Journal of
Ship&Ocean
Technology

SOTECH Vol. 9, No. 1, pp. 64~71, 2005

Preliminary Research on the Uncertainty Estimation in the Probabilistic Designs

Byung D. Youn[1] and Jae-Hwan Lee[2]

[1] Dpartment of Mechanical Engineering, University of Detroit Mercy, Detroit, U.S.A.
[2] Department of Naval Architecture and Ocean Eng., Chungnam National University, Daejeon, Korea; E-mail: jaelee@cnu.ac.kr

Abstract

In probabilistic design, the challenge is to estimate the uncertainty propagation, since outputs of subsystems at lower levels could constitute inputs of other systems or at higher levels of the multilevel systems. Three uncertainty propagation estimation techniques are compared in this paper in terms of numerical efficiency and accuracy: root sum square (linearization), distribution-based moment approximation, and Taguchi-based integration. When applied to reliability-based design optimization (RBDO) under uncertainty, it is investigated which type of applications each method is best suitable for. Two nonlinear analytical examples and one vehicle crashworthiness for side-impact simulation example are employed to investigate the unique features of the presented techniques for uncertainty propagation. This study aims at helping potential users to identify appropriate techniques for their applications in the multilevel design.
Keywords: reliability based optimization, uncertainty propagation, experimental design, random variables

1 Introduction

Probabilistic design does not only entail the difficulty of formulating and solving non-deterministic optimization problems; it is also quite challenging to model the mechanism of uncertainty propagation throughout the multilevel hierarchy. Outputs of subsystems at lower levels could constitute inputs of other systems or at higher levels of the multilevel systems which refers to the optimization process of large, complex engineering systems that are decomposed into a hierarchy of subsystems. It is thus necessary to estimate the statistical information of these outputs (which could be also inputs of subsystems at higher levels) with adequate accuracy without requiring a huge amount of raw data. Analytical target cascading (ATC) formulation enables the use of first-order Taylor series (Greenwood and Chase 1990) for approximating nonlinear responses. In response to these new requirements, the ATC formulation has been extended to solve probabilistic design optimization problems. The coupled interactions between subsystems need to be taken into consideration to achieve consistent designs. ATC is a methodology that takes these interactions into account during the early stages of the design optimization process and the ATC consistency constraints do not allow large deviations from the incumbent expansion point (which are the mean values of the design variables) during the optimization process. In this manner, not only nonlinear responses can be linearized, but also they can be considered as normally distributed if all the random variables they depend on were also normally distributed. Although large approximation errors of expected values for the nonlinear responses are avoided, the convergence rate of the ATC process can be low since many iterations involving small "steps" may be necessary. In addition, this estimation technique may exhibit relatively large errors when approximating higher-order statistical moments (Youn and Choi 2004).

This paper considers two alternative methods for estimating statistical moments of nonlinear responses of random variables in order to prepare the multilevel design. The first method generates approximate probability density functions to be numerically integrated. The second method uses numerical quadrature rules motivated by Taguchi-type experimental designs (Youn and Choi 2004). The scope of this paper is to investigate the stability, accuracy, and efficiency of these two methods when applied on simulation-based, reliability optimization problems, and to determine which type of applications each method is best suitable for. And this could be the basis for the reliability optimization of multilevel system design.

2 Probabilistic design process: reliability-based design optimization

As a parametric design process, the RBDO model can be generally defined (Youn and Choi 2004) as

$$\begin{aligned} &\min\ \text{Cost}(\mathbf{d}) \\ &\text{s.t. } P(G_i(\mathbf{X}) \le 0) - \Phi(-\beta_t) \le 0,\ i = 1, \cdots, NP \\ &\quad \mathbf{d}^L \le \mathbf{d} \le \mathbf{d}^U,\ \mathbf{d} \in R^{NDV} \text{ and } \mathbf{X} \in R^{NRV} \end{aligned} \tag{1}$$

where $\mathbf{d} = \mu(\mathbf{X})$ is the design vector, $\mathbf{X}$ is the random vector, and the probabilistic constraints are described by the performance function $G_i(X)$ with $G_i(X) < 0$ as a failure, their probabilistic models, and their prescribed confidence level β_t.

Through inverse transformation, the probabilistic constraint in equation (1) can be further expressed in two distinct forms as:

$$\beta_{s_i} = \left(-\Phi^{-1}\left(F_{G_i}(0)\right)\right) \ge \beta_t \tag{2}$$

$$G_{p_i} = F_{G_i}^{-1}\left(\Phi(-\beta_t)\right) \ge 0 \tag{3}$$

where β_{s_i} and G_{P_i} are respectively referred to as the safety reliability index and the probabilistic performance measure for the ith probabilistic constraint. Using the reliability index, equation (3) is then employed to describe the probabilistic constraint in equation (1), i.e., the so-called reliability index approach (RIA). Similarly, equation (4) can replace the probabilistic constraint in equation (1) with the performance measure, referred to as the performance measure approach (PMA).

There are three major advantages in using PMA as compared to RIA (Youn and Choi 2004). First, it is found that PMA is inherently robust and more effective when the probabilistic constraint is either very much feasible or very much infeasible. Second, and more significantly, PMA always yields a solution, whereas RIA may not yield solutions for certain types of distributions, such as Gumbel or uniform distributions. Third, it is also found that PMA is more effective than RIA when the response surface method (RSM) is used for RBDO. Therefore, PMA is only presented in this study, rather than employing ineffective RIA.

First-order reliability analysis in PMA

The first-order reliability analysis in PMA can be formulated as the inverse of the first-order reliability analysis in RIA. The first-order probabilistic performance measure G_p is obtained from an optimization problem with an n-dimensional explicit sphere constraint in U-space, defined as

$$\begin{aligned} \text{To find } \mathbf{u}^*_{\beta_t},\ \ \text{minize} \quad & G(\mathbf{U}) \\ \text{subject to} \quad & \|\mathbf{U}\| = \beta_t \end{aligned}, \qquad (4)$$

The optimum point on a target reliability surface is identified as the most probable point (MPP) $\mathbf{u}^*_{\beta_t}$.

3 Uncertainty propagation techniques

The solution of a probabilistic design problem requires information on the distribution and moments of the random design variables and parameters. Typically, this information is given or postulated at the bottom level of a probabilistic multilevel system design problem. However, since the outputs of lower-level problems constitute inputs to higher-level problems, we must propagate the uncertainty information as accurate as possible to solve the higher-level problems and the overall multilevel design problem. In this section we present two alternative techniques for estimating uncertainty propagation.

3.1 Advanced mean value based distribution generation and moment estimation

The main idea of this technique is to perform a reliability analysis on the output response (i.e., the nonlinear response of the random variables) using first order reliability method (FORM) for a sufficiently large range of reliability targets, e.g., from $\beta = 4$ (with corresponding probability of failure $P_f = \Phi(-\beta) = 0.00003$) to $\beta = -4$ (with $P_f = \Phi(-\beta) = 0.99997$). Once the most probable point is found by the hybrid mean value (HMV) method (Youn and Choi 2004), the output response is evaluated at this point to provide the "corrected" function value for the corresponding probability of failure (Wu et al 1990). With the cumulative density function (CDF) available, one can then differentiate numerically to obtain the probability density function (PDF) (Du and Chen 2001). Central differences are used to obtain second-order accurate approximations. Finally, numerical integration is performed using spline interpolation to estimate response values that lie between the available "discrete" points of the PDF, to compute moments. As will be shown later by means of preliminary numerical results, this method is quite accurate. However, it can be inefficient depending on how the "β-range" is "discretized".

3.2 Taguchi-based integration and moment estimation

3.2.1 Output statistical moment modeling: numerical integration on input domain

One purpose of statistical moment estimation stems from the robust design optimization, which attempts to minimize the quality loss (Chandra 2001, Taguchi et al 1989), which is a function of the statistical mean and standard deviation. Several methods are proposed to estimate the first two statistical moments of the output response. Analytically, the statistical moments are expressed in an integration form as

$$
\begin{aligned}
E[R] &= \int_{-\infty}^{\infty} R(\mathbf{x}) f_{\mathbf{x}}(\mathbf{x}) d\mathbf{x} \\
E[(R(\mathbf{x}) - \mu_R)]^k &= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} (R(\mathbf{x}) - \mu_R)^k f_{\mathbf{x}}(\mathbf{x}) d\mathbf{x}
\end{aligned} \tag{5}
$$

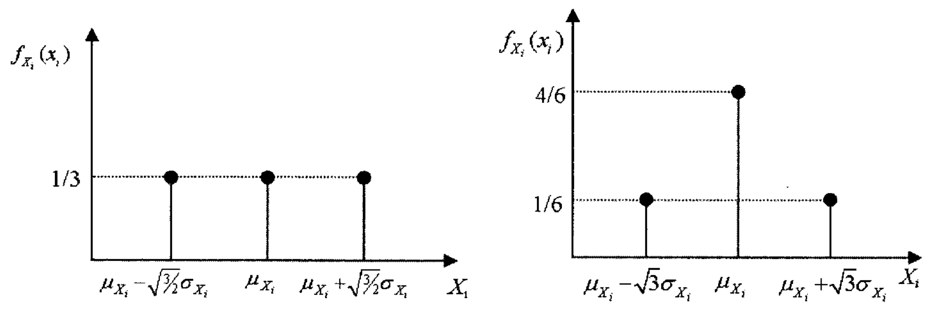
Using numerical integration, the statistical moments of output response are approximated through numerical integration on the input domain as

$$
\begin{aligned}
E[R] &\cong \overline{\mu}_R \\
&= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} R(\mathbf{x}) \prod_{i=1}^{n} f_{X_i}(x_i) dx_1 \cdots dx_n \\
&= \sum_{j_1=1}^{m} w_{j_1} \cdots \sum_{j_n=1}^{m} w_{j_n} R(\mu_1 + \alpha_{j_1}, \cdots, \mu_n + \alpha_{j_n}) \\
E[R(\mathbf{x}) - \mu_R]^k &\cong \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} (R(x) - \mu_R)^k \prod_{i=1}^{n} f_{X_i}(x_i) dx_1 \cdots dx_n \\
&= \sum_{j_1=1}^{m} w_{j_1} \cdots \sum_{j_n=1}^{m} w_{j_n} \left[R(\mu_1 + \alpha_{j_1}, \cdots, \mu_n + \alpha_{j_n}) - \overline{\mu}_R \right]^k
\end{aligned} \tag{6}
$$

For application, Taguchi (1978, 1989) proposed an experimental design approach for statistical tolerance design with a three-level (m=3) factorial experiment, which are composed of low, center, and high levels as $\{w_1, w_2, w_3, a_1, a_2, a_3\} = \{1/3, 1/3, 1/3, -\sqrt{3/2}, 0, \sqrt{3/2}\}$.

Three-level factorial experiment is modified by D'Errico and Zaino(1988) by employing distinctive weights at different levels as $\{w_1, w_2, w_3, a_1, a_2, a_3\} = \{1/6, 4/6, 1/6, -\sqrt{3}, 0, \sqrt{3}\}$.

Thus, the modified three-level factorial experiment improved numerical accuracy in estimating the statistical moments of output response. In numerical integration, three weights for X_i are used to approximate the probability density of X_i at three different probability levels. From the statistical point of view, the modified three-level factorial experiment is meaningful, since many random input variables follows the rule of high density near the mean and low density at the tail of statistical distribution, as shown in figure 1.



(a) Taguchi Method (1978) (b) D'Errico and Zaino Method (1988)

Figure 1: Three-level numerical integration on the input domain

In the experimental method, the computation of the moment could be very expensive for a large number of design and/or random parameters, since the number of function evaluations or experiments required is $N = 3^n$ where n is a number of design and random parameters. Thus, this method is not used in this paper.

3.2.2 Output statistical moment modeling: numerical integration on output domain

In Section 3.2.1, statistical moments of output response are estimated through numerical integration on the input domain, making it very expensive for reliability-based robust design optimization. In this paper, the proposed method directly identifies uncertainty propagation using numerical integration on the output domain. Unlike equation (5), the statistical moment calculation is carried out by

$$\begin{aligned} E[R]^1 &= \int_{-\infty}^{\infty} r f_R(r)dr = \mu_R \\ E[(R-\mu_R)]^k &= \int_{-\infty}^{\infty} (r-\mu_R)^k f_R(r)dr \end{aligned} \tag{7}$$

where $f_R(r)$ is a probability density function of *R*. To approximate the statistical moments of R accurately, N-point numerical quadrature technique can be used as

$$\begin{aligned} E[R]^1 &= \mu_R \cong \sum_{i=1}^{N} w_i r_i \quad \text{and} \\ E[(R-\mu_R)]^k &\cong \sum_{i=1}^{N} w_i (r_i - \mu_R)^k \quad \text{for} \quad 2 \le k \le 5 \end{aligned} \tag{8}$$

At minimum, the three-point integration (N=3) is required to maintain a good accuracy in estimating first two statistical moments. By solving equation (8), three levels and weights on the output domain are obtained as $\{r_1, r_2, r_3\} = \{r_{\beta=-\sqrt{3}}, r(\mu_x), r_{\beta=+\sqrt{3}}\}$ and $\{w_1, w_2, w_3\} = \{1/6, 4/6, 1/6\}$ as shown in figure 2. In general, upper and lower levels are not symmetrically located, as shown in figure 2.

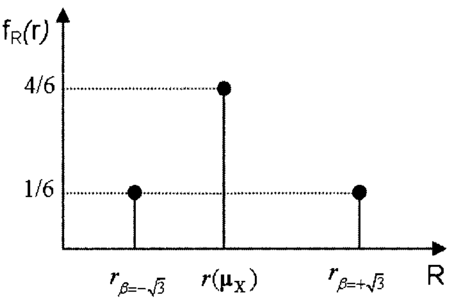


Figure 2: Three-level numerical integration on the output domain

Using the three-level numerical integration on the output domain, the first two statistical moments in equation (7), the mean and standard variation of the output response are approximated to be

$$E[R]^1 = \mu_R \cong \frac{1}{6} r_{\beta=-\sqrt{3}} \frac{4}{6} r(\boldsymbol{\mu}_X) + \frac{1}{6} r_{\beta=+\sqrt{3}}$$

$$E[R-\mu_R]^2 = \sigma_R^2 = \int_{-\infty}^{\infty} (r-\mu_R)^2 f_R(r) dr \qquad (9)$$

$$\cong \frac{1}{6}\left(r_{\beta=-\sqrt{3}} - \mu_R\right)^2 + \frac{1}{6}\left(r_{\beta=+\sqrt{3}} - \mu_R\right)^2$$

Since the statistical moments of output response are estimated through a numerical integration on the output (or performance) domain, this method is called performance moment integration (PMI) method. In the PMI method, $r_{\beta=-\sqrt{3}}$ and $r_{\beta=+\sqrt{3}}$ are obtained through reliability analyses (Wu 1990, Du 2001) at $\beta = \pm\sqrt{3}$ confidence levels. In this paper, HMV method is used for reliability analysis.

4 Numerical examples

Two nonlinear analytical examples and one vehicle crashworthiness for side-impact simulation example are used to demonstrate the aforementioned techniques. For abbreviation purposes, the method presented in Section 3.1 is called distribution-based method (DBM) in this paper. Monte Carlo simulation (MCS) with one million samples and root sum square (RSS) method are used for numerical comparison. Experimental methods (Taguchi 1989) are not used for comparison because it would be too expensive even though it would be as accurate as MCS. Statistical non-normality of the response functions is represented by skewness and kurtosis. Skewness is a measure of symmetry of probability density function (a normal distribution has a skewness value of 0). Kurtosis is a measure of relative peakness/flatness of probability density function to normal distribution, which has a kurtosis value of 3.

The first analytical example, the response is

$$R_1(\mathbf{X}) = 1 - X_1^2 X_2 / 20 \qquad (10)$$

For this example, the input random parameters are modeled as $X_i \sim N(5.0, 0.3)$ for i=1,2. As shown in table 1 and figure 3, the probabilistic distribution of the first response is close to a normal distribution with a moderate rate of skewness and kurtosis. Thus, RSS, DBM, and performance moment integration (PMI) show overall a good accuracy in estimating the first two statistical moments of responses.

The second analytical example response is

$$R_2(\mathbf{X}) = -e^{X_1-7} - X_2 + 10 \qquad (11)$$

The input random parameters are modeled as $X_i \sim N(6.0, 0.8)$ for i=1,2. As shown in table 1, the RSS method yields a large approximation error of 107% for the second moment, whereas the DBM and PMI methods are much more accurate for both the mean and standard deviation.

The last single-level example R_3 is the pubic force from a side impact simulation (Youn Choi, Yang and Gu), which is modeled with input uncertainties of Gumbel distribution and 10% coefficient of variation. Even though the stochastic response is highly skewed with large kurtosis, the PMI method seems to predict the first two statistical moments accurately whereas the RSS could yield larger errors. DBM results are not available for this example.

Table 1: Single-level examples

	Mean				Standard Deviation				Skew.	Kurt.
	RSS	DBM	PMI	MCS	RSS	DBM	PMI	MCS		
R1	-5.2500	-5.286	-5.286	-5.2719	0.8385	0.842	0.8411	0.8405	-0.26	3.11
Error, %	0.415	-0.259	-0.259	--	-0.238	0.17	0.071	--	--	--
R2	3.6321	3.6029	3.6082	3.4937	1.9386	0.9013	0.8800	0.9349	-0.57	7.13
Error, %	3.961	3.125	3.277	--	107.4	-3.593	-5.872	--		
R3	-1.4100	N/A	-1.4135	-1.4291	0.0632	N/A	0.0685	0.0708	-0.99	4.93
Error, %	1.337	N/A	1.092	--	-10.73	N/A	-3.248	--	--	--

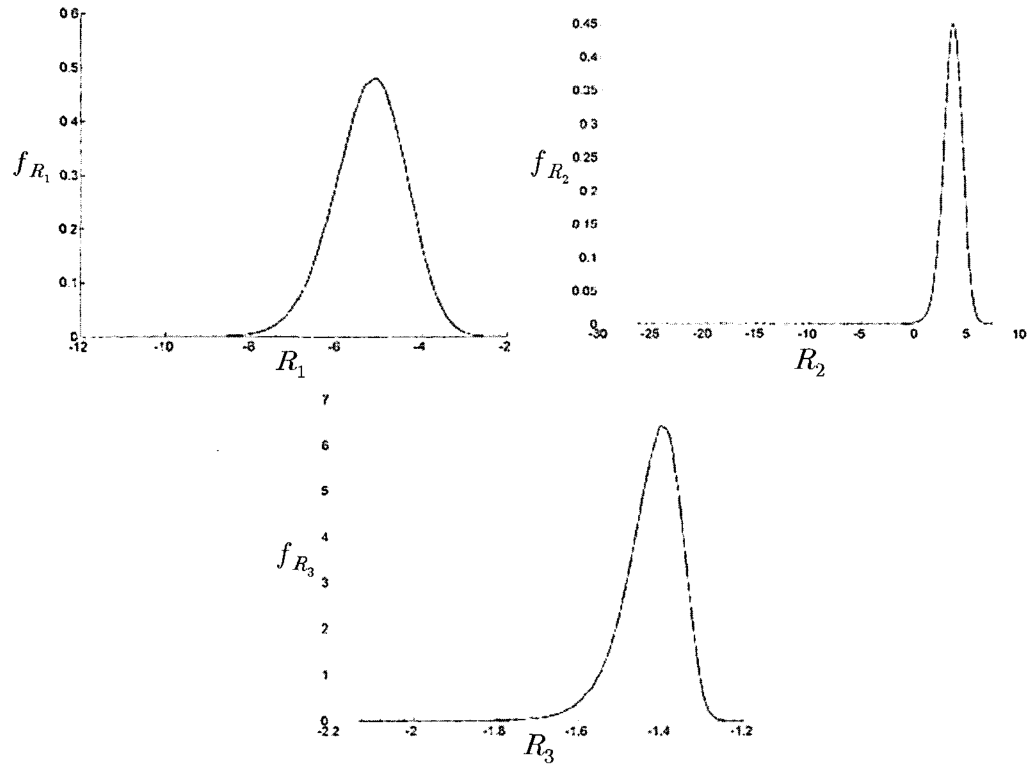


Figure 3: PDF of R_1 (left), R_2 (right), and R_3 (bottom)

5 Discussion and conclusions

Two alternative techniques for estimating uncertainty propagation in probabilistic design of multilevel systems were presented in the paper. The first method generates approximate probability density functions, which are then integrated numerically to obtain statistical moments (DBM). The second method uses numerical quadrature rules to estimate statistical moments of output response (PMI). The methods were successfully applied to model the uncertainty propagation mechanism by estimating statistical moments efficiently and accurately. The scope of this paper was to investigate the stability, accuracy, and efficiency of these two methods when applied on simulation-based, multilevel system design optimization problems, and to determine which type of applications each method is best suited for. It was found that both DBM and PMI estimate statistical moments accurately for nonlinear responses with high skewness and kurtosis. Thus, PMI and DBM successfully carried out the probabilistic design optimization of multilevel hierarchical system. The methods were compared to the RSS method and Monte Carlo simulation was performed to compare the estimation of statistical moments. PMI and DBM are more accurate to assess statistical moments than RSS. PMI can be useful for many nonlinear engineering systems, since it is computationally inexpensive yet accurate. On the other hand, DBM can be more accurate but is computationally more expensive. Finally, DBM should be used when the probabilistic design of multilevel systems requires generating distributions of nonlinear responses, i.e., when moments are not adequate to model propagation of uncertainties.

Acknowledgments

This research was supported by ASERC of KOSEF and CNU Academic Foundation. The support is really acknowledged.

References

Chandra, M. J. 2001. Statistical Quality Control, CRC Press, Chapter 3, Boca Raton, FL.

D'Errico, J.R. and N.A. Zaino Jr. 1988. Statistical tolerancing using a modification of taguchi's method. Technometrics, **30**, **4**, 397-405.

Du, X. and W. Chen. 2001. A most probable point-based method for efficient uncertainty analysis. J. Design and Manufacturing Automation **1**, **1-2**, 47-66.

Greenwood, W.H. and K.W. Chase. 1990. Root sum squares tolerance analysis with nonlinear problems. ASME J. Engineering for Industry, **112**, 382-384.

Taguchi, G. 1978. Performance analysis design. International J. Production Research, **16**, **6**, 521-530.

Taguchi, G., E. Elsayed and T. Hsiang. 1989. Quality Engineering in Production Systems McGraw-Hill, Chapters 2 and 3, NY, U.S.A.

Wu, Y.T., H.R. Millwater and T.A. Cruse. 1990. Advanced probabilistic structural analysis method of implicit performance functions. AIAA Journal, **28**, **9**.

Youn, B.D., K.K. Choi. 2004. An investigation of nonlinearity of reliability-based design optimization approaches. J. Mechanical Design, ASME, **126**, **3**, 403-411.

Youn, B.D., K.K. Choi. 2004. A new response surface methodology for reliability-based design optimization. Computers and Structures, **82**, **2**, 241-256.

Youn, B.D. and K.K. Choi. 2004. Performance moment integration (PMI) method for reliability-based robust design optimization. To appear in Proceedings of 2004 ASME Design Engineering Technical Conferences: 30th Design Automation Conference, UT, U.S.A.

Youn, B.D. and K.K. Choi. 2004. Selecting probabilistic approaches for reliability-based design optimization. AIAA Journal, **42**, **1**.

Youn, B.D., K.K. Choi, R.J. Yang and L. Gu. Reliability-based design optimization for crashworthiness of side impact. To appear in the J. Structural and Multidisciplinary Optimization.