $\mathbf{q}=\bar\xi^n$, we obtain from the sum of (22) and (23) that

$$\frac{1}{2\Delta t}(\|\theta^n\|^2-\|\theta^{n-1}\|^2)+\|\bar\xi^n\|^2 \;\le\; \frac{1}{2\delta_1}\|\frac{u_t^n+u_t^{n-1}}{2}-u_t^{n-\frac{1}{2}}\|^2+\frac{\delta_1}{2}\|\bar\theta^n\|^2$$
$$+\frac{1}{2\delta_2}\|\frac{u^n-u^{n-1}}{\Delta t}-u_t^{n-\frac{1}{2}}\|^2+\frac{\delta_2}{2}\|\bar\theta^n\|^2+\frac{Ch^4}{2}\|\mathbf{p}^{n-\frac{1}{2}}\|_2^2+\frac{1}{2}\|\bar\xi^n\|^2$$

for each $1\le n\le N$.
Applying Lemma 4.2 and letting $\delta>0$ such that $1-\frac{\Delta t}{2}\delta>\frac{1}{2}$ with $\delta=\delta_1+\delta_2$, we have

$$\|\theta^n\|^2 \;\le\; C\;(\;\|\theta^{n-1}\|^2+(\Delta t)^4\int_{t_{n-1}}^{t_n}\|\frac{\partial^3 u}{\partial t^3}\|^2dt+h^4\|\mathbf{p}^{n-\frac{1}{2}}\|_2^2\;),$$

Considering $u_h(\cdot,0) = i_h g = i_h u(\cdot,0)$ and adding all equations for $n = 1, 2, ..., m \ \leq N$,

$$\|\theta^m\|^2 \ \leq \ C \ (\ (\Delta t)^2 (\int_0^{t_m} \|\frac{\partial^3 u}{\partial t^3}\|^2 dt)^{\frac{1}{2}} + h^2 (\sum_{j=1}^{m} \|\mathbf{p}^{j-\frac{1}{2}}\|_2^2)^{\frac{1}{2}} \ )^2,$$

where $t_m = m\Delta t \leq N\Delta t = T$.

Next, we consider

$$(24) \quad (\frac{\xi^n - \xi^{n-1}}{\Delta t}, \mathbf{q}) + (\frac{\theta^n - \theta^{n-1}}{\Delta t}, \mathrm{div}\mathbf{q}) = (\frac{\mathbf{p}^{n-\frac{1}{2}} - j_h \mathbf{p}^{n-\frac{1}{2}}}{\Delta t}, \mathbf{q}), \ \forall \mathbf{q} \in H_h^0$$

instead of the second equation (23). And since

$$(25) \qquad (\mathbf{p_h}^0 - j_h \mathbf{p}^0, \mathbf{q}) + (u_h^0 - i_h u^0, \mathrm{div}\mathbf{q}) = (\mathbf{p}^0 - j_h \mathbf{p}^0, \mathbf{q}),$$

let $\mathbf{q} = \mathbf{p_h}^0 - j_h \mathbf{p}^0$, then $\|\mathbf{p_h}(\cdot,0) - j_h \mathbf{p}(\cdot,0)\| \leq Ch^2 \|\mathbf{p}(\cdot,0)\|_2$.
From the sum of (22) with $v = \frac{\theta^n - \theta^{n-1}}{\Delta t}$ and (24) with $\mathbf{q} = \bar{\xi}^n$, we yield

$$\begin{aligned}
\|\frac{\theta^n - \theta^{n-1}}{\Delta t}\|^2 + \frac{1}{2\Delta t}(\|\xi^n\|^2 - \|\xi^{n-1}\|^2) &\leq \frac{1}{2}\|\frac{u_t^n + u_t^{n-1}}{2} - u_t^{n-\frac{1}{2}}\|^2 \\
&+ \frac{1}{2}\|\frac{\theta^n - \theta^{n-1}}{\Delta t}\|^2 + \frac{1}{2}\|\frac{u^n - u^{n-1}}{\Delta t} - u_t^{n-\frac{1}{2}}\|^2 \\
&+ \frac{1}{2}\|\frac{\theta^n - \theta^{n-1}}{\Delta t}\|^2 + \frac{1}{\Delta t}(\frac{\epsilon^{-1} h^4}{2}\|\mathbf{p}^{n-\frac{1}{2}}\|_2^2 + \frac{\epsilon}{2}\|\bar{\xi}^n\|^2).
\end{aligned}$$

Choosing $\epsilon > 0$ such that $1 - \frac{\epsilon}{2} > 0$, we have

$$\|\xi^n\|^2 \ \leq \ C \ (\ \|\xi^{n-1}\|^2 + (\Delta t)^4 \int_{t_{n-1}}^{t_n} \|\frac{\partial^3 u}{\partial t^3}\|^2 dt + h^4 \|\mathbf{p}^{n-\frac{1}{2}}\|_2^2 \ ),$$

Adding all equations for each m with $1 \leq m \leq N$,

$$\|\xi^m\|^2 \leq \ C \ (\ (\Delta t)^2 (\int_0^{t_m} \|\frac{\partial^3 u}{\partial t^3}\|^2 dt)^{\frac{1}{2}} + h^2 (\|\mathbf{p}(\cdot,0)\|_2 + (\sum_{j=1}^{m} \|\mathbf{p}^{j-\frac{1}{2}}\|_2^2)^{\frac{1}{2}}) \ )^2.$$

This completes the proof.

□

We use proper postprocessing method to get global superconvergence. For this purpose, we assume that $\mathcal{T}_h$ has been obtained from $\mathcal{T}_{2h}$ by dividing each element of $\mathcal{T}_{2h}$ into four congruent rectangles $\tau = \sum_{i=1}^4 K_i \in \mathcal{T}_{2h}$

with $K_i \in \mathcal{T}_h$. Then we can define two postprocessing operators as follows.

$$(26) \qquad \left\{ \begin{array}{l} J_{2h}\mathbf{p} \in Q_{1,1}(\tau) \times Q_{1,1}(\tau), \\ \int_{l_i} (J_{2h}\mathbf{p} - \mathbf{p}) \cdot \mathbf{n} ds = 0, \\ \; i = 1, 2, ..., 8 \ , \end{array} \right. \qquad \left\{ \begin{array}{l} I_{2h}u \in Q_1(\tau), \\ \int_{K_i} (I_{2h}u - u) = 0, \\ \; i = 1, ..., 4 \ , \end{array} \right.$$

where $l_i (i = 1, 2, ..., 8)$ is sides of K_1, K_2, K_3, K_4 which are composed of boundary of $\partial\tau$ and $\mathbf{n}$ is outward unit normal to l_i. It is easy to check that

$$(27) \left\{ \begin{array}{l} J_{2h} j_h = J_{2h}, \\ \|J_{2h}\mathbf{q}\| \leq c\|\mathbf{q}\|, \ \forall \mathbf{q} \in H_h^0(\Omega), \\ \|J_{2h}\mathbf{p} - \mathbf{p}\| \leq ch^2\|\mathbf{p}\|_2 \ , \end{array} \right. \qquad \left\{ \begin{array}{l} I_{2h} i_h = I_{2h}, \\ \|I_{2h}v\| \leq c\|v\|, \ \forall v \in V_h, \\ \|I_{2h}u - u\| \leq ch^2\|u\|_2 \ . \end{array} \right.$$

Corollary 4.1. *We have the global L_2-superconvergence for $Q_0 - Q_{1,0} \times Q_{0,1}$ element.*

$$\|I_{2h}u_h - u\| + \|J_{2h}\mathbf{p_h} - \mathbf{p}\| \leq \ C \ [\ (\Delta t)^2 (\int_0^T \| \frac{\partial^3 u}{\partial t^3} \|^2 dt)^{\frac{1}{2}} + h^2 (\|\mathbf{p}(\cdot, 0)\|_2 + \max_{1 \leq j \leq N} \|\mathbf{p}^{j-\frac{1}{2}}\|_2 + \|\mathbf{p}\|_2 + \|u\|_2) \ ],$$

where $N\Delta t = T$.

5. Numerical results

In this section we examine the superconvergence phenomena. Consider the parabolic problem

$$\begin{aligned} \frac{\partial u}{\partial t} - \mathrm{div}(\nabla u(\mathbf{x}, t)) &= f(\mathbf{x}, t) \text{ in } \Omega \times [0, T), \\ \nabla u \cdot \mathbf{n} &= 0 \text{ on } \partial\Omega \times [0, T), \\ u(\cdot, 0) &= 0 \text{ in } \Omega \times \{0\} \end{aligned}$$

with the exact solution $u(x, y, t) = t(\cos(\pi x) cos(\pi y) + 1)$, where $\Omega = [0, 1] \times [0, 1]$ and $f(x, y, t) = 1 + \cos(\pi x)\cos(\pi y) + 2\pi^2 t \cos(\pi x)\cos(\pi y)$.

Let $\mathcal{T}_h = \{K\}$ be a rectangular partition of Ω. To solve this problem by the Crank-Nicolson mixed finite element method, we divide Ω into M^2 squares uniformly using $(M-1)$ vertical lines and $(M-1)$ horizontal lines and take basis functions on $Q_0-Q_{1,0}\times Q_{0,1}$ element. We can choose the basis functions on reference rectangle $\hat{K}_{ref}=[-1,1]^2$ such that

$$\begin{cases} \hat{\phi}=1, \\ \hat{\psi}_1^x=(\frac{1-x}{2},\ 0),\ \hat{\psi}_2^x=(\frac{1+x}{2},\ 0),\ \hat{\psi}_1^y=(0,\ \frac{1-y}{2}),\ \hat{\psi}_2^y=(0,\ \frac{1+y}{2}). \end{cases}$$

Then we let

$$u_h=\sum_{i=1}^{M^2}\alpha_i\psi_i(x),\ \mathbf{p_h}=\sum_{j=1}^{2M(M-1)}\beta_j\phi_j(x).$$

Applying it into (17)∼(19), we get $(M^2+2M(M-1))\times(M^2+2M(M-1))$ matrix and $M^2+2M(M-1)$ load vector.

$$\begin{pmatrix} A & B \\ B^T & D \end{pmatrix}\begin{pmatrix} \tilde{u} \\ \tilde{p} \end{pmatrix}=\begin{pmatrix} F \\ G \end{pmatrix}$$

Since $\tilde{p}=-D^{-1}B^T\tilde{u}+D^{-1}G$, we first solve u_h from $(A-BD^{-1}B^T)\tilde{u}=F-BD^{-1}G$ by lumping technique. And then, we gain $\mathbf{p_h}$.

All programs were written in Matlab, and ran on PC. We use two point Gauss quadrature to evaluate the integrals. If K is rectangle with four vertices (x_1,y_1), (x_2,y_1), (x_1,y_2) and (x_2,y_2),

$$\int\int_K f(x,y)dxdy\approx\frac{h^2}{4}\sum_{i,j=1}^{2}\omega_{i,j}f(x_i,y_j),$$

where $\omega_{i,j}=1$, x_i and y_j $(i,j=1,2)$ are two Gauss points.

Table 5.1 L_2 Error Estimate for Mixed Approximate Solution

$\Delta t=0.1$	$t=0.1$		$t=1.0$	
$M,\ (h=1/M)$	$\|u-u_h\|$	$\|\mathbf{p}-\mathbf{p_h}\|$	$\|u-u_h\|$	$\|\mathbf{p}-\mathbf{p_h}\|$
4	0.0157389202	0.0502744069	0.1585319879	0.5102217316
8	0.0079793558	0.0251694639	0.0799553806	0.2527217829
16	0.0040033968	0.0125896729	0.0400548251	0.1260281101
32	0.0020034127	0.0062954834	0.0200367543	0.0629713514

Table 5.2 Convergence Order

$\Delta t=0.1$	$t=0.1$		$t=1.0$	
M_1-M_2	u	$\mathbf{p}$	u	$\mathbf{p}$
$4-8$	0.9800	0.9981	0.9875	1.0136
$8-16$	0.9950	0.9994	0.9972	1.0038
$16-32$	0.9988	0.9999	0.9993	1.0010

Applying the postprocessing technique (26) to u_h and $\mathbf{p_h}$, we can get the postprocessed solution $I_{2h}u_h$ and $J_{2h}\mathbf{p_h}$ on each $\mathcal{T}_{2h}$ element.

Table 5.3 L_2 Error Estimate for Postprocessed Solution

$\Delta t = 0.1$	$t = 0.1$		$t = 1.0$	
$M,\ (h = 1/M)$	$\|u - I_{2h}u_h\|$	$\|\mathbf{p} - J_{2h}\mathbf{p_h}\|$	$\|u - I_{2h}u_h\|$	$\|\mathbf{p} - J_{2h}\mathbf{p_h}\|$
4	0.0015575989	0.0073846048	0.0261724987	0.1209839702
8	0.0003390042	0.0015352651	0.0062273735	0.0279609947
16	0.0000810741	0.0003620325	0.0015332437	0.0068303653
32	0.0000200288	0.0000891003	0.0003817688	0.0016973053

Table 5.4 Convergence Order showing Superconvergence Phenomena

$\Delta t = 0.1$	$t = 0.1$		$t = 1.0$	
$M_1 - M_2$	u	$\mathbf{p}$	u	$\mathbf{p}$
$4 - 8$	2.1999	2.2660	2.0714	2.1133
$8 - 16$	2.0640	2.0843	2.0220	2.0334
$16 - 32$	2.0172	2.0226	2.0058	2.0087

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