

## RANK INEQUALITIES OVER SEMIRINGS

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ABSTRACT. Inequalities on the rank of the sum and the product of two matrices over semirings are surveyed. Preferences are given to the factor rank, row and column ranks, term rank, and zero-term rank of matrices over antinegative semirings.

### 1. Introduction

During the past century a lot of literature has been devoted to investigations of semirings. Briefly, a semiring is essentially a ring where only the zero element is required to have an additive inverse. Therefore, all rings are also semirings. Moreover, among semirings there are such combinatorially interesting systems as the Boolean algebra of subsets of a finite set (with addition being union and multiplication being intersection), nonnegative integers and reals (with the usual arithmetic), fuzzy scalars (with fuzzy arithmetic), etc. Matrix theory over semirings is an object of much study in the last decades, see for example [9]. In particular, many authors have investigated various rank functions for matrices over semirings and their properties, see [1, 3, 6, 7, 8, 12] and references there in.

There are classical inequalities for the rank function  $\rho$  of sums and products of matrices over fields, see, for example [10, 11]:

The *rank-sum inequalities*:

$$|\rho(A) - \rho(B)| \leq \rho(A + B) \leq \rho(A) + \rho(B);$$

*Sylvester's laws*:

$$\rho(A) + \rho(B) - n \leq \rho(AB) \leq \min\{\rho(A), \rho(B)\}$$

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and the *Frobenius inequality*:

$$\rho(AB) + \rho(BC) \leq \rho(ABC) + \rho(B),$$

where  $A, B, C$  are real or complex conformal matrices.

These inequalities may or may not hold when  $\mathcal{S}$  is not a field.

In the present paper we compare different rank functions for matrices over semirings. For these rank functions we investigate the semiring versions of the above mentioned classical inequalities for the sum and product of matrices. Numerous examples are given to illustrate the behaviour of rank functions under consideration. In particular, we show that our bounds are exact and the best possible. Namely not only pairs of matrices satisfying the case of equality are presented but for any given  $r$  and  $s$  it is proved that there exist matrices  $A$  and  $B$  of ranks  $r$  and  $s$  respectively such that the equality holds.

Our paper is organized as follows. In section 2 we collect all necessary definitions and notations. In section 3 we compare different semiring rank functions for a given matrix. In subsequent sections, we investigate upper and lower bounds on these introduced rank functions for sums and products of matrices: section 4 is devoted to factor rank, section 5 is devoted to term rank, section 6 is devoted to zero-term rank, and section 7 is devoted to row and column rank functions.

## 2. Definitions and notations

DEFINITION 2.1. A semiring,  $\mathcal{S}$ , consists of a nonempty set  $\mathcal{S}$  and two binary operations, addition and multiplication, such that:

- $\mathcal{S}$  is an Abelian monoid under addition (identity denoted by 0);
- $\mathcal{S}$  is a semigroup under multiplication (identity, if any, denoted by 1);
- multiplication is distributive over addition on both sides;
- $s0 = 0s = 0$  for all  $s \in \mathcal{S}$ .

In this paper we will always assume that there is a multiplicative identity 1 in  $\mathcal{S}$ .

DEFINITION 2.2. A semiring is called antinegative if no nonzero element has an additive inverse.

DEFINITION 2.3. Let  $\mathcal{S}$  be a set of subsets of a given set  $M$  containing the empty set and  $M$ , the sum of two subsets is their union, and the product is their intersection. If  $\mathcal{S}$  is closed under addition (unions) and multiplication (intersections),  $M$  is a semiring. The zero element is

the empty set and the identity element is the whole set  $M$ . We call these types of semirings, or any semiring which is isomorphic to these, Boolean semirings.

It is straightforward to see that a Boolean semiring is commutative and antinegative. If  $\mathcal{S}$  consists of only the empty subset and  $M$ ,  $M \neq \emptyset$ , then it is called a binary Boolean semiring (or  $\{0, 1\}$ -semiring) and is denoted by  $\mathbf{B}$ .

DEFINITION 2.4. A semiring is called a chain semiring if the set  $\mathcal{S}$  is totally ordered with universal lower and upper bounds and the operations are defined by  $a + b = \max\{a, b\}$  and  $a \cdot b = \min\{a, b\}$ .

Let  $\mathcal{M}_{m,n}(\mathcal{S})$  denote the set of  $m \times n$  matrices with entries from the semiring  $\mathcal{S}$ . Throughout we assume that  $m \leq n$ . The matrix  $I_n$  is the  $n \times n$  identity matrix,  $J_{m,n}$  is the  $m \times n$  matrix of all ones,  $O_{m,n}$  is the  $m \times n$  zero matrix. We omit the subscripts when the sizes of matrices is obvious from the context, and we write  $I$ ,  $J$ , and  $O$ , respectively. The matrix  $E_{i,j}$ , called a *cell*, denotes the matrix with exactly one nonzero entry, that being a one in the  $(i, j)$  entry. Let  $R_i$  denote the matrix whose  $i^{\text{th}}$  row is all ones and is zero elsewhere, and  $C_j$  denote the matrix whose  $j^{\text{th}}$  column is all ones and is zero elsewhere. Let  $U_k$  denote the  $k \times k$  matrix of all ones above and on the main diagonal,  $L_k$  denote  $k \times k$  strictly lower triangular matrix of ones. A *line* of matrix  $A$  is a row or column of the matrix  $A$ . We denote by  $A \oplus B$  the block-diagonal matrix of the form  $\begin{bmatrix} A & O \\ O & B \end{bmatrix}$ . Note that in this sense the operation  $\oplus$  is not commutative. We say that the matrix  $A$  *dominates* the matrix  $B$  if and only if  $b_{i,j} \neq 0$  implies that  $a_{i,j} \neq 0$ , and we write  $A \geq B$  or  $B \leq A$ . If  $A$  and  $B$  are  $(0, 1)$ -matrices and  $A \geq B$  we let  $A \setminus B$  denote the matrix  $C$  where

$$c_{i,j} = \begin{cases} 0 & \text{if } b_{i,j} = 1 \\ a_{i,j} & \text{otherwise} \end{cases} .$$

Below we recall well-known rank concepts for matrices over semirings. The detailed information on this subject can be found in [1, 2, 4, 8].

DEFINITION 2.5. The matrix  $A \in \mathcal{M}_{m,n}(\mathcal{S})$ ,  $A \neq O$  is said to be of factor rank  $k$  ( $\text{rank}(A) = k$ ) if  $k$  is the smallest positive integer such that there exist matrices  $B \in \mathcal{M}_{m,k}(\mathcal{S})$  and  $C \in \mathcal{M}_{k,n}(\mathcal{S})$  such that  $A = BC$ . The factor rank the zero matrix,  $O$ , shall be equal to 0.

PROPOSITION 2.6. *The factor rank of  $A$  is equal to the minimum number of factor rank-1 matrices whose sum is  $A$ .*

*Proof.* For  $\text{rank}(A) = k$  and  $A = BC$ ,  $B \in \mathcal{M}_{m,k}(\mathcal{S})$  and  $C \in \mathcal{M}_{k,n}(\mathcal{S})$  we have  $A = \sum_{i=1}^k \mathbf{b}^i \mathbf{c}_i$  where  $\mathbf{b}^i$  denotes the  $i^{\text{th}}$  column of  $B$  and  $\mathbf{c}_i$  denotes the  $i^{\text{th}}$  row of  $C$ . Thus the factor rank of  $A$  is at least the minimum number of factor rank-1 matrices whose sum is  $A$ . If  $A = \sum_{i=1}^l \mathbf{b}^i \mathbf{c}_i$  where  $\mathbf{b}^i \in \mathcal{M}_{m,1}(\mathcal{S})$  and  $\mathbf{c}_i \in \mathcal{M}_{1,n}(\mathcal{S})$  then  $A = BC$

where  $B = [\mathbf{b}^1, \mathbf{b}^2, \dots, \mathbf{b}^l]$  and  $C = \begin{bmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \\ \vdots \\ \mathbf{c}_l \end{bmatrix}$ . Thus, the factor rank of  $A$

is at most the minimum number of factor rank-1 matrices whose sum is  $A$ . The proposition follows.  $\square$

DEFINITION 2.7. A matrix  $A \in \mathcal{M}_{m,n}(\mathcal{S})$  is said to be of term rank  $k$  ( $t(A) = k$ ) if the minimum number of lines needed to include all nonzero elements of  $A$  is equal to  $k$ .

Let us denote by  $t_c(A)$  the least number of columns needed to include all nonzero elements of  $A$  and by  $t_r(A)$  the least number of rows needed to include all nonzero elements of  $A$ .

DEFINITION 2.8. A generalized diagonal of a matrix  $A \in \mathcal{M}_{m,n}(\mathcal{S})$  is a set of  $\min\{m, n\}$  entries of  $A$  such that no row or column contains two of these entries.

PROPOSITION 2.9. [4, Theorem 1.2.1] *The term rank of  $A$  is the maximum number of nonzero entries in some generalized diagonal of  $A$ .*

DEFINITION 2.10. The matrix  $A \in \mathcal{M}_{m,n}(\mathcal{S})$  is said to be of zero-term rank  $k$  ( $z(A) = k$ ) if the minimum number of lines needed to include all zero elements of  $A$  is equal to  $k$ .

A vector space is usually only defined over fields or division rings, and modules are generalizations of vector spaces defined over rings. We generalize the concept of vector spaces to *semiring vector spaces* defined over arbitrary semirings.

DEFINITION 2.11. Given a semiring  $\mathcal{S}$ , we define a semiring vector space,  $V(\mathcal{S})$ , to be a nonempty set with two operations, addition and scalar multiplication such that  $V(\mathcal{S})$  is closed under addition and scalar multiplication, addition is associative and commutative, and such that for all  $\mathbf{u}$  and  $\mathbf{v}$  in  $V(\mathcal{S})$  and  $r, s \in \mathcal{S}$

1. There exists a  $\mathbf{0}$  such that  $\mathbf{0} + \mathbf{v} = \mathbf{v}$ ,
2.  $\mathbf{1}\mathbf{v} = \mathbf{v} = \mathbf{v}\mathbf{1}$ ,
3.  $r s \mathbf{v} = r(s \mathbf{v})$ ,
4.  $(r + s)\mathbf{v} = r\mathbf{v} + s\mathbf{v}$ , and
5.  $r(\mathbf{u} + \mathbf{v}) = r\mathbf{u} + r\mathbf{v}$ .

DEFINITION 2.12. A set of vectors,  $\mathcal{S}$ , from a semiring vector space,  $V(\mathcal{S})$  is called linearly independent if there is no vector in  $\mathcal{S}$  that can be expressed as a nontrivial linear combination of the others. The set is linearly dependent if it is not independent.

Note that, unlike vectors over fields, there are several ways to define independence, we will use the definition above.

DEFINITION 2.13. A collection,  $B$ , of linearly independent vectors is said to be a basis of the semiring vector space  $V(\mathcal{S})$  if its linear span is  $V(\mathcal{S})$ . The dimension of  $V(\mathcal{S})$  is a minimal number of vectors in any basis of  $V(\mathcal{S})$ .

DEFINITION 2.14. The matrix  $A \in \mathcal{M}_{m,n}(\mathcal{S})$  is said to be of row(resp. column) rank  $k$  ( $r(A) = k$ ) if the dimension of the linear span of the rows(resp. columns) of  $A$  is equal to  $k$ .

DEFINITION 2.15. The matrix  $A \in \mathcal{M}_{m,n}(\mathcal{S})$  is said to be of spanning row(resp. column) rank  $k$  ( $sr(A) = k$ ) if the minimal number of rows(resp. columns) that span all rows (resp. columns) of  $A$  is  $k$ .

DEFINITION 2.16. The matrix  $A \in \mathcal{M}_{m,n}(\mathcal{S})$  is said to be of maximal row(resp. column) rank  $k$  ( $mr(A) = k$ ) if it has  $k$  linearly independent rows(resp. columns) and any  $(k + 1)$  rows(resp. columns) are linearly dependent.

### 3. Rank relations

If the semiring  $\mathcal{S}$  is a subsemiring of a field  $\mathcal{F}$ , let  $\rho(A)$  denote the rank of the matrix  $A$  as an element of  $\mathcal{M}_{m,n}(\mathcal{F})$ . If the semiring  $\mathcal{S}$  coincides with the field  $\mathcal{F}$ , then,  $\rho(A) = \text{rank}(A) = r(A) = c(A) = sr(A) = sc(A) = mr(A) = mc(A)$ . Note that  $\rho(A)$  is invariant of the field chosen to contain  $\mathcal{S}$ . Over more general semirings, the situation is more complicated. Namely, the following inequalities are true for matrices with entries from arbitrary semirings:

PROPOSITION 3.1. *Let  $A \in \mathcal{M}_{m,n}(\mathcal{S})$  then*

1.  $\text{rank}(A) \leq \min\{r(A), c(A)\}$ ;
2.  $r(A) \leq sr(A) \leq mr(A)$ ;  $c(A) \leq sc(A) \leq mc(A)$ ;
3.  $\text{rank}(A) \leq t(A)$ ;
4. If  $\mathcal{S}$  is a subsemiring of a field  $\mathcal{F}$  then  $\rho(A) \leq \text{rank}(A)$ .

*Proof.* 1. See [3, Lemma 2.3].

2. Follows directly from the definitions.

3. Let  $A \in \mathcal{M}_{m,n}(\mathcal{S})$ . Suppose that the term rank of  $A$  is  $k$ . By König's Theorem, see [4, Chapter 1.2], there exist permutation matrices  $P \in \mathcal{M}_{m,m}(\mathcal{S})$  and  $Q \in \mathcal{M}_{n,n}(\mathcal{S})$  such that

$$PAQ = \begin{bmatrix} A_1 & A_2 \\ A_3 & O_{(m-r) \times (n-s)} \end{bmatrix},$$

where  $A_1 \in \mathcal{M}_{r,s}(\mathcal{S})$ ,  $A_2 \in \mathcal{M}_{r,n-s}(\mathcal{S})$ ,  $A_3 \in \mathcal{M}_{m-r,s}(\mathcal{S})$ , and  $r + s = k$ . Now, define

$$X = \begin{bmatrix} I_r & O_{r \times s} \\ O_{(m-r) \times r} & A_3 \end{bmatrix} \text{ and } Y = \begin{bmatrix} A_1 & A_2 \\ I_s & O_{s \times (n-s)} \end{bmatrix}.$$

Then  $PAQ = XY$  and for  $B = P^tX$  and  $C = YQ^t$  we have  $A = BC$ , and  $B \in \mathcal{M}_{m,k}(\mathcal{S})$ ,  $C \in \mathcal{M}_{k,n}(\mathcal{S})$ . Thus  $\text{rank}(A) \leq k = t(A)$ .

4. It is well-known that the factorization of  $A$  into the product of  $m \times \rho(A)$  and  $\rho(A) \times n$  matrices exists over  $\mathcal{F}$  and there is no factorization of  $A$  into the product of  $m \times k$  and  $k \times n$  matrices for  $k < \rho(A)$ . Thus  $\text{rank}(A) \geq \rho(A)$  since  $\mathcal{S}$  is a subsemiring of  $\mathcal{F}$ .  $\square$

### EXAMPLE 3.2.

1. Note that the inequality  $\max\{r(A), c(A)\} \leq t(A)$  does not hold over any antinegative semiring  $\mathcal{S}$ , since for

$$A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} \in \mathcal{M}_{3,4}(\mathcal{S})$$

one has that  $r(A) = t(A) = 3$ ,  $c(A) = 4$ .

2. The inequality  $\min\{r(A), c(A)\} \leq t(A)$  does not hold over  $\mathbf{Z}^+$ , since for

$$A = \begin{bmatrix} 3 & 5 & 7 \\ 5 & 0 & 0 \\ 7 & 0 & 0 \end{bmatrix} \in \mathcal{M}_{3,3}(\mathbf{Z}^+)$$

one has that  $r(A) = c(A) = 3$ ,  $t(A) = 2$ .

3. The spanning column rank may actually exceed the column rank over some semirings. For example, we consider  $A = (3 - \sqrt{7}, \sqrt{7} - 2) \in \mathcal{M}_{1,2}(\mathbf{Z}[\sqrt{7}]^+)$ . Thus  $sc(A) = 2$  since  $3 - \sqrt{7} \neq \alpha(\sqrt{7} - 2)$  and  $\alpha(3 - \sqrt{7}) \neq \sqrt{7} - 2$  in  $\mathbf{Z}[\sqrt{7}]^+$ . However,  $c(A) = 1$  since  $1 = (3 - \sqrt{7}) + (\sqrt{7} - 2)$  generates the column space of  $A$ , see [8] for the details.
4. The maximal column rank may actually exceed the spanning column rank over some semirings. For example, we consider  $A = (4 - \sqrt{7}, \sqrt{7} - 2, 1) \in \mathcal{M}_{1,3}(\mathbf{Z}[\sqrt{7}]^+)$ . Thus  $sc(A) = 1$ , since 1 spans all columns of  $A$ . However similar to the previous example one can see that  $mc(A) = 2$ .

REMARK 3.3. If  $\mathcal{S} = \mathbf{B}$  is a binary Boolean semiring then  $z(A) = t(J \setminus A)$  for all  $A \in \mathcal{M}_{m,n}(\mathcal{S})$ .

#### 4. The factor rank

The inequalities for factor rank differ according to the type of arithmetic within the semiring. Let us show that the standard lower bound for the rank of sum of two matrices is not valid in general.

EXAMPLE 4.1. Let  $\mathcal{S}$  be a Boolean semiring. Then the inequality  $\text{rank}(A + B) \geq |\text{rank}(A) - \text{rank}(B)|$  need not hold.

Let us consider  $A = K_7 = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}$  and  $B = I_7$

then,  $1 = \text{rank}(J_7) = \text{rank}(A + B)$  but over any Boolean semiring  $\text{rank}(K_7) \leq 5$  due to the factorization

$$K_7 = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

(see [5] for more details). Thus  $|\text{rank}(A) - \text{rank}(B)| = \text{rank}(B) - \text{rank}(A) \geq 7 - 5 = 2 > 1 = \text{rank}(A + B)$ .

However, the following bounds are true.

PROPOSITION 4.2. *Let  $\mathcal{S}$  be an antinegative semiring,  $A, B \in \mathcal{M}_{m,n}(\mathcal{S})$ . Then*

1.  $\text{rank}(A + B) \leq \min\{\text{rank}(A) + \text{rank}(B), m, n\}$ ;
2.  $\text{rank}(A + B) \geq \begin{cases} \text{rank}(A) & \text{if } B = O \\ \text{rank}(B) & \text{if } A = O \\ 1 & \text{if } A \neq O \text{ and } B \neq O \end{cases}$ .

*These bounds are exact, the upper bound is the best possible and the lower bound is the best possible over Boolean semirings.*

*Proof.* 1. By Proposition 2.6 we have the first inequality. To prove that this bound is exact and the best possible, for each pair  $(r, s)$ ,  $0 \leq r, s \leq n$  consider the matrices  $A_r = I_r \oplus O_{n-r}$  and  $B_s = O_{n-s} \oplus I_s$  in the case  $m = n$ . It is routine to generalize the above example to the case  $m \neq n$ .

2. Since  $A + B = O$  if and only if both  $A = O$  and  $B = O$ , we have  $\text{rank}(A + B) \geq 1$  unless  $A = B = O$ , and clearly if  $A = O$ ,  $\text{rank}(A + B) = \text{rank}(B)$  and the second inequality is established. For the exactness in the second inequality, let  $A = B = E_{1,1}$ . To show that this bound is the best possible, consider the following family of matrices: for each pair  $(r, s)$ ,  $0 \leq r, s \leq m$  consider the matrices

$$A_r = \begin{bmatrix} U_r & J_{r,n-r} \\ J_{m-r,r} & J_{m-r,n-r} \end{bmatrix}$$

and

$$B_s = \begin{bmatrix} J_{s,n-s} & L_s + I_s \\ J_{m-s,n-s} & J_{m-s,s} \end{bmatrix}.$$

Then  $\text{rank}(A_r) = r$ ,  $\text{rank}(B_s) = s$  and over Boolean semirings  $\text{rank}(A_r + B_s) = 1$  since over Boolean semirings  $A_r + B_s = J_{m,n}$ . □

Let us see that the lower bound for the factor rank of a product of two matrices is not parallel to Sylvester’s lower inequality.

EXAMPLE 4.3. Let  $\mathcal{S}$  be a Boolean semiring,

$$A = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 1 & \cdots & \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

and  $B = A^t$ , then  $\text{rank}(A) = \text{rank}(B) = n$  and  $\text{rank}(AB) = 1 \not\geq \text{rank}(A) + \text{rank}(B) - n = n$  since  $AB = J$ .

The parallel proposition for the Sylvester inequalities is:

PROPOSITION 4.4. *Let  $\mathcal{S}$  be an antinegative semiring,  $A \in \mathcal{M}_{m,n}(\mathcal{S})$  and  $B \in \mathcal{M}_{n,r}(\mathcal{S})$ . Then*

1.  $\text{rank}(AB) \leq \min\{\text{rank}(A), \text{rank}(B)\}$ ,
2. Assume  $\mathcal{S}$  has no zero divisors, then

$$\text{rank}(AB) \geq \begin{cases} 0 & \text{if } \text{rank}(A) + \text{rank}(B) \leq n \\ 1 & \text{if } \text{rank}(A) + \text{rank}(B) > n \end{cases} .$$

These bounds are exact, the upper bound is the best possible and the lower bound is the best possible over Boolean semirings that have no zero divisors.

*Proof.* 1. The first inequality is verified as if the semiring were a field.

2. To prove the second part, let us note that if  $\text{rank}(A) + \text{rank}(B) \leq n$  it is possible that  $AB = O$ , for example if  $A = E_{1,1}$  and  $B = O_1 \oplus X$  for any matrix  $X \in \mathcal{M}_{n-1,r-1}(\mathcal{S})$ . However, if  $\text{rank}(A) + \text{rank}(B) > n$ ,  $AB \neq O$ . To see this, suppose that  $AB = O$ , then for some permutation matrix  $Q$ ,  $AQ = [A_1|O]$  where  $A_1$  has  $k$  columns and  $Q^tB = \begin{bmatrix} O \\ B_1 \end{bmatrix}$  where  $B_1$  has at most  $n - k$  rows, since there are no zero divisors in  $\mathcal{S}$ . But then,  $\text{rank}(A) + \text{rank}(B) \leq k + (n - k) \leq n$ , a contradiction. For exactness one can take matrices from Example 4.3. In the case  $m = n = k$  in order to show that this bound is the best possible we consider the family of matrices

$$A_r = \begin{bmatrix} L_r & O_{r,n-r} \\ J_{n-r,r} & O_{n-r,n-r} \end{bmatrix}$$

and

$$B_s = \begin{bmatrix} U_s & J_{s,n-s} \\ O_{n-s,s} & O_{n-s,n-s} \end{bmatrix}$$

for each pair  $(r, s)$ ,  $0 \leq r, s \leq n$ . Then  $\text{rank}(A_r) = r$ ,  $\text{rank}(B_s) = s$  and  $\text{rank}(A_r B_s) = 1$  if  $r, s \neq 0$ . It is routine to generalize the above example to the case  $m \neq n \neq k$ . □

EXAMPLE 4.5. The triple  $(A, I, A^t)$ , where  $A$  is as in Example 4.3, provides a counterexample to Frobenius inequality in the Boolean case.

If  $\mathcal{S}$  is a subsemiring of  $\mathfrak{R}^+$ , the positive real numbers, better bounds can be found. However the standard lower bound for the rank of sum of two matrices is not valid in this case either.

EXAMPLE 4.6. Let  $\mathcal{S}$  be a subsemiring of  $\mathfrak{R}^+$ . Then the inequality  $\text{rank}(A + B) \geq |\text{rank}(A) - \text{rank}(B)|$  need not hold.

Let  $r, s \geq 4$  and  $s < n - 4$ . Let us consider

$$A' = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 2 & 2 \end{bmatrix}$$

and

$$B' = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

Note that  $\text{rank}(A') = 4$ ,  $\rho(A') = 3$ ,  $\text{rank}(B') = \rho(B') = 1$  and  $\text{rank}(A' + B') = \rho(A' + B') = 2$ .

Let

$$A = \begin{bmatrix} A' & O_{4,r-4} & O_{4,n-r} \\ O_{r-4,4} & L_{r-4} + I_{r-4} & O_{r-4,n-r} \\ O_{m-r,4} & O_{m-r,r-4} & O_{m-r,n-r} \end{bmatrix}$$

and

$$B = \begin{bmatrix} B' & O_{4,1} & O_{4,s-1} & O_{4,n-s-4} \\ O_{s-1,4} & O_{s-1,1} & U_{s-1} & O_{s-1,n-s-4} \\ O_{m-s-3,4} & O_{m-s-3,1} & O_{m-s-3,s-1} & O_{m-s-3,n-s-4} \end{bmatrix}.$$

Here we have that  $\text{rank}(A) = r$ ,  $\rho(A) = r - 1$ ,  $\text{rank}(B) = \rho(B) = s$  and  $\text{rank}(A + B) = |r - s| - 1 < |\text{rank}(A) - \text{rank}(B)|$  if  $\mathcal{S}$  is a subsemiring of  $\mathfrak{R}^+$ . Note if  $r = s + 3$ , reversing the roles of  $A'$  and  $B'$  in  $A$  and  $B$  also gives the corresponding example.

PROPOSITION 4.7. Let  $\mathcal{S} \subseteq \mathfrak{R}^+$ ,  $A, B \in \mathcal{M}_{m,n}(\mathcal{S})$ . Then

1.  $\text{rank}(A + B) \leq \min\{\text{rank}(A) + \text{rank}(B), m, n\}$ ,
2.  $\text{rank}(A + B) \geq |\rho(A) - \rho(B)|$ .

These bounds are exact and the best possible.

*Proof.* 1. By Proposition 4.2 we have the first inequality.

2. By Proposition 3.1.4 the second inequality follows. For the exactness one can take  $A = E_{1,1} + E_{1,2} + E_{2,1}$ ,  $B = E_{2,2}$ . In order to prove that this bound is the best possible for each pair  $(r, s)$ ,  $0 \leq r, s \leq m$  we consider the family of matrices

$$A_r = \begin{bmatrix} L_r + I_r & O_{r,n-r} \\ O_{m-r,r} & O_{m-r,n-r} \end{bmatrix}$$

and

$$B_s = \begin{bmatrix} 0 & U_s & O_{s,n-s} \\ 0 & O_{m-s,s} & O_{m-s,n-s} \end{bmatrix}.$$

Then  $\text{rank}(A_r) = r = \rho(A_r)$ ,  $\text{rank}(B_s) = s = \rho(B_s)$  and  $\text{rank}(A_r + B_s) = |s - r|$ .  $\square$

In the following example we see that the standard lower bounds for the Sylvester and Frobenius inequalities also are not valid in the case  $\mathcal{S} \subseteq \mathfrak{R}^+$ .

EXAMPLE 4.8. Let  $\mathcal{S} = \mathfrak{R}^+$  and let us consider  $A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 4 \\ 1 & 1 & 4 & 0 \end{bmatrix}$

and  $B = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ , then  $\rho(A) = 3$ ,  $\text{rank}(A) = 4$ ,  $\rho(B) =$

$\text{rank}(B) = 2$  and  $AB = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 1 & 1 & 2 & 2 \\ 2 & 2 & 4 & 4 \\ 2 & 2 & 4 & 4 \end{bmatrix}$ , so  $\text{rank}(AB) = 1$ . Thus,

$1 = \text{rank}(AB) \not\geq \text{rank}(A) + \text{rank}(B) - n = 4 + 2 - 4 = 2$  and  $6 = 4 + 2 = \text{rank}(AI) + \text{rank}(IB) \not\leq \text{rank}(AIB) + \text{rank}(I) = 1 + 4 = 5$ .

However the multiplicative upper bounds and the following generalizations of lower bounds are true.

PROPOSITION 4.9. Let  $\mathcal{S} \subseteq \mathfrak{R}^+$ ,  $A \in \mathcal{M}_{m,n}(\mathcal{S})$ ,  $B \in \mathcal{M}_{n,k}(\mathcal{S})$ . Then

1.  $\text{rank}(AB) \leq \min\{\text{rank}(A), \text{rank}(B)\}$ ,
2.  $\text{rank}(AB) \geq \begin{cases} 0 & \text{if } \rho(A) + \rho(B) \leq n, \\ \rho(A) + \rho(B) - n & \text{if } \rho(A) + \rho(B) > n \end{cases}$ .

These bounds are exact and the best possible.

*Proof.* 1. The justification for the upper bounds is the same as if the semiring were a field.

2. The lower bounds are the bounds for any real matrices by Proposition 3.1.4. For the exactness in both cases one can take  $A = E_{11}$ ,  $B = I$ . For the proof that lower bound is the best possible we consider

$$A_r = \begin{bmatrix} I_r & O_{r,n-r} \\ O_{n-r,r} & O_{n-r,n-r} \end{bmatrix}$$

and

$$B_s = \begin{bmatrix} O_{n-s,n-s} & O_{n-s,s} \\ O_{s,n-s} & I_s \end{bmatrix}$$

for each pair  $(r, s)$ ,  $0 \leq r, s \leq n$  in the case  $m = n = k$ . Then  $\text{rank}(A_r) = r$ ,  $\text{rank}(B_s) = s$ ,  $\text{rank}(A_r B_s) = 0$  if  $r + s \leq n$ , and  $\text{rank}(A_r B_s) = r + s - n$  if  $r + s > n$ . It is routine to generalize the above example to the case  $m \neq n \neq k$ .  $\square$

PROPOSITION 4.10. *Let  $\mathcal{S} \subseteq \mathbb{R}^+$ ,  $A \in \mathcal{M}_{m,n}(\mathcal{S})$ ,  $B \in \mathcal{M}_{n,k}(\mathcal{S})$ , and  $C \in \mathcal{M}_{k,l}(\mathcal{S})$ . Then*

$$\rho(AB) + \rho(BC) \leq \text{rank}(ABC) + \text{rank}(B).$$

*This bound is exact and the best possible.*

*Proof.* We have  $\rho(AB) + \rho(BC) \leq \rho(ABC) + \rho(B)$  by Frobenius' inequality, and  $\rho(ABC) + \rho(B) \leq \text{rank}(ABC) + \text{rank}(B)$  since  $\text{rank}(X) \geq \rho(X)$  for all nonnegative real matrices  $A$  by Proposition 3.1.4. For the exactness take  $A = B = C = E_{1,1}$ . To prove that this bound is the best possible we consider the following family of matrices: in the case  $m = n = k = l$  for given  $r, s$  let us take

$$A_r = \begin{bmatrix} I_r & O_{r,n-r} \\ O_{n-r,r} & O_{n-r,n-r} \end{bmatrix}, \quad C_s = \begin{bmatrix} I_s & O_{s,n-s} \\ O_{n-s,s} & O_{n-s,n-s} \end{bmatrix}, \text{ and}$$

$$B_{r,s} = \begin{bmatrix} I_t & O_{t,n-t} \\ O_{n-t,t} & O_{n-t,n-t} \end{bmatrix}$$

where  $t$  is the greatest integer less than  $\frac{r+s}{2}$ . Then, if  $r \leq s$ ,  $\rho(A_r B_{r,s}) + \rho(B_{r,s} C_s) = r + t = \text{rank}(A_r B_{r,s} C_s) + \text{rank}(B_{r,s})$  and if  $s \leq r$  then  $\rho(A_r B_{r,s}) + \rho(B_{r,s} C_s) = s + t = \text{rank}(A_r B_{r,s} C_s) + \text{rank}(B_{r,s})$ . It is routine to generalize the above example to the case  $m \neq n \neq k$ ,  $n \neq l$ .  $\square$

### 5. The term rank

The following inequalities are true for the term rank:

PROPOSITION 5.1. *Let  $\mathcal{S}$  be an arbitrary antinegative semiring. For any matrices  $A, B \in \mathcal{M}_{m,n}(\mathcal{S})$  we have:*

$$t(A + B) \leq \min\{t(A) + t(B), m, n\}.$$

*This bound is exact and the best possible.*

*Proof.* This inequality follows directly from the definition of term rank. The substitution  $A_r = I_r \oplus O_{n-r}$ ,  $B_s = O_{n-s} \oplus I_s$  for each pair  $(r, s)$ ,  $0 \leq r, s \leq n$  shows that this bound is exact and the best possible in the case  $m = n$ . It is routine to generalize this example to the case  $m \neq n$ .  $\square$

EXAMPLE 5.2. A nontrivial additive lower bound for the term rank of a sum does not hold over an arbitrary semiring. It is enough to take  $A = B = J_{m,n}$  over a field whose characteristic is equal to 2. Then  $t(A + B) = t(0) = 0$ .

However for antinegative semirings there is a lower bound for the term rank of a sum which is better than the one for fields or arbitrary semirings. Namely, the following is true.

PROPOSITION 5.3. *Let  $\mathcal{S}$  be an antinegative semiring. For any matrices  $A, B \in \mathcal{M}_{m,n}(\mathcal{S})$  the following inequality holds:*

$$t(A + B) \geq \max\{t(A), t(B)\}.$$

*This bound is exact and the best possible.*

*Proof.* This inequality follows from the antinegativity of  $\mathcal{S}$ , i.e.,  $a+b \neq 0$  for any  $a, b \in \mathcal{S}$ ,  $a \neq 0$ , and the definition of the term rank. To prove that this bound is exact and the best possible we consider the matrices  $A_r = I_r \oplus O_{n-r}$ ,  $B_s = I_s \oplus O_{n-s}$  for each pair  $(r, s)$ ,  $0 \leq r, s \leq n$  in the case  $m = n$ . It is routine to generalize this example to the case  $m \neq n$ .  $\square$

EXAMPLE 5.4. A nontrivial multiplicative lower bound does not hold over an arbitrary semiring. It is enough to take  $A = B = J_n$  over a field whose characteristic is a divisor of  $n$ . Then  $t(AB) = t(nJ_n) = 0$ .

Over an antinegative semiring the Sylvester lower bound holds:

PROPOSITION 5.5. *Let  $\mathcal{S}$  be an antinegative semiring without zero divisors. Then for any  $A \in \mathcal{M}_{m,n}(\mathcal{S})$ ,  $B \in \mathcal{M}_{n,k}(\mathcal{S})$  the following inequality holds:*

$$t(AB) \geq \begin{cases} 0 & \text{if } t(A) + t(B) \leq n, \\ t(A) + t(B) - n & \text{if } t(A) + t(B) > n. \end{cases}$$

*This bound is exact and the best possible.*

*Proof.* Let  $A \in \mathcal{M}_{m,n}(\mathcal{S})$ ,  $B \in \mathcal{M}_{n,k}(\mathcal{S})$  be arbitrary matrices,  $t(A) = t_A$ ,  $t(B) = t_B$ . Then  $A$  and  $B$  have generalized diagonals with  $t_A$  and  $t_B$  nonzero elements, respectively. Denote them by  $D_A$  and  $D_B$ ,

respectively. Then  $AB \geq D_A D_B$  since  $\mathcal{S}$  is antinegative. Since the product of two generalized diagonal matrices, which have  $t_A$  and  $t_B$  nonzero entries, respectively, has at least  $t_A + t_B - n$  nonzero entries, the inequality follows.

In order to show that this bound is exact and the best possible for each pair  $(r, s)$ ,  $0 \leq r, s \leq n$  let us take  $A_r = I_r \oplus O_{n-r}$ ,  $B_s = O_{n-s} \oplus I_s$  in the case  $m = n$ . It is routine to generalize this example to the case  $m \neq n$ .  $\square$

**EXAMPLE 5.6.** The inequality  $t(AB) \leq \min(t(A), t(B))$  does not hold. It is enough to take  $A = C_1$ ,  $B = R_1$ . Then  $t(AB) = t(J_n) = n > 1$ .

However the following inequality is true

**PROPOSITION 5.7.** *Let  $\mathcal{S}$  be an antinegative semiring. Then for any  $A \in \mathcal{M}_{m,n}(\mathcal{S})$ ,  $B \in \mathcal{M}_{n,k}(\mathcal{S})$  the inequality  $t(AB) \leq \min(t_r(A), t_c(B))$  holds. This is exact and the best possible bound.*

*Proof.* This inequality is a direct consequence of the definition of the term rank and antinegativity. The exactness follows from Example 5.6. In order to prove that this bound is the best possible, for each pair  $(r, s)$ ,  $0 \leq r \leq m$ ,  $0 \leq s \leq k$ , consider the family of matrices  $A_r = E_{1,1} + \dots + E_{r,1}$  and  $B_s = E_{1,1} + \dots + E_{1,s}$ .  $\square$

**EXAMPLE 5.8.** For an arbitrary semiring, the triple  $(C_1, R_1, 0)$  is a counterexample to the term rank version of the Frobenius inequality, since  $t(C_1 R_1) + t(R_1 0) = n > t(C_1 R_1 0) + t(R_1) = 1$ . However if  $\mathcal{S}$  is a subsemiring of  $\mathfrak{R}^+$  the following obvious version is true:

$$\rho(AB) + \rho(BC) \leq t(ABC) + t(B)$$

## 6. Zero-term rank

**PROPOSITION 6.1.** *Let  $\mathcal{S}$  be an antinegative semiring. For  $A, B \in \mathcal{M}_{m,n}(\mathcal{S})$  one has that*

$$0 \leq z(A + B) \leq \min\{z(A), z(B)\}$$

*These bounds are exact and the best possible.*

*Proof.* The lower bound follows from the definition of the zero-term rank function.

In order to check that this is exact and the best possible for each pair  $(r, s)$ ,  $0 \leq r, s \leq \min\{m, n\}$  let us consider the family of matrices

$A_r = J \setminus (\sum_{i=1}^r E_{i,i})$ ,  $B_s = J \setminus (\sum_{i=1}^s E_{i,i+1})$  if  $s < \min\{m, n\}$  and  $B_s = J \setminus (\sum_{i=1}^{s-1} E_{i,i+1} + E_{s,1})$  if  $s = \min\{m, n\}$ . Then  $z(A_r) = r, z(B_s) = s$  by definition and  $z(A_r + B_s) = 0$  by antinegativity.

The upper bound follows directly from the definition of zero-term rank and from the antinegativity of  $\mathcal{S}$ . For the proof of its exactness let us take  $A = J$  and  $B = 0$ . In order to check that this bound is the best possible we consider the following family of matrices: for each pair  $(r, s)$ ,  $0 \leq r, s \leq \min\{m, n\}$  let us consider the matrices  $A_r = J \setminus (\sum_{i=1}^r E_{i,i})$  and  $B_s = J \setminus (\sum_{i=1}^s E_{i,i})$ .  $\square$

PROPOSITION 6.2. *Let  $\mathcal{S}$  be an antinegative semiring without zero divisors. For  $A \in \mathcal{M}_{m,n}(\mathcal{S})$ ,  $B \in \mathcal{M}_{n,k}(\mathcal{S})$  one has that*

$$0 \leq z(AB) \leq \min\{z(A) + z(B), k, m\}$$

*These bounds are exact and the best possible for  $n > 2$ .*

*Proof.* The lower bound follows from the definition of the zero-term rank function. In order to show that this bound is exact and the best possible let us consider the family of matrices: for each pair  $(r, s)$ ,  $0 \leq r \leq \min\{m, n\}$ ,  $0 \leq s \leq \min\{k, n\}$ , we take  $A_r = J \setminus (\sum_{i=1}^r E_{i,i})$ ,  $B_s = J \setminus (\sum_{i=1}^s E_{i,i+1})$  if  $s < \min\{k, n\}$  and  $B_s = J \setminus (\sum_{i=1}^{s-1} E_{i,i+1} + E_{s,1})$  if  $s = \min\{k, n\}$ . Then  $z(A_r) = r, z(B_s) = s$  by definition and if  $n > 2$  then  $A_r B_s$  does not have zero elements by antinegativity. Thus  $z(A_r B_s) = 0$ .

The upper bound follows directly from the definition of zero-term rank and from the antinegativity of  $\mathcal{S}$ .

In order to show that this bound is exact and the best possible let us consider the family of matrices: for each pair  $(r, s)$ ,  $0 \leq r \leq \min\{m, n\}$ ,  $0 \leq s \leq \min\{k, n\}$ , we take  $A_r = J \setminus (\sum_{i=1}^r R_i)$  and  $B_s = J \setminus (\sum_{i=1}^s C_i)$ .  $\square$

EXAMPLE 6.3. The triple  $(C_1, I, R_1)$  is a counterexample to the zero-term rank version of the Frobenius inequality, since  $z(C_1) + z(R_1) = 2n - 2 > z(C_1 R_1) + z(I) = n$  for  $n > 2$ .

## 7. Row and column ranks

The standard upper bound in the additive inequalities is not valid for row and column ranks.

EXAMPLE 7.1. Consider

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 2 & 2 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \\ 0 & 6 & 2 \\ 0 & 4 & 6 \end{bmatrix}.$$

Then it is easy to see that  $r(A) = r(B) = sr(A) = sr(B) = mr(A) = mr(B) = 2$ . However,

$$r(A+B) = r \begin{bmatrix} 1 & 2 & 2 \\ 0 & 3 & 0 \\ 1 & 1 & 2 \\ 1 & 6 & 2 \\ 0 & 5 & 6 \end{bmatrix} = 5 = sr(A+B) = mr(A+B)$$

over  $\mathbf{Z}^+$ .

PROPOSITION 7.2. Let  $\mathcal{S}$  be an antinegative semiring. Then for  $0 \neq A, B \in \mathcal{M}_{m,n}(\mathcal{S})$  one has that

$$1 \leq c(A+B), r(A+B), sr(A+B), sc(A+B), mr(A+B), mc(A+B)$$

These bounds are exact over any antinegative semiring and the best possible over Boolean semirings.

*Proof.* These inequalities follow directly from the definition of these rank functions and the condition that  $A, B \neq 0$ . For the exactness one can take  $A = B = E_{1,1}$ . Let  $\mathcal{S}$  be a Boolean semiring. For each pair  $(r, s)$ ,  $0 \leq r, s \leq m$  we consider the matrices  $A_r = J \setminus (\sum_{i=1}^r E_{i,i})$ ,

$B_s = J \setminus (\sum_{i=1}^s E_{i,i+1})$  if  $s < m$  and  $B_s = J \setminus (\sum_{i=1}^{s-1} E_{i,i+1}) + E_{s,1}$  if  $s = m$ .

Then

$$c(A_r) = r(A_r) = sr(A_r) = sc(A_r) = mr(A_r) = mc(A_r) = r,$$

$$c(B_s) = r(B_s) = sr(B_s) = sc(B_s) = mr(B_s) = mc(B_s) = s$$

by definition and  $A_r + B_s = J$  has row and column ranks equal to 1. Thus, these bounds are the best possible over Boolean semirings.  $\square$

PROPOSITION 7.3. Let  $\mathcal{S}$  be a subsemiring in  $\mathfrak{R}^+$ . Then for  $A, B \in \mathcal{M}_{m,n}(\mathcal{S})$  one has that

$$c(A+B), r(A+B), sr(A+B), sc(A+B), \\ mr(A+B), mc(A+B) \geq |\rho(A) - \rho(B)|$$

These bounds are exact and the best possible.

*Proof.* These inequalities follow directly from the fact that  $\rho(X) \leq r(X), c(X)$  for all  $X \in \mathcal{M}_{m,n}(\mathfrak{R}^+)$ , Proposition 3.1.2, and corresponding inequalities for matrices with coefficients from the field  $\mathfrak{R}$ . For the proof of exactness consider matrices  $A = E_{1,1} + \dots + E_{n-1,n-1}$ ,  $B = J \setminus A$ . In order to show that these bounds are the best possible one can take the family of matrices  $A_r, B_s$  that show that the lower bound in Proposition 4.7 is the best possible.  $\square$

**PROPOSITION 7.4.** *Let  $\mathcal{S}$  be an antinegative semiring without zero divisors. For  $0 \neq A \in \mathcal{M}_{m,n}(\mathcal{S})$ ,  $0 \neq B \in \mathcal{M}_{n,k}(\mathcal{S})$  and  $c(A) + r(B) > n$ , one has that*

$$1 \leq c(AB), r(AB), sr(AB), sc(AB), mr(AB), mc(AB)$$

*These bounds are exact over any antinegative semiring without zero divisors and the best possible over Boolean semirings without zero divisors.*

*Proof.* For an arbitrary antinegative semiring, if  $c(A) = i$  and  $r(B) = j$  then  $A$  has at least  $i$  nonzero columns while  $B$  has at least  $j$  nonzero rows. Thus, if  $i + j > n$ ,  $AB \neq O$  and hence these bounds are established. For the proof of exactness let us take  $A = B = E_{1,1}$ .

Let  $\mathcal{S}$  be a semiring with Boolean arithmetic and without zero divisors. In the case  $m = n = k$  for each pair  $(r, s)$ ,  $1 \leq r, s \leq n$  let us consider the matrices  $A_r = \sum_{i=1}^r E_{i,i} + \sum_{i=1}^m E_{i,1}$ ,  $B_s = \sum_{i=1}^s E_{i,i} + \sum_{i=1}^n E_{1,i}$ . Then

$$c(A_r) = r(A_r) = sr(A_r) = sc(A_r) = mr(A_r) = mc(A_r) = r,$$

$$c(B_s) = r(B_s) = sr(B_s) = sc(B_s) = mr(B_s) = mc(B_s) = s$$

by definition and  $A_r B_s = J$ . Thus  $c(A_r B_s) = r(A_r B_s) = sr(A_r B_s) = sc(A_r B_s) = mr(A_r B_s) = mc(A_r B_s) = 1$ . It is routine to generalize this example to the case  $m \neq n \neq k$ .  $\square$

Note that the condition that  $c(A) + r(B) > n$  is necessary because for  $A = I_k \oplus O$  and  $B = O \oplus I_j$  we have  $AB = O$  whenever  $k + j \leq n$ .

**PROPOSITION 7.5.** *Let  $\mathcal{S}$  be a subsemiring in  $\mathfrak{R}^+$ . Then for  $A \in \mathcal{M}_{m,n}(\mathcal{S})$   $B \in \mathcal{M}_{n,k}(\mathcal{S})$  one has that*

$$c(AB), sc(AB), mc(AB), r(AB), sr(AB), mr(AB) \geq \begin{cases} 0 & \text{if } \rho(A) + \rho(B) \leq n, \\ \rho(A) + \rho(B) - n & \text{if } \rho(A) + \rho(B) > n \end{cases}$$

*These bounds are exact and the best possible.*

*Proof.* These inequalities follow directly from the fact that  $\rho(X) \leq r(X), c(X)$  for all  $X \in \mathcal{M}_{m,n}(\mathfrak{R}^+)$ , Proposition 3.1.2, and corresponding inequalities for matrices with coefficients from the field  $\mathfrak{R}$ . For the exactness one can take  $A = B = I$ . In order to show that these bounds are the best possible one can take the family of matrices  $A_r, B_s$  that show that the lower bound in Proposition 4.9 is the best possible.  $\square$

The following example, given in [8] for the spanning column rank, shows that standard analogs for upper bound of the rank of product of two matrices do not work for row and column ranks.

EXAMPLE 7.6. Let  $A = (3, 7, 7) \in \mathcal{M}_{1,3}(\mathbf{Z}^+)$ ,  $B = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ .

Then  $c(A) = sc(A) = mc(A) = 2$ ,  $c(B) = sc(B) = mc(B) = 3$ , and  $c(AB) = c(3, 10, 17) = sc(AB) = mc(AB) = 3$ .

However, the following upper bounds are true due to [8].

PROPOSITION 7.7. *Let  $\mathcal{S}$  be an antinegative semiring. For  $A \in \mathcal{M}_{m,n}(\mathcal{S})$ ,  $B \in \mathcal{M}_{n,k}(\mathcal{S})$ , one has that*

$$c(AB) \leq c(B), sc(AB) \leq sc(B), mc(AB) \leq mc(B),$$

$$r(AB) \leq r(A), sr(AB) \leq sr(A), mr(AB) \leq mr(A)$$

*These bounds are exact and the best possible.*

*Proof.* The inequality  $sc(AB) \leq sc(B)$  is proved in [8, Formula 2.4]. The other inequalities can be proved in essentially the same way. For the proof that these bounds are exact and the best possible, consider  $A_r = I_r \oplus O_{n-r}$  and  $B_s = I_s \oplus O_{n-s}$  for each pair  $r, s$ ,  $1 \leq r, s \leq n$  in the case  $m = n$ . It is routine to generalize this example to the case  $m \neq n$ .  $\square$

EXAMPLE 7.8. The triple  $(A, I, B)$ , where  $A$  and  $B$  are from Proposition 7.4 is a counterexample to the corresponding Frobenius inequalities if  $r$  and  $s$  are chosen such that  $r + s > n + 1$ .

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