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FREE LIE SUPERALGEBRAS AND THE REPRESENTATIONS OF $\mathfrak{gl}(m,n)$ AND $\mathfrak{q}(n)$

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ABSTRACT. Let $\mathscr{L}$ be the free Lie superalgebra generated by a $\mathbb{Z}_2$-graded vector space V over $\mathbb{C}$. Suppose that $\mathfrak{g}$ is a Lie superalgebra $\mathfrak{gl}(m,n)$ or $\mathfrak{q}(n)$. We study the $\mathfrak{g}$-module structure on the kth homogeneous component $\mathscr{L}_k$ of $\mathscr{L}$ when V is the natural representation of $\mathfrak{g}$. We give the multiplicities of irreducible representations of $\mathfrak{g}$ in $\mathscr{L}_k$ by using the character of $\mathscr{L}_k$. The multiplicities are given in terms of the character values of irreducible (projective) representations of the symmetric groups.

1. Introduction

Let $\mathscr{L}$ be the free Lie algebra generated by a vector space V over a field k. If V is a representation of a group G (finite or infinite), then $\mathscr{L}$ becomes a representation of G, and its homogeneous component $\mathscr{L}_k (k \geq 1)$ is a submodule of $\mathscr{L}$. Hence, it is natural to ask how $\mathscr{L}$(or $\mathscr{L}_k$) decomposes into irreducible representations of G whenever it is semisimple.

Consider V as a representation of its full linear group. For simplicity, assume that $k = \mathbb{C}$. Let $V = \mathbb{C}^m$ be an m-dimensional vector space over $\mathbb{C}$, which is the natural representation of $GL(m)$, or $\mathfrak{gl}(m)$. The k-fold tensor product of V is a $GL(m)$-module and decomposes into irreducible polynomial representations which are parameterized by the partitions λ of k with length $\ell(\lambda) \leq m$. Let V^λ be the corresponding irreducible representation. It is well-known that the multiplicity of V^λ in $V^{\otimes k}$ is given by the dimension of the irreducible representation S^λ of the symmetric group S_k corresponding to λ ([30, 36]).

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Let $\mathscr{L}$ be the free Lie algebra generated by V and $\mathscr{L}_k (k \geq 1)$ its kth homogeneous component. Then $\mathscr{L}_k$ is a finite dimensional $GL(m)$-submodule of $V^{\otimes k}$. In [5], Brandt gave the character of $\mathscr{L}_k$(i.e. the trace of $\mathrm{diag}(x_1, \cdots, x_m)$ on $\mathscr{L}_k$)

$$ch\mathscr{L}_k = \frac{1}{k}\sum_{d|k} \mu(d) p_d(x)^{k/d} \tag{1.1}$$

where $p_d(x) = x_1^d + \cdots + x_m^d$ is the dth power symmetric function. Then from (1.1) and the Frobenius formula, the multiplicity of V^λ in $\mathscr{L}_k$ is given by

$$\frac{1}{k}\sum_{d|k} \mu(d) \chi^\lambda_{S_k}(\sigma_{(d^{k/d})}), \tag{1.2}$$

(which was first given by Wever [37]) where $\chi^\lambda_{S_k}$ is the character of S^λ and $\sigma_{(d^{k/d})}$ is an element of cycle type $(d^{k/d})$ in S_k (see also [20, 21]). Also, various module structures of free Lie algebras have been studied in more general cases where V is a representation of an arbitrary group and the base field may have positive characteristic (see for example [6, 7, 9]).

In this paper, we will study the module structures of free Lie superalgebras generated by a representation of a Lie superalgebra. (see [16] for a general exposition on Lie superalgebras): The main interests in this paper are the super-analogues of (1.1) and (1.2). Let V be a $\mathbb{Z}_2$-graded vector space over $\mathbb{C}$, and $\mathscr{L}$ the free Lie superalgebra generated by V. We study the module structure of the kth homogeneous component $\mathscr{L}_k$ when V is the natural representation of a Lie superalgebra $\mathfrak{gl}(m,n)$ or $\mathfrak{q}(n)$. As the main results, we describe the multiplicity of each irreducible representation in $\mathscr{L}_k$.

This paper is organized as follows. In section 2, we introduce the character of $\mathscr{L}_k$ given in [26]. We give here an alternative proof for the character of $\mathscr{L}_k$ using the homological methods (cf. [19]). The character of $\mathscr{L}_k$ is written as a linear combination of power super symmetric functions (Theorem 2.1). In section 3, we review some basic facts on the (super) symmetric functions and the irreducible characters of S_k and its double cover $\widetilde{S}_k$. Then, in section 4, we derive the multiplicities of irreducible representations in $\mathscr{L}_k$, combining the character of $\mathscr{L}_k$ and the expansion of power (super) symmetric functions into hook Schur functions and Schur Q-functions respectively (Theorem 4.1 and 4.3). The character values of the symmetric groups appear naturally as in the case of (1.2). In section 5, we consider the case when $\mathscr{L}$ is the free Lie algebra generated by $V = \mathbb{C}^n$. We describe the multiplicities of each irreducible polynomial representation of $\mathfrak{sp}(n)$ (n : even) and $\mathfrak{so}(n)$ in $\mathscr{L}_k$ for $1 \le k \le n$, using the character values of the Brauer algebras (Proposition 5.3).

Throughout this paper, we assume that the ground field is $\mathbb{C}$.

2. Free Lie superalgebras and characters

In this section, we will derive the character of the homogeneous component of a free Lie superalgebra.

A $\mathbb{Z}_2$-graded vector space $\mathscr{L} = \mathscr{L}_0 \oplus \mathscr{L}_1$ is a *Lie superalgebra* if there is a bilinear map $[\,,\,] : \mathscr{L} \times \mathscr{L} \longrightarrow \mathscr{L}$ such that

(i) $[\mathscr{L}_a, \mathscr{L}_b] \subset \mathscr{L}_{a+b}$
(ii) $[x, y] = -(-1)^{ab}[y, x]$
(iii) $[x, [y, z]] = [[x, y], z] + (-1)^{ab}[y, [x, z]]$

for $a, b \in \mathbb{Z}_2$ and $x \in \mathscr{L}_a$, $y \in \mathscr{L}_b$.

Let Γ be a countable abelian semigroup (under addition). We assume that every element in Γ can be written as a sum of elements in Γ in only finitely many ways, which we call the *finiteness condition on* Γ.

Let $V = \bigoplus_{\alpha \in \Gamma} V_\alpha$ be a Γ-graded vector space where each $V_\alpha = V_{(\alpha,0)} \oplus V_{(\alpha,1)}$ is a finite dimensional $\mathbb{Z}_2$-graded vector space with $\dim V_{(\alpha,0)} = m_\alpha$ and $\dim V_{(\alpha,1)} = n_\alpha$. Note that $V = \bigoplus_{(\alpha,a) \in \Gamma \times \mathbb{Z}_2} V_{(\alpha,a)}$ is also $(\Gamma \times \mathbb{Z}_2)$-graded. Set $V_a = \bigoplus_{\alpha \in \Gamma} V_{(\alpha,a)}$ for $a \in \mathbb{Z}_2$.

Consider the free Lie superalgebra $\mathscr{L} = \mathscr{L}_0 \oplus \mathscr{L}_1$ generated by V. The universal enveloping algebra of $\mathscr{L}$ is isomorphic to the tensor algebra $T(V) = \bigoplus_{k \ge 0} V^{\otimes k}$ generated by V (as $\mathbb{Z}_2$-graded algebras), and $\mathscr{L}$ can be embedded into $T(V)$ in such a way that $[x, y] = x \otimes y - (-1)^{ab} y \otimes x$ for $x \in V_a$ and $y \in V_b$ ($a, b \in \mathbb{Z}_2$). Since $T(V)$ is Γ-graded, $\mathscr{L}$ is also Γ-graded, that is, $\mathscr{L} = \bigoplus_{\alpha \in \Gamma} \mathscr{L}_\alpha$. Note that the dimension of $\mathscr{L}_\alpha$ ($\alpha \in \Gamma$) is finite from the finiteness condition on Γ.

For each $\alpha \in \Gamma$, we set

$$\mathscr{L}_{(\alpha,0)} = \mathscr{L}_\alpha \cap \mathscr{L}_0, \qquad \mathscr{L}_{(\alpha,1)} = \mathscr{L}_\alpha \cap \mathscr{L}_1. \tag{2.1}$$

Since $\mathscr{L}_\alpha = \mathscr{L}_{(\alpha,0)} \oplus \mathscr{L}_{(\alpha,1)}$, $\mathscr{L}$ is a $(\Gamma \times \mathbb{Z}_2)$-graded Lie superalgebra.

Set $G = \prod_{\alpha \in \Gamma} G_\alpha \subset GL(V)$ where $G_\alpha = GL(V_{(\alpha,0)}) \times GL(V_{(\alpha,1)})$. For each $(\alpha, a) \in \Gamma \times \mathbb{Z}_2$, $\mathscr{L}_{(\alpha,a)}$ is a G-module where the action is given by

$$g \cdot [v_1[v_2[\cdots[v_{k-1}, v_k]\cdots]]] = [gv_1[gv_2[\cdots[gv_{k-1}, gv_k]\cdots]]] \tag{2.2}$$

for $g \in G$ and $v_i \in V$ $(1 \leq i \leq k)$.

For each $\gamma \in \Gamma$, consider the following variables

$$\begin{aligned} x^\gamma &= (x_1^\gamma, \cdots, x_{m_\gamma}^\gamma) \in (\mathbb{C}^\times)^{m_\gamma}, \\ y^\gamma &= (y_1^\gamma, \cdots, y_{n_\gamma}^\gamma) \in (\mathbb{C}^\times)^{n_\gamma}. \end{aligned} \tag{2.3}$$

For $(\alpha, a) \in \Gamma \times \mathbb{Z}_2$, we define the *character of* $\mathscr{L}_{(\alpha,a)}$

$$\mathrm{ch}\mathscr{L}_{(\alpha,a)} = \mathrm{tr}((\mathrm{diag}(x^\gamma, y^\gamma))_{\gamma\in\Gamma}|\mathscr{L}_{(\alpha,a)}). \tag{2.4}$$

From the finiteness condition on Γ, $\mathrm{ch}\mathscr{L}_{(\alpha,a)}$ is a polynomial in x^γ, y^γ's. The character of $\mathscr{L}_\alpha$ can be defined in the same way, and it is given by

$$\mathrm{ch}\mathscr{L}_\alpha = \mathrm{ch}\mathscr{L}_{(\alpha,0)} + \mathrm{ch}\mathscr{L}_{(\alpha,1)}. \tag{2.5}$$

Next, we define the *set of partitions of* α $(\alpha \in \Gamma)$ to be

$$P(\alpha) = \{\, s = (s_\gamma)_{\gamma\in\Gamma} \mid s_\gamma \in \mathbb{Z}_{\geq 0}, \sum_{\gamma\in\Gamma} s_\gamma \gamma = \alpha \,\}. \tag{2.6}$$

It is a finite set from the finiteness condition on Γ. For $s \in P(\alpha)$, we write $|s| = \sum s_\gamma$ and $s! = \prod s_\gamma!$.

Now, we can state the formula for the character of $\mathscr{L}_\alpha$ (see [26]). We give here an alternative proof based on the homological method used in [19].

THEOREM 2.1. ([26]) *For $\alpha \in \Gamma$, we have*

$$\mathrm{ch}\mathscr{L}_\alpha = \sum_{\substack{d>0 \\ d\beta=\alpha}} \frac{1}{d}\mu(d) \sum_{s\in P(\beta)} \frac{(|s|-1)!}{s!} \prod_{\gamma\in\Gamma} p_d(x^\gamma, y^\gamma)^{s_\gamma}, \tag{2.7}$$

where μ is the Möbius function and

$$p_d(x^\gamma, y^\gamma) = \sum_{i=1}^{m_\gamma} (x_i^\gamma)^d - \sum_{j=1}^{n_\gamma} (-y_j^\gamma)^d \tag{2.8}$$

for $d \geq 1$.

Proof. First, we will compute $\mathrm{ch}\mathscr{L}_{(\alpha,a)}$ for $(\alpha, a) \in \Gamma \times \mathbb{Z}_2$. It is already given in [19] as the special case of a more general formula (see (2.17) in section 2). But for the reader's convenience and self-containedness, we give a proof restricting the arguments in [19] to the case of free Lie superalgebras.

For $k \geq 0$, let $C_k(\mathscr{L}) = \bigoplus_{p+q=k} \Lambda^p(\mathscr{L}_0) \otimes S^q(\mathscr{L}_1)$ where $\Lambda^p(\mathscr{L}_0)$ is the pth alternating space of L_0 and $S^q(\mathscr{L}_1)$ is the qth symmetric space of $\mathscr{L}_1$ (note that $C_0(\mathscr{L}) = \mathbb{C}$). The homology modules $H_k(\mathscr{L}) = H_k(\mathscr{L}, \mathbb{C})$ are determined by the following complex:

(2.9)

$$\cdots \longrightarrow C_k(\mathscr{L}) \xrightarrow{d_k} C_{k-1}(\mathscr{L}) \xrightarrow{d_{k-1}} \cdots \longrightarrow C_1(\mathscr{L}) \xrightarrow{d_1} C_0(\mathscr{L}) \longrightarrow 0,$$

where the differentials $d_k : C_k(\mathscr{L}) \to C_{k-1}(\mathscr{L})$ are given by

(2.10)

$$\begin{aligned} & d_k((x_1 \wedge \cdots \wedge x_p) \otimes (y_1 \cdots y_q)) \\ &= \sum_{1 \leq s < t \leq p} (-1)^{s+t}([x_s, x_t] \wedge x_1 \wedge \cdots \wedge \widehat{x_s} \wedge \cdots \wedge \widehat{x_t} \wedge \cdots \wedge x_p) \\ &\qquad \otimes (y_1 \cdots y_q) \\ &+ \sum_{s=1}^{p} \sum_{t=1}^{q} (-1)^s (x_1 \wedge \cdots \wedge \widehat{x_s} \wedge \cdots \wedge x_p) \otimes ([x_s, y_t] y_1 \cdots \widehat{y_t} \cdots y_q) \\ &- \sum_{1 \leq s < t \leq q} ([y_s, y_t] \wedge x_1 \wedge \cdots \wedge x_p) \otimes (y_1 \cdots \widehat{y_s} \cdots \widehat{y_t} \cdots y_q) \end{aligned}$$

for $k \geq 2$, $x_i \in \mathscr{L}_0$, $y_j \in \mathscr{L}_1$ and $d_1 = 0$ (cf. [8],[11]).

From the $(\Gamma \times \mathbb{Z}_2)$-grading of $\mathscr{L}$, $C_k(\mathscr{L})$ and $H_k(\mathscr{L})$ are $(\Gamma \times \mathbb{Z}_2)$-graded vector spaces, and for each $(\alpha, a) \in \Gamma \times \mathbb{Z}_2$, we can define $\mathrm{ch} C_k(\mathscr{L})_{(\alpha,a)}$ and $\mathrm{ch} H_k(\mathscr{L})_{(\alpha,a)}$ as in (2.4). Set

$$\begin{aligned} \mathrm{ch} C_k(\mathscr{L}) &= \sum_{(\alpha,a) \in \Gamma \times \mathbb{Z}_2} \mathrm{ch} C_k(\mathscr{L})_{(\alpha,a)} u^\alpha v^a, \\ \mathrm{ch} H_k(\mathscr{L}) &= \sum_{(\alpha,a) \in \Gamma \times \mathbb{Z}_2} \mathrm{ch} H_k(\mathscr{L})_{(\alpha,a)} u^\alpha v^a, \end{aligned} \tag{2.11}$$

where u^α is a formal variable satisfying $u^\alpha u^\beta = u^{\alpha+\beta}$ $(\alpha, \beta \in \Gamma)$, and v is a variable commuting with u^α satisfying $v^2 = 1$. By the Euler-Poincaré principle, we have

$$\sum_{k \geq 0} (-1)^k \mathrm{ch} C_k(\mathscr{L}) = \sum_{k \geq 0} (-1)^k \mathrm{ch} H_k(\mathscr{L}). \tag{2.12}$$

On the other hand, the left hand side of the above equation can be written as follows.

$$\begin{aligned}(2.13)\qquad &\sum_{k\geq 0}(-1)^k \mathrm{ch} C_k(\mathscr{L}) \\ &= \prod_{(\alpha,a)\in\Gamma\times\mathbb{Z}_2} \exp\left(-\sum_{r\geq 1}\frac{(-1)^a}{r}\mathrm{tr}(g^r|\mathscr{L}_{(\alpha,a)})u^{r\alpha}(-v)^{ra}\right),\end{aligned}$$

where $g = (\mathrm{diag}(x^\gamma, y^\gamma))_{\gamma\in\Gamma} \in G$. Since $H_k(\mathscr{L}) = V$ when $k = 1$ and $H_k(\mathscr{L}) = 0$ otherwise (see Corollary 3.2 in [19]), we obtain the *twisted denominator identity of* $\mathscr{L}$;

$$\begin{aligned}(2.14)\qquad &\prod_{(\alpha,a)\in\Gamma\times\mathbb{Z}_2} \exp\left(-\sum_{r\geq 1}\frac{(-1)^a}{r}\mathrm{tr}(g^r|\mathscr{L}_{(\alpha,a)})u^{r\alpha}(-v)^{ra}\right) \\ &= 1 - \sum_{\beta\in\Gamma}(p_1(x^\beta)u^\beta + p_1(y^\beta)u^\beta v),\end{aligned}$$

where $p_1(x^\beta) = \sum_{i=1}^{m_\beta} x_i^\beta$ and $p_1(y^\beta) = \sum_{j=1}^{n_\beta} y_j^\beta$.

For each $(\beta, b) \in \Gamma \times \mathbb{Z}_2$, we set

$$(2.15)\qquad P(\beta,b) = \{\, s = (s_{\gamma,a})_{(\gamma,a)\in\Gamma\times\mathbb{Z}_2} \mid s_{\gamma,a} \in \mathbb{Z}_{\geq 0}, \sum s_{\gamma,a}(\gamma,a) = (\beta,b)\,\}$$

and write $|s| = \sum s_{\gamma,a}$ and $s! = \prod s_{\gamma,a}!$.

Taking the formal logarithm on the inverse of (2.14), we get

$$\begin{aligned}(2.16)\qquad &\sum_{(\alpha,a)\in\Gamma\times\mathbb{Z}_2}\sum_{r\geq 1}\frac{(-1)^a}{r}\mathrm{tr}(g^r|\mathscr{L}_{(\alpha,a)})u^{r\alpha}(-v)^{ra} \\ &= \sum_{(\beta,b)\in\Gamma\times\mathbb{Z}_2} W_g(\beta,b)u^\beta(-v)^b,\end{aligned}$$

where

$$(2.17)\qquad W_g(\beta,b) = \sum_{s\in P(\beta,b)} \frac{(|s|-1)!}{s!}\prod_{\gamma\in\Gamma} p_1(x^\gamma)^{s_{\gamma,0}}(-p_1(y^\gamma))^{s_{\gamma,1}}.$$

By comparing the coefficients of $u^\alpha(-v)^a$ on both sides and applying the Möbius inversion, we obtain

$$(2.18)\qquad (-1)^a \mathrm{ch}\mathscr{L}_{(\alpha,a)} = \sum_{\substack{d>0 \\ d(\beta,b)=(\alpha,a)}} \frac{1}{d}\mu(d)W_{g^d}(\beta,b).$$

Now, we have

$$\begin{aligned}\mathrm{ch}\mathscr{L}_{(\alpha,0)} &= \sum_{\substack{d\beta=\alpha\\ d:\text{even}}} \frac{1}{d}\mu(d)\{W_{g^d}(\beta,0)+W_{g^d}(\beta,1)\}\\ &\quad+\sum_{\substack{d\beta=\alpha\\ d:\text{odd}}} \frac{1}{d}\mu(d)W_{g^d}(\beta,0),\end{aligned}\tag{2.19}$$

and

$$\mathrm{ch}\mathscr{L}_{(\alpha,1)} = -\sum_{\substack{d\beta=\alpha\\ d:\text{odd}}} \frac{1}{d}\mu(d)W_{g^d}(\beta,1).\tag{2.20}$$

For any $\beta\in\Gamma$ and $d\geq 1$, it is straightforward to check that

$$\begin{aligned}W_{g^d}(\beta,0)&+(-1)^dW_{g^d}(\beta,1)\\ &=\sum_{s\in P(\beta)}\frac{(|s|-1)!}{s!}\prod_{\gamma\in\Gamma}p_d(x^\gamma,y^\gamma)^{s_\gamma},\end{aligned}\tag{2.21}$$

where $p_d(x^\gamma,y^\gamma)=\sum_{i=1}^{m_\gamma}(x_i^\gamma)^d-\sum_{j=1}^{n_\gamma}(-y_j^\gamma)^d$.

Therefore, we get

$$\begin{aligned}\mathrm{ch}\mathscr{L}_\alpha &= \mathrm{ch}\mathscr{L}_{(\alpha,0)}+\mathrm{ch}\mathscr{L}_{(\alpha,1)}\\ &=\sum_{d\beta=\alpha}\frac{1}{d}\mu(d)\{W_{g^d}(\beta,0)+(-1)^dW_{g^d}(\beta,1)\}\\ &=\sum_{d\beta=\alpha}\frac{1}{d}\mu(d)\sum_{s\in P(\beta)}\frac{(|s|-1)!}{s!}\prod_{\gamma\in\Gamma}p_d(x^\gamma,y^\gamma)^{s_\gamma}.\end{aligned}\tag{2.22}$$

□

If we take g as the identity element in G, then the character of $\mathscr{L}_\alpha$ yields the dimension of $\mathscr{L}_\alpha$:

$$\sum_{d\beta=\alpha}\frac{1}{d}\mu(d)\sum_{s\in P(\beta)}\frac{(|s|-1)!}{s!}\prod_{\gamma\in\Gamma}(m_\gamma-(-1)^dn_\gamma)^{s_\gamma},\tag{2.23}$$

which is a generalization of the Witt's dimension formula for free Lie algebras (see [25, 26], also compare with the formula in [18]).

In particular, suppose that $\Gamma=\mathbb{N}$ and $V_k=0$ for $k\geq 2$, that is, $V=V_1$. Let $\dim V_{(1,0)}=m$ and $\dim V_{(1,1)}=n$. Then $\mathscr{L}=\bigoplus_{k\geq 1}\mathscr{L}_k$ is an $\mathbb{N}$-graded free Lie superalgebra. For $n\geq 1$, we have $P(n)=\{n\}$. The character of $\mathscr{L}_k$ is the trace of $\mathrm{diag}(x,y)\in GL(m)\times GL(n)$ on $\mathscr{L}_k$

for $x = (x_1, \cdots, x_m) \in (\mathbb{C}^\times)^m$ and $y = (y_1, \cdots, y_m) \in (\mathbb{C}^\times)^n$. From (2.7), we have

$$\mathrm{ch}\mathscr{L}_k = \frac{1}{k}\sum_{d|k}\mu(d)p_d(x,y)^{k/d}, \tag{2.24}$$

where $p_d(x,y) = \sum_{i=1}^m x_i^d - \sum_{j=1}^n (-y_j)^d$ for $d \geq 1$. We will use the above formula for the main results in this paper.

REMARK 2.2. Though the proof of (2.7) in [26] uses only PBW theorem, the importance of the homological method (or Euler-Poincaré principle) we used here lies in the fact that it can be used when we consider the characters of other class of Lie (super)algebras, for example, generalized Kac-Moody (super)algebras. In fact, when $\mathscr{L}$ is a graded Lie algebra with a group action, Kac and Kang gave a formula for characters of $\mathscr{L}$ (or traces on $\mathscr{L}$) by using the homology of Lie algebras [17]. This was also generalized to the case of graded Lie superalgebras by Kang and the author in [19]. See also [22].

3. Symmetric functions and characters of the symmetric groups

In this section, we give a brief review on the (projective) representations of the symmetric groups, and their characters which are closely related with the theory of symmetric functions (see [13, 24] for a general and detailed exposition).

A *partition of* k ($k \geq 1$) is a finite non-increasing sequence $\lambda = (\lambda_1, \cdots, \lambda_r)$ of positive integers whose sum is k. We write $\lambda \vdash k$, and denote by $\mathscr{P}(k)$, the set of partitions of k. Each λ_i is called a *part* of λ, and r is called the *length* of λ, denoted by $\ell(\lambda)$. We also write $\lambda = (1^{m_1}, 2^{m_2}, \cdots)$ where m_i is the number of the parts of λ equal to i ($i \geq 1$). The *conjugate of* λ is the partition $\lambda' = (\lambda'_1, \lambda'_2, \cdots)$ where λ'_i is the number of parts in λ which are no less than i.

3.1. Characters of the symmetric groups

Fix an integer $m \geq 1$. Let S_m be the symmetric group. It is generated by the transposition $\sigma_i = (i\ i+1)$ for $1 \leq i \leq m-1$. Let $x_1, \cdots, x_m$ be variables. S_m acts on $\mathbb{Z}[x_1, \cdots, x_m]$ by permuting the indices of the variables. Consider $\Lambda_m = \mathbb{Z}[x_1, \cdots, x_m]^{S_m}$ the ring of symmetric functions in m variables. For each partition λ with $\ell(\lambda) \leq m$, the *Schur* *function* corresponding to λ is the polynomial $s_\lambda(x_1, \cdots, x_m) = s_\lambda(x)$ given by

$$(3.1) \qquad s_\lambda(x) = \det(h_{\lambda_i - i + j}(x))_{1 \le i,j \le m},$$

where $h_k(x)$ is defined by $\sum_{k \ge 0} h_k(x) t^k = \prod_{i=1}^m (1 - x_i t)^{-1}$ (we assume that $\lambda_k = 0$ for $k > \ell(\lambda)$). Then $\{ s_\lambda(x) \mid \ell(\lambda) \le m \}$ is a $\mathbb{Z}$-basis of Λ_m. If we multiply two Schur functions, we can write it as a linear combination of Schur functions again

$$(3.2) \qquad s_\mu(x) s_\nu(x) = \sum_\lambda N_{\mu\nu}^\lambda s_\lambda(x),$$

where the coefficients $N_{\mu\nu}^\lambda$ are called the *Littlewood-Richardson coefficients.*

For each $k \ge 1$, set $p_k(x_1, \cdots, x_m) = p_k(x) = x_1^k + \cdots + x_m^k$. The *power symmetric function* corresponding to a partition λ, is the polynomial $p_\lambda(x_1, \cdots, x_m) = p_\lambda(x)$ given by

$$(3.3) \qquad p_\lambda(x) = p_{\lambda_1}(x) \cdots p_{\lambda_r}(x).$$

It is known that $\mathbb{Q} \otimes_{\mathbb{Z}} \Lambda_m = \mathbb{Q}[x_1, \cdots, x_m]^{S_m}$ is generated by $p_k(x)$ $(k \ge 1)$ and hence spanned by $\{ p_\lambda(x) \mid \lambda : \text{a partition} \}$.

Note that the irreducible representations of S_k are parameterized by $\mathscr{P}(k)$. For $\lambda \in \mathscr{P}(k)$, let $\chi_{S_k}^\lambda$ be the irreducible character corresponding to λ. For $\mu = (\mu_1, \cdots, \mu_r) \in \mathscr{P}(k)$, let s_i $(1 \le i \le r)$ be a cycle of length μ_i for $1 \le i \le r$, which are mutually disjoint. Then $\sigma_\mu = s_1 \cdots s_r$ is a permutation of cycle type μ. Since a character is determined by the values at the conjugacy classes of S_k, it suffices to know $\chi_{S_k}^\lambda(\sigma_\mu)$. These values are determined by the coefficient of $s_\lambda(x)$ in the expansion of $p_\mu(x)$ into Schur functions, which is known as the *Frobenius formula*:

THEOREM 3.1. ([10]) *For each $\mu \in \mathscr{P}(k)$, we have*

$$(3.4) \qquad p_\mu(x) = \sum_{\substack{\lambda \vdash k \\ \ell(\lambda) \le m}} \chi_{S_k}^\lambda(\sigma_\mu) s_\lambda(x).$$

Fix $n \ge 1$, and let $y_1, \cdots, y_n$ be variables. Let $\Lambda_{m/n}$ be the ring of polynomials $f(x, y)$ in $m + n$ variables $x_1, \cdots, x_m$ and $y_1, \cdots, y_n$ with integral coefficients satisfying

(1) f is invariant under the action of $S_m \times S_n$,
(2) when we put $x_m = -y_n = t$, the resulting polynomial is independent of t.

We call $\Lambda_{m/n}$ the *ring of super symmetric functions* in $m+n$ variables $x_1, \cdots, x_m$ and $y_1, \cdots, y_n$. For each partition $\lambda = (\lambda_1 \cdots, \lambda_r)$, the *hook Schur function* corresponding to λ is the polynomial $hs_\lambda(x,y)$ given by

$$hs_\lambda(x,y) = \det(h_{\lambda_i - i + j}(x,y))_{1 \le i,j \le r}, \tag{3.5}$$

where $h_k(x,y)$ is defined by $\sum_{k \ge 0} h_k(x,y) t^k = \prod_{i=1}^m (1 - x_i t)^{-1} \prod_{j=1}^n (1 + y_j t)$. It is known that $hs_\lambda(x,y) \neq 0$ if and only if λ is (m,n)-*hook shaped*, that is, $\lambda_{m+1} \le n$. Note that $\{ hs_\lambda(x,y) \,|\, \lambda : (m,n)\text{-hook shaped} \}$ is a $\mathbb{Z}$-basis of $\Lambda_{m/n}$ (see [24, 27]).

Set $p_k(x,y) = \sum_{i=1}^m x_i^k - \sum_{j=1}^n (-y_j)^k$ $(k \ge 1)$. For each partition $\lambda = (\lambda_1 \cdots, \lambda_r)$, define

$$p_\lambda(x,y) = p_{\lambda_1}(x,y) \cdots p_{\lambda_r}(x,y), \tag{3.6}$$

which is called the *power super symmetric function* corresponding to λ. In [34], it was shown that $\mathbb{Q} \otimes_\mathbb{Z} \Lambda_{m/n}$ generated by $p_k(x,y)$ $(k \ge 1)$, and hence spanned by $\{ p_\lambda(x,y) \,|\, \lambda : \text{a partition} \}$.

Then, we have the same relation between $\{ hs_\lambda(x,y) \,|\, \lambda : \text{(m,n)-hook shaped} \}$ and $\{ p_\lambda(x,y) \,|\, \lambda : \text{a partition} \}$, which can be proved in a standard way.

PROPOSITION 3.2. *For* $\lambda, \mu \in \mathscr{P}(k)$, *we have*

$$p_\mu(x,y) = \sum_{\lambda \vdash k} \chi^\lambda_{S_k}(\sigma_\mu) hs_\lambda(x,y). \tag{3.7}$$

Proof. Let $z_1, \cdots, z_r$ be variables where r is sufficiently large. Following the arguments in [24] (see Section 4), we obtain the identities

$$\sum_\lambda hs_\lambda(x,y) s_\lambda(z) = \frac{\prod_{k=1}^r \prod_{j=1}^n (1 + y_j z_k)}{\prod_{k=1}^r \prod_{i=1}^m (1 - x_i z_k)} = \sum_\lambda \frac{1}{z_\lambda} p_\lambda(x,y) p_\lambda(z), \tag{3.8}$$

where the sum is taken over all partitions and z_μ is the number of elements in the centralizer of σ_μ in S_k $(\mu \vdash k)$. Note that the above equation can be viewed as a linear combination of $s_\lambda(z)$ and $p_\lambda(z)$ over the ring $\Lambda_{m/n}$.

From (3.4) and the orthogonality of characters, we have for $\lambda \in \mathscr{P}(k)$

$$s_\lambda(z) = \sum_\mu \frac{1}{z_\mu} \chi^\lambda_{S_k}(\sigma_\mu) p_\mu(z). \tag{3.9}$$

Hence, we have

$$\sum_\mu \frac{1}{z_\mu} \Big(\sum_\lambda \chi^\lambda_{S_k}(\sigma_\mu) hs_\lambda(x,y)\Big) p_\mu(z) = \sum_\lambda \frac{1}{z_\lambda} p_\lambda(x,y) p_\lambda(z), \tag{3.10}$$

where $\lambda, \mu \vdash k$. Since $p_\mu(z)$ are linearly independent for $r \gg k$, we obtain (3.7). □

REMARK 3.3. The formula (3.4) and (3.7) can be interpreted from the Schur-Weyl duality [4, 32, 31, 36]. For example, when $V = V_0 = \mathbb{C}^m$, there is a right S_k-action on $V^{\otimes k}$ given by

$$(v_1 \otimes \cdots \otimes v_k) \cdot \sigma = v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)} \tag{3.11}$$

for $v_i \in V$ and $\sigma \in S_k$. It commutes with a left $\mathfrak{gl}(m)$-action on $V^{\otimes k}$, and $\mathfrak{gl}(m)$ and S_k give full centralizers of each other. By taking a trace of an element $\sigma_\mu \times \mathrm{diag}(x_1, \cdots, x_m)$ on $V^{\otimes k}$, we can recover (3.4). For a general review on the duality theorems of various algebras and their relations with the Frobenius formula, the readers are referred to [1].

3.2. Spin characters of the double cover of the symmetric groups

For a given partition λ, we say that λ is *strict* if all parts of λ are distinct, and λ is *even* (resp. *odd*) if all parts of λ are even (resp. odd).

Let Γ_m be the subring of Λ_m generated by $q_k(x)$ $(k \geq 1)$ where $q_k(x)$ is defined by $\sum_{k \geq 0} q_k(x) t^k = \prod_{i=1}^m (1 - tx_i)^{-1}(1 + tx_i)$.

For each partition $\lambda = (\lambda_1, \cdots, \lambda_m)$, the *Schur Q-function* corresponding to λ is the polynomial $Q_\lambda(x)$ given by

$$Q_\lambda(x) = 2^{\ell(\lambda)} \sum_{\sigma \in S_m / S_m^\lambda} \sigma \left(x_1^{\lambda_1} \cdots x_m^{\lambda_m} \prod_{\lambda_i > \lambda_j} \frac{(x_i + x_j)}{(x_i - x_j)} \right), \tag{3.12}$$

where S_m^λ is the subgroup of permutations σ such that $\lambda_{\sigma(i)} = \lambda_i$ for all $1 \leq i \leq m$. Then $Q_\lambda(x) = 0$ unless λ is strict, and $\{ Q_\lambda(x) \mid \lambda :$ strict, $\ell(\lambda) \leq m \}$ forms a $\mathbb{Z}$-basis of Γ_m. Also, $\mathbb{Q} \otimes_{\mathbb{Z}} \Gamma_m$ is generated by $p_k(x)$ for $k = 1, 3, 5 \cdots$, and hence spanned by $\{ p_\lambda(x) \mid \lambda : \text{odd} \}$.

Let $\mathcal{A}_k$ be the associative algebra generated by τ_i's $(1 \leq i \leq k-1)$ satisfying the following relations:

$$\begin{aligned} &\tau_i^2 = -1, \quad (1 \leq i \leq k-1), \\ &\tau_i \tau_{i+1} \tau_i = \tau_{i+1} \tau_i \tau_{i+1} \quad (1 \leq i \leq k-2), \\ &\tau_i \tau_j = -\tau_j \tau_i \quad (1 \leq i, j \leq k-1, |i-j| > 1). \end{aligned} \tag{3.13}$$

Let $\widetilde{S}_k \subset \mathcal{A}_k$ be the group generated by -1 and τ_i $(1 \leq i \leq k-1)$. There exists a surjective homomorphism $\pi : \widetilde{S}_k \to S_k$ given by $\pi(\tau_i) = \sigma_i$ and $\pi(-1) = 1$ with $\ker \pi = \{1, -1\}$ which is central. Hence $\widetilde{S}_k$ is a double cover of S_k and $\mathcal{A}_k$ is a twisted group algebra of S_k. A representation of $\mathcal{A}_k$ corresponds to a projective representation of S_k, which is also called a *spin representation* of $\widetilde{S}_k$.

Set $\mathscr{OP}(k) = \{ \lambda \in \mathscr{P}(k) \,|\, \lambda : \text{odd} \}$ and $\mathscr{DP}(k) = \{ \lambda \in \mathscr{P}(k) \,|\, \lambda : \text{strict} \}$. Let $\mathscr{DP}^+(k)$ (resp. $\mathscr{DP}^-(k)$) be the set of strict partitions of k such that the number of even parts is even (resp. odd).

We consider $\mathcal{A}_k$ as a $\mathbb{Z}_2$-graded algebra (or superalgebra) with $\deg(\tau_i) = 1$ $(1 \leq i \leq k-1)$. Then for each $\lambda \in \mathscr{DP}(k)$, there exists a $\mathbb{Z}_2$-graded irreducible representation T^λ of $\mathcal{A}_k$, whose characters $\chi^\lambda_{\mathcal{A}_k}$ give all the irreducible characters of $\mathcal{A}_k$ (see [15, 38] for a more detailed exposition).

For $\mu = (\mu_1, \cdots, \mu_r) \in \mathscr{P}(k)$, set $\tau_\mu = t_1 \cdots t_r$ where

$$t_i = \tau_{\mu_1+\cdots+\mu_{i-1}+1} \cdots \tau_{\mu_1+\cdots+\mu_i-1}$$

for $1 \leq i \leq r$. Since $\chi^\lambda_{\mathcal{A}_k}(\tau_\mu) = 0$ unless $\mu \in \mathscr{OP}(k)$, it suffices to know $\chi^\lambda_{\mathcal{A}_k}(\tau_\mu)$ for $\mu \in \mathscr{OP}(k)$, and it is given by the coefficient of $Q_\lambda(x)$ in the expansion of $p_\mu(x)$ $(\mu \in \mathscr{OP}(k))$ into Schur Q-functions:

THEOREM 3.4 ([15, 30]). *For each partition $\mu \in \mathscr{OP}(k)$, we have*

$$p_\mu(x) = \sum_{\lambda \in \mathscr{DP}(k)} \left(\frac{1}{\sqrt{2}}\right)^{\ell(\mu)+\ell(\lambda)+\varepsilon(\lambda)} \chi^\lambda_{\mathcal{A}_k}(\tau_\mu) Q_\lambda(x), \tag{3.14}$$

where $\varepsilon(\lambda) = (1 \mp 1)/2$ if $\lambda \in \mathscr{DP}^\pm(k)$.

REMARK 3.5. The above formula is originally due to Schur ([30]), and our description of his result in the language of $\mathbb{Z}_2$-graded algebras is due to [15]. In [33], Sergeev established a duality relation of the Lie superalgebra $\mathfrak{q}(n)$ and the twisted group algebra of the hyperoctahedral group H_k. From this, we have a similar formula where the spin characters of $\widetilde{S}_k$ are replaced by the characters of the twisted group algebra of H_k. The formula (3.14) can also be deduced from the duality relation of $\mathfrak{q}(n)$ and the twisted group algebra of S_k ([38]).

4. Multiplicities of irreducible representations in $\mathscr{L}_n$

Let $\mathscr{L}$ be a Lie superalgebra. A $\mathbb{Z}_2$-graded vector space V is called an *$\mathscr{L}$-module* if there exists a bilinear map $L \times V \longrightarrow V$, $(x, v) \mapsto x \cdot v$ such that

(1) $x \cdot v \in V_{a+b}$ for $x \in \mathscr{L}_a$, $v \in V_b$ $(a, b \in \mathbb{Z}_2)$
(2) $[x, y] \cdot v = x \cdot (y \cdot v) - (-1)^{ab} y \cdot (x \cdot v)$ for $x \in \mathscr{L}_a$ and $y \in \mathscr{L}_b$.

Let V and W be $\mathbb{Z}_2$-graded vector spaces. Then $V \otimes W$ is also a $\mathbb{Z}_2$-graded vector space where $(V \otimes W)_a = \bigoplus_{b+c=a} V_b \otimes W_c$. Moreover, if they are $\mathscr{L}$-modules, then $V \otimes W$ becomes an $\mathscr{L}$-module where the action of $\mathscr{L}$ is given by

$$x \cdot (v \otimes w) = (x \cdot v) \otimes w + (-1)^{ab} v \otimes (x \cdot w), \tag{4.1}$$

where $x \in \mathscr{L}_a$, $v \in V_b$ and $w \in W$.

4.1. Decomposition as a $\mathfrak{gl}(m,n)$-module

Suppose that V is a $\mathbb{Z}_2$-graded vector space with $V_0 = \mathbb{C}^m$ and $V_1 = \mathbb{C}^n$.

Let $\mathfrak{gl}(m,n)$ be the space of all $(m+n) \times (m+n)$ matrices. We may view an element of $\mathfrak{gl}(m,n)$ as an endomorphism of V. For $a \in \mathbb{Z}_2$, set

$$\mathfrak{gl}(m,n)_a = \{ X \in \mathfrak{gl}(m,n) \mid X(V_b) \subset V_{a+b} \text{ for } b \in \mathbb{Z}_2 \}. \tag{4.2}$$

Then $\mathfrak{gl}(m,n) = \mathfrak{gl}(m,n)_0 \oplus \mathfrak{gl}(m,n)_1$ is a $\mathbb{Z}_2$-graded Lie superalgebra, called the *general linear Lie superalgebra*, with the superbracket defined by

$$[X,Y] = XY - (-1)^{ab} YX \tag{4.3}$$

for $X \in \mathfrak{gl}(m,n)_a$, $Y \in \mathfrak{gl}(m,n)_b$, and $a, b \in \mathbb{Z}_2$.

By left multiplication, V becomes a $\mathfrak{gl}(m,n)$-module, which is called the *natural representation*. For $k \geq 1$, $V^{\otimes k}$ is a $\mathfrak{gl}(m,n)$-module with the action given by

$$X \cdot (v_1 \otimes \cdots \otimes v_k) = \sum_{i=1}^{k} (-1)^{a(\sum_{j<i} a_j)} v_1 \otimes \cdots \otimes (X \cdot v_i) \otimes \cdots \otimes v_k, \tag{4.4}$$

for $X \in \mathfrak{gl}(m,n)_a$ and $v_i \in V_{a_i}$ $(1 \leq i \leq k)$.

Let $\mathscr{L}$ be the free Lie superalgebra generated by V and $\mathscr{L}_k$ the kth homogeneous component. Note that $\mathscr{L}_k$ is a subspace of $V^{\otimes k}$ and it is a $\mathfrak{gl}(m,n)$-submodule of $V^{\otimes k}$ where the action of $\mathfrak{gl}(m,n)$ is induced from $V^{\otimes k}$:

$$\begin{aligned} &X \cdot [v_1, [v_2, [\cdots [v_{k-1}, v_k] \cdots]]] \\ &= \sum_{i=1}^{k} (-1)^{a(\sum_{j<i} a_j)} [v_1, [\cdots [X \cdot v_i [\cdots [v_{k-1}, v_k] \cdots]]]], \end{aligned} \tag{4.5}$$

for $X \in \mathfrak{gl}(m,n)_a$ and $v_i \in V_{a_i}$ $(1 \leq i \leq k)$.

For $k \geq 1$, it is known that $V^{\otimes k}$ is completely reducible as a $\mathfrak{gl}(m,n)$-module and its irreducible components are parameterized by the (m,n)-hook shaped partitions of k. For each (m,n)-hook shaped partition

λ of k, let V^λ be the corresponding irreducible representation. Let $x = (x_1, \cdots, x_m) \in (\mathbb{C}^\times)^m$ and $y = (y_1, \cdots, y_n) \in (\mathbb{C}^\times)^n$ be variables. The character of V^λ, i.e. the trace of diag(x, y) on V^λ, is

$$\mathrm{ch}V^\lambda = hs_\lambda(x, y), \tag{4.6}$$

where $hs_\lambda(x, y)$ is the hook Schur function corresponding to λ (see [4] for the above arguments).

Note that $\mathrm{ch}V^{\otimes k} = p_1(x, y)^k = p_{(1^k)}(x, y)$. From (3.7), we have

$$\mathrm{ch}V^{\otimes k} = \sum_\lambda \chi^\lambda_{S_k}(1) hs_\lambda(x, y), \tag{4.7}$$

where the sum is over all (m, n)-hook shaped partitions of k. Hence, the multiplicity of V^λ in $V^{\otimes k}$ is equal to $\chi^\lambda_{S_k}(1)$.

THEOREM 4.1. *For $k \geq 1$, let $\mathscr{L}_k$ be the kth homogeneous component of the free Lie superalgebra generated by the natural representation V of $\mathfrak{gl}(m, n)$. As a $\mathfrak{gl}(m, n)$-module, $\mathscr{L}_k$ is completely reducible, and for each (m, n)-hook shaped partition λ of k, the multiplicity of V^λ in $\mathscr{L}_k$ is equal to*

$$\frac{1}{k} \sum_{d|k} \mu(d) \chi^\lambda_{S_k}(\sigma_{(d^{k/d})}). \tag{4.8}$$

Proof. Since $\mathscr{L}_k$ is a $\mathfrak{gl}(m, n)$-submodule of $V^{\otimes k}$, it is completely reducible. In terms of characters, we have

$$ch\mathscr{L}_k = \sum_\lambda m_\lambda hs_\lambda(x, y), \tag{4.9}$$

where the sum is over all (m, n)-hook shaped partitions of k, and m_λ is the multiplicity of V^λ in $\mathscr{L}_k$. On the other hand, we have

$$\begin{aligned}
\mathrm{ch}\mathscr{L}_k &= \frac{1}{k} \sum_{d|k} \mu(d) p_d(x, y)^{k/d} \qquad \text{by (2)} \\
&= \frac{1}{k} \sum_{d|k} \mu(d) \left(\sum_{\lambda \vdash k} \chi^\lambda_{S_k}(\sigma_{(d^{k/d})}) hs_\lambda(x, y) \right) \qquad \text{by (3.7)} \\
&= \sum_{\lambda \vdash k} \left(\frac{1}{k} \sum_{d|k} \mu(d) \chi^\lambda_{S_k}(\sigma_{(d^{k/d})}) \right) hs_\lambda(x, y).
\end{aligned} \tag{4.10}$$

Since $\{ hs_\lambda(x, y) \mid \lambda : (m, n)\text{-hook shaped} \}$ is linearly independent in $\mathbb{Q} \otimes_{\mathbb{Z}} \Lambda_{m/n}$, we obtain the result by comparing (4.9) and (4.10). □

Consider the action of S_k on $V^{\otimes k}$ given by

$$(4.11)\quad (v_1 \otimes \cdots \otimes v_k) \cdot \sigma_i = (-1)^{a_i a_{i+1}} (v_1 \otimes \cdots \otimes v_{i+1} \otimes v_i \otimes \cdots \otimes v_k),$$

where $v_j \in V_{a_j}$ $(1 \le j \le k)$. It defines a right S_k-module structure on $V^{\otimes k}$ and commutes with the left action of $\mathfrak{gl}(m,n)$. With these two commuting actions, Berele and Regev established the Schur-Weyl duality ([4]). On the other hand, let $\mathrm{Ind}_{C_k}^{S_k}\theta$ be the induced representation of a faithful representation θ of a cyclic subgroup C_k of order k. Then, for each (m,n)-hook shaped partition λ of k, it is not difficult to see that the multiplicity of the Specht module S^λ in $\mathrm{Ind}_{C_k}^{S_k}\theta$ is equal to (4.8). Therefore, from the Schur-Weyl duality, we have

$$(4.12)\qquad \mathscr{L}_k \cong V^{\otimes k} \otimes_{\mathbb{C}[S_k]} \mathrm{Ind}_{C_k}^{S_k}\theta,$$

as $\mathfrak{gl}(m,n)$-modules. When $\mathscr{L}$ is a free Lie algebra (or $V_1 = 0$), this was given by Klyachko ([20]).

REMARK 4.2. In [19], the multiplicity m_λ is given in a recursive form in terms of character values of the symmetric groups, and hence expressed in a rather complicated way. But in this paper, we express the character of $\mathscr{L}_k$ in terms of power super symmetric functions directly (2), and use the Frobenius formula (3.7) to obtain a closed form of the multiplicities. Some generalizations of the formula (1.2) using the theory of symmetric functions can be found in [9, 14].

4.2. Decomposition as a $\mathfrak{q}(n)$-module

In this subsection, we assume that $m = n$, i.e. $V = \mathbb{C}^n \oplus \mathbb{C}^n$. Let $\mathfrak{q}(n)$ be the Lie subsuperalgebra of $\mathfrak{gl}(n,n)$ consisting of all matrices of the form $\begin{pmatrix} A & B \\ B & A \end{pmatrix}$ where A and B are $n \times n$ matrices. Then V is a $\mathfrak{q}(n)$-module called the *natural representation.*

For $k \ge 1$, let us consider the k-fold tensor product of V. Then, as in the case of $\mathfrak{gl}(m,n)$, $V^{\otimes k}$ is completely reducible as a $\mathfrak{q}(n)$-module, and its irreducible components are parameterized by $\mathscr{DP}(k)$.

For each $\lambda \in \mathscr{DP}(k)$, let U^λ be the corresponding irreducible representation. The character of U^λ, i.e. the trace of $\mathrm{diag}(x,x)$ $(x \in (\mathbb{C}^\times)^n)$ on U^λ, is given by

$$(4.13)\qquad \mathrm{ch}U^\lambda = (\sqrt{2})^{d(\lambda)-\ell(\lambda)} Q_\lambda(x),$$

where $Q_\lambda(x)$ is the Schur Q-function corresponding to λ and $d(\lambda) = (1-(-1)^{\ell(\lambda)})/2$ (see [33] for the above arguments).

Note that the trace of $\mathrm{diag}(x,x)$ $(x \in (\mathbb{C}^\times)^n)$ on $V^{\otimes k}$ is equal to $2^k p_1(x)^k = 2^k p_{(1^k)}(x)$. From (3.14), we have

$$2^k p_{(1^k)}(x) = \sum_{\lambda \in \mathscr{DP}(k)} (\sqrt{2})^{k-\ell(\lambda)-\varepsilon(\lambda)} \chi^\lambda_{\mathcal{A}_k}(1) Q_\lambda(x), \tag{4.14}$$

which implies that the multiplicity of U^λ in $V^{\otimes k}$ is $(\sqrt{2})^{k-d(\lambda)-\varepsilon(\lambda)}\chi^\lambda_{\mathcal{A}_k}(1)$.

THEOREM 4.3. *For $k \geq 1$, let $\mathscr{L}_k$ be the kth homogeneous component of the free Lie superalgebra generated by the natural representation V of $\mathfrak{q}(n)$. As a $\mathfrak{q}(n)$-module, $\mathscr{L}_k$ is completely reducible, and for each $\lambda \in \mathscr{DP}(k)$, the multiplicity of U^λ in $\mathscr{L}_k$ is equal to*

$$\frac{1}{k} \sum_{\substack{d|k \\ d:\text{odd}}} \mu(d) \chi^\lambda_{\mathcal{A}_k}(\tau_{(d^{k/d})}) (\sqrt{2})^{(k/d)-d(\lambda)-\varepsilon(\lambda)}. \tag{4.15}$$

Proof. Since $\mathscr{L}_k$ is a $\mathfrak{q}(n)$-submodule of $V^{\otimes k}$, it is completely reducible. In terms of characters, we have

$$ch \mathscr{L}_k = \sum_{\lambda \in DP(k)} m_\lambda \mathrm{ch} U^\lambda, \tag{4.16}$$

where m_λ is the multiplicity of U^λ in $\mathscr{L}_k$.

On the other hand, we have

(4.17)

$$\begin{aligned}
&\mathrm{ch}\mathscr{L}_k \\
&= \frac{1}{k} \sum_{d|k} \mu(d) p_d(x,x)^{k/d} \\
&= \frac{1}{k} \sum_{\substack{d|k \\ d:\text{odd}}} \mu(d) 2^{k/d} p_d(x)^{k/d} \quad \text{by (2.24)} \\
&= \frac{1}{k} \sum_{\substack{d|k \\ d:\text{odd}}} \mu(d) \left(\sum_{\lambda \in \mathscr{DP}(k)} (\sqrt{2})^{(k/d)-d(\lambda)-\varepsilon(\lambda)} \chi^\lambda_{\mathcal{A}_k}(\tau_{(d^{k/d})}) \mathrm{ch} U^\lambda \right) \quad \text{by (3.14)} \\
&= \sum_{\lambda \in \mathscr{DP}(k)} \left(\frac{1}{k} \sum_{\substack{d|k \\ d:\text{odd}}} \mu(d) (\sqrt{2})^{(k/d)-d(\lambda)-\varepsilon(\lambda)} \chi^\lambda_{\mathcal{A}_k}(\tau_{(d^{k/d})}) \right) \mathrm{ch} U^\lambda.
\end{aligned}$$

Since $\{ Q_\lambda(x) \mid \lambda \in \mathscr{DP}(k) \}$ is linearly independent in $\mathbb{Q} \otimes_{\mathbb{Z}} \Gamma_m$, we obtain the result by comparing (4.16) and (4.17). □

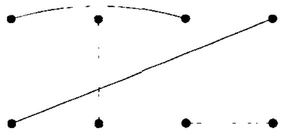
5. Remarks on free Lie algebras

Suppose that $V = \mathbb{C}^n$. Let $\mathscr{L}$ be the free Lie algebra generated by V. Let $\mathfrak{g} = \mathfrak{sp}(n)$ (n : even) or $\mathfrak{so}(n)$ be a subalgebra of $\mathfrak{gl}(n)$. For each λ with $\ell(\lambda) \leq n$, the irreducible $\mathfrak{gl}(n)$-module V^λ decomposes into irreducible $\mathfrak{g}$-modules with the multiplicities given in terms of the Littlewood-Richardson coefficients (see (5.3)). Hence, by (1.2), the multiplicity of each irreducible $\mathfrak{g}$-module in $\mathscr{L}_k$ is given in terms of the character values of S_k and the Littlewood-Richardson coefficients.

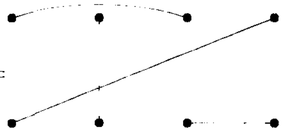
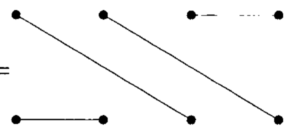
In this section, we show that the multiplicity of each irreducible $\mathfrak{g}$-module in $\mathscr{L}_k$ for $1 \leq k \leq n$ can be simplified in terms of the character values of the Brauer algebras, which is analogous to (1.2).

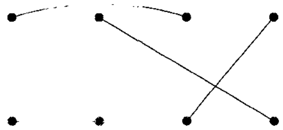
5.1. Characters of the Brauer algebras

Fix $f \geq 1$. Consider two rows each of which consists of f vertices. An *f-diagram* is a graph with the above $2f$ vertices and f edges where each vertex belongs to exactly one edge. For example, the following is a 4-diagram.



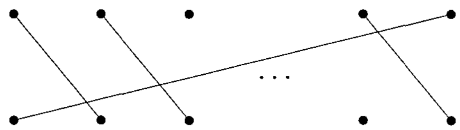
Let z be an indeterminate. Given two f-diagrams d_1 and d_2, we associate an f-diagram d obtained by (i) placing d_2 below d_1, (ii) identifying f vertices in the bottom row of d_1 with f vertices in the top row in d_2. Then we define $d_1 d_2$ to be the d multiplied by z^c where c is the number of cycles appearing in the middle row. For example, if

$d_1 =$  $d_2 =$  , then

$d_1 d_2 = \; z$ 

The *Brauer algebra* $D_f(z)$ is an associative $\mathbb{C}(z)$-algebra spanned by all f-diagrams whose multiplication is described above. Note that the multiplicative identity is an f-diagram consisting of exactly f vertical edges. Then $D_f(z)$ is a semisimple $\mathbb{C}(z)$-algebra and the irreducible representations of $D_f(z)$ are indexed by $\bigcup_{0\leq k\leq [f/2]} \mathscr{P}(f-2k)$ (see [35]).

For $d \in D_f(z)$ and $d' \in D_{f'}(z)$, by a natural embedding of $D_f(z) \otimes D_{f'}(z)$ into $D_{f+f'}(z)$, we may view $d \otimes d'$ as an element in $D_{f+f'}(z)$. Let e be a 2-diagram with 2 horizontal edges. For $k \geq 2$, let γ_k be the k-diagram of the following form



and we set γ_1 to be the identity in $D_1(z)$. Note that the symmetric group S_f can be embedded into $D_f(z)$ where σ_i $(1 \leq i \leq f-1)$ corresponds to the element $\gamma_1^{\otimes i-1} \otimes \gamma_2 \otimes \gamma_1^{\otimes f-i-1}$.

For each partition $\mu = (\mu_1, \cdots, \mu_r)$, let $\gamma_\mu = \gamma_{\mu_1} \otimes \cdots \otimes \gamma_{\mu_r}$. In [28], Ram has shown that the characters of $D_f(z)$ are completely determined by the values at $e^{\otimes h} \otimes \gamma_\mu$ where $2h + |\mu| = f$, and computed the irreducible characters of $D_f(z)$ in terms of the characters of the symmetric groups:

THEOREM 5.1 ([28]). *Let $\lambda \in \mathscr{P}(f-2k)$ $(0 \leq k \leq [f/2])$, and $\chi^\lambda_{D_f(z)}$ the irreducible character of $D_f(z)$ corresponding to λ. For $e^{\otimes h} \otimes \gamma_\mu$ ($2h + |\mu| = f$), we have*

$$\chi^\lambda_{D_f(z)}(e^{\otimes h} \otimes \gamma_\mu) = z^h \sum_{\nu \vdash f-2h} \Big(\sum_{\substack{\eta \vdash 2k-2h \\ \eta:\text{even}}} N^\nu_{\lambda\eta} \Big) \chi^\nu_{S_{f-2h}}(\sigma_\mu), \tag{5.1}$$

where $N^\nu_{\lambda\eta}$ are the Littlewood-Richardson coefficients given in (3.2).

REMARK 5.2. The formula (5.1) was obtained from the duality relation of the orthogonal groups and the Brauer algebras. Following the same arguments in [28] by using the duality relation between the symplectic groups and the Brauer algebras, it is not difficult to see that

$$\chi^{\lambda'}_{D_f(z)}(e^{\otimes h} \otimes \gamma_\mu) = (-1)^{|\mu|-\ell(\mu)} z^h \sum_{\nu \vdash f-2h} \Big(\sum_{\substack{\eta \vdash 2k-2h \\ \eta':\text{even}}} N^\nu_{\lambda\eta} \Big) \chi^\nu_{S_{f-2h}}(\sigma_\mu). \tag{5.2}$$

Both (5.1) and (5.2) will be used for our computations.

5.2. Decomposition as $\mathfrak{sp}(n)$ and $\mathfrak{so}(n)$-modules

In this subsection, we assume that $V_1 = 0$, and hence $\mathscr{L}$ is the free Lie algebra generated by $V = V_0 = \mathbb{C}^n$. Suppose that $\mathfrak{g} = \mathfrak{sp}(n)$ (n : even) or $\mathfrak{so}(n) \subset \mathfrak{gl}(n)$.

By restriction, $\mathscr{L}_k$ is a representation of $\mathfrak{g} \subset \mathfrak{gl}(n)$. Note that $V^{\otimes k}$ decomposes into polynomial representations parameterized by the partitions $\mu \in \mathscr{P}(k-2i)$ ($i = 0, \cdots, [k/2]$) satisfying $\ell(\mu) \leq [n/2]$ (see [32, 31, 36]). We denote by W^μ the corresponding representation. If $\mathfrak{g} = \mathfrak{so}(n)$ and $\ell(\mu) = n/2$ (n : even), then W^μ is a sum of two irreducible representations of the same dimension. Otherwise, W^μ is irreducible. For each partition λ with $\ell(\lambda) \leq n$, let V^λ be the irreducible polynomial representation of $\mathfrak{gl}(n)$. When restricted to a representation of $\mathfrak{g}$, it decomposes into W^μ's and the multiplicity of W^μ in V^λ is given by

$$\begin{cases} \sum_{\nu:\text{even}} N^\lambda_{\mu\nu} & \text{if } \mathfrak{g} = \mathfrak{so}(n), \\ \sum_{\nu':\text{even}} N^\lambda_{\mu\nu} & \text{if } \mathfrak{g} = \mathfrak{sp}(n), \end{cases} \tag{5.3}$$

(see [23]). Now, combining (5.1), (5.2) and (5.3), we can describe the multiplicities of irreducible $\mathfrak{g}$-modules in $\mathscr{L}_k (k \leq n)$:

PROPOSITION 5.3. *For $k \geq 1$, let $\mathscr{L}_k$ be the kth homogeneous component of the free Lie algebra generated by $V = \mathbb{C}^n$. Then $\mathscr{L}_k$ is completely reducible as a $\mathfrak{g}$-module. And*

(a) *if $\mathfrak{g} = \mathfrak{so}(n)$ and $k \leq n$, then for each $\mu \in \mathscr{P}(k-2i)$ ($i = 0, \cdots, [k/2]$) with $\ell(\mu) \leq [n/2]$, the multiplicity of W^μ in $\mathscr{L}_k$ is*

$$\frac{1}{k} \sum_{d|k} \mu(d) \chi^\mu_{D_k(z)}(\gamma_{(d^{k/d})}). \tag{5.4}$$

(b) *if $\mathfrak{g} = \mathfrak{sp}(n)$ (n : even) and $k \leq n$, then for each $\mu \in \mathscr{P}(k-2i)$ ($i = 0, \cdots, [k/2]$) with $\ell(\mu) \leq n/2$, the multiplicity of W^μ in $\mathscr{L}_k$ is*

$$\frac{1}{k} \sum_{d|k} \mu(d) \chi^{\mu'}_{D_k(z)}(\gamma_{(d^{k/d})})(-1)^{k-(k/d)}. \tag{5.5}$$

Proof. (a) As a $\mathfrak{gl}(n)$-module, we have $\mathscr{L}_k = \bigoplus_{\substack{\lambda \vdash k \\ \ell(\lambda) \leq n}} (V^\lambda)^{\oplus m_\lambda}$ where

$$m_\lambda = \frac{1}{k} \sum_{d|k} \mu(d) \chi^\lambda_{S_k}(\sigma_{(d^{k/d})}). \tag{5.6}$$

Since $k \leq n$, the condition $\ell(\lambda) \leq n$ in the decomposition of $\mathscr{L}_k$ is sufficient. Hence, for each $\mu \in \mathscr{P}(k-2i)$ $(i = 0, \cdots, [k/2])$ with $\ell(\mu) \leq [n/2]$, the multiplicity of W^μ in $\mathscr{L}_k$ is

$$\begin{aligned}
&\sum_{\lambda \vdash k} \left(\left(\sum_{\nu:\text{even}} N_{\mu\nu}^{\lambda} \right) \frac{1}{k} \sum_{d|k} \mu(d) \chi_{S_k}^{\lambda}(\sigma_{(d^{k/d})}) \right) \\
(5.7) \qquad &= \frac{1}{k} \sum_{d|k} \mu(d) \left(\sum_{\lambda \vdash k} \left(\sum_{\nu:\text{even}} N_{\mu\nu}^{\lambda} \right) \chi_{S_k}^{\lambda}(\sigma_{(d^{k/d})}) \right) \\
&= \frac{1}{k} \sum_{d|k} \mu(d) \chi_{D_k(z)}^{\mu}(\gamma_{(d^{k/d})}) \quad \text{by (5.1).}
\end{aligned}$$

(b) The proof for $\mathfrak{g} = \mathfrak{sp}(n)$ is almost the same as in (a) except using (5.2). □

REMARK 5.4. (1) Proposition 5.3 can be obtained directly by the analogues of Frobenius formula (see Corollary 4.5 and Theorem 4.6 in [28]).

(2) By (5.1) and (5.2), it follows that if $\mu \vdash k$, then $\chi_{D_k(z)}^{\mu}(\gamma_{(d^{k/d})})$ in (5.4) and (5.5) is equal to $\chi_{S_k}^{\mu}(\sigma_{(d^{k/d})})$.

(3) We would like to remark one more application of the Brauer algebras to the decomposition of free Lie algebras. If $\mathscr{L}$ is the free Lie algebra generated by $V^{\otimes p} \otimes (V^*)^{\otimes q}$, then as a $\mathfrak{gl}(n)$-module ($\dim V = n$), $\mathscr{L}_k$ decomposes into rational irreducible representations, whose characters are given by rational Schur functions. In this case, a Frobenius formula is given in [12], and by using (1.1), the multiplicities of irreducible representations in $\mathscr{L}_k$ can be expressed in terms of the character values of a subalgebra $D_{p,q}(z)$ of $D_k(z)$ consisting of (p,q)-diagrams (cf. [2]).

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