International Journal of Fuzzy Logic and Intelligent Systems, vol. 5, no. 3, September 2005 pp. 259-262

Initial L-fuzzy quasi-uniform structures

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Abstract

We prove the existence of initial L-fuzzy (quasi-)uniform structures. From this fact, we can define subspaces and products of L-fuzzy (quasi-)uniform spaces.

Key Words : L-fuzzy (quasi-)uniform structures, initial L-fuzzy (quasi-)uniform structures, subspaces and products of L-fuzzy (quasi-)uniform spaces.

1. Introduction

Sostak [11] introduced the notion of L-fuzzy topological spaces as a generalization of $[0,1]$-topological spaces [2]. Moreover, Samanta [10] introduced the concept of $[0,1]$-fuzzy uniform spaces as an expansion of Hutton L-uniform spaces [5]. Kim [7] defined an L-fuzzy (resp. quasi-)uniform space in a somewhat different view of the definition of Samanta [10].

In this paper, we will prove the existence of initial L-fuzzy (quasi-)uniform structures. From this fact, we can define subspaces and products of L-fuzzy (quasi-)uniform spaces.

2. Preliminaries

In this paper, let L be a nonempty set. Let $L=(L,\le,\vee,\wedge,')$ be a completely distributive lattice with an order-reversing involution ', 0 and 1 denote the least and the greatest element in L. For $\alpha\in L$, $\underline{\alpha}(x)=\alpha$ for each $x\in L$ and $L_1=L-\{1\}$.

Let Ω_X denote the family of all functions $f: L^X\to L^X$ with the following properties:

(1) $f(\underline{0})=\underline{0}$, $\mu\le f(\mu)$, for every $\mu\in L^X$,

(2) $f(\bigvee\mu_i)=\bigvee f(\mu_i)$, for $\mu_i\in L^X$.

For $f,g\in\Omega_X$, we define, for all $\mu\in L^X$,

$$f^{-1}(\mu)=\bigwedge\{\rho\in L^X \mid f(\underline{1}-\rho)\le\underline{1}-\mu\},$$

$$(f\triangledown g)(\mu)=\bigwedge\{f(\mu_1)\vee g(\mu_2)\mid \mu_1\vee\mu_2=\mu\},$$

$$(f\circ g)(\mu)=f(g(\mu)).$$

Then $f^{-1}, f\triangledown g, f\circ g\in\Omega_X$.

Lemma 2.1([5-8]) For every $f,g,h,f_1,g_1\in\Omega_X$, the following properties hold:

(1) If $f\le f_1, g\le g_1$, then $f\triangledown g\le f_1\triangledown g_1$.

(2) $f\triangledown g\le f$, $f\triangledown g\le g$, $(f\triangledown g)\triangledown h=f\triangledown(g\triangledown h)$ and $f\triangledown f=f$.

(3) $(f^{-1})^{-1}=f$.

(4) $f\le g$ iff $f^{-1}\le g^{-1}$.

(5) $(g\circ f)^{-1}=f^{-1}\circ g^{-1}$.

(6) $(f\triangledown g)^{-1}=f^{-1}\triangledown g^{-1}$.

(7) For $\rho\in L^X$, we define $f_\rho: L^X\to L^X$ as follows:

$$f_\rho(\lambda)=\begin{cases}\underline{0} & \text{if } \lambda=\underline{0},\\ \rho & \text{if } \underline{0}\ne\lambda\le\rho,\\ \underline{1} & \text{otherwise}.\end{cases}$$

Then:

(a) $f_\rho\in\Omega_X$ and $f_\rho^{-1}=f_{\underline{1}-\rho}$.

(b) $f_\rho\circ f_\rho=f_\rho$ and $(f_\rho\triangledown f_\mu)\circ(f_\rho\triangledown f_\mu)=f_\rho\triangledown f_\mu$

(c) $f_{\underline{1}}=f_{\underline{1}}^{-1}$, $f\triangledown f_{\underline{1}}=f$ and $f\le f_{\underline{1}}$ for all $f\in\Omega_X$.

Definition 2.2 ([7]) A function $U:\Omega_X\to L$ is said to be an L-fuzzy quasi-uniformity on X if it satisfies the following conditions:

(QU1) $U(f_1\triangledown f_2)\ge U(f_1)\wedge U(f_2)$, for all $f_1,f_2\in\Omega_X$.

(QU2)} For $f\in\Omega_X$, we have

$$\bigvee\{U(f_1)\mid f_1\circ f_1\le f\}\ge U(f).$$

(QU3) If $f_1\ge f$, then $U(f_1)\ge U(f)$.

(QU4) There exists $f\in\Omega_X$ such that $U(f)=1$.

The pair (X,U) is said to be an L-fuzzy quasi-uniform space.

An L-fuzzy quasi-uniform space (X,U) is called an L-fuzzy uniform space if it satisfies:

(U) For $f\in\Omega_X$, $\bigvee\{U(f_1)\mid f_1\le f^{-1}\}\ge U(f)$.

Let U_1 and U_2 be L-fuzzy (resp.quasi-)uniformities on X . We say U_1 is finer than U_2.(or U_2 is coarser than U_1) , denoted by $U_2\le U_1$, iff for any $f\in\Omega_X$, $U_2(f)\le U_1(f)$.

Let (X,U) be an L-fuzzy quasi-uniform space. We define for $f\in\Omega_X$, $U^{-1}(f)=U(f^{-1})$ From Lemma 2.1, we easily show that U^{-1} is an L-fuzzy quasi-uniformity on X.

Manuscript received Aug. 8, 2005; revised Sep. 9, 2005.

Lemma 2.3 ([5-8]) Let $\psi: X\to Y$ be a function. For each $f\in\Omega_Y$, a function $\psi^{-1}(f): L^X\to L^X$ is defined by, for all $\mu\in L^X$,

$$\psi^{-1}(f)(\mu)=(\psi^{\leftarrow}\circ f\circ\psi^{\rightarrow})(\mu)=\psi^{\leftarrow}(f(\psi^{\rightarrow}(\mu))).$$

For $f, f_1, f_2\in\Omega_Y$, we have the following properties.

(1) $\psi^{-1}(f)\in\Omega_X$.

(2) If $f_1\le f_2$, then $\psi^{-1}(f_1)\le\psi^{-1}(f_2)$.

(3) $\psi^{-1}(f_1)\circ\psi^{-1}(f_2)\le\psi^{-1}(f_1\circ f_2)$ with equality if ψ is onto.

(4) $(\psi^{-1}(f))^{-1}=\psi^{-1}(f^{-1})$.

(5) $\psi^{-1}(f_1)\nabla\psi^{-1}(f_2)=\psi^{-1}(f_1\nabla f_2)$

(6) $\psi^{\rightarrow}((\psi^{-1}(f))^{-1}(\lambda))\le f^{-1}(\psi^{\rightarrow}(\lambda))$, for all $\lambda\in L^X$.

(7) $\psi^{-1}(f_\rho)=f_{\psi^{-1}(\rho)}$.

Definition 2.4 ([7]) Let (X, U) and (Y, V) be L-fuzzy (quasi-)uniform spaces. A function $\psi: (X, U)\to(Y, V)$ is LF-uniformly continuous if $V(f)\le U(\psi^{-1}(f))$, for every $f\in\Omega_Y$.

Theorem 2.5([7]) Let $(X, U), (Y, V)$ and (Z, W) be L-fuzzy (quasi-)uniform spaces. If $\psi: (X, U)\to(Y, V)$ and $\phi: (Y, V)\to(Z, W)$ are LF-uniformly continuous, then $\phi\circ\psi: (X, U)\to(Z, W)$ is LF-uniformly continuous.

Theorem 2.6([7]) Let (X, U) and (Y, V) be L-fuzzy (quasi-)uniform spaces. If $\psi: (X, U)\to(Y, V)$ is LF-uniformly continuous, then $\psi: (X, U^{-1})\to(Y, V^{-1})$ is LF-uniformly continuous.

3. Initial L-fuzzy quasi-uniformity structures

Theorem 3.1 Let $\{(X_k, V_k)\mid k\in\Gamma\}$ be a family of L-fuzzy (resp. quasi-)uniform spaces, X a set and for each $k\in\Gamma$, $\psi_k: X\to X_k$ a function. We define a function $U:\Omega_X\to L$ by

$$U(f)=\bigvee\{\bigwedge_{i=1}^{n} V_{k_i}(f_{k_i})\mid \nabla_{i=1}^{n}\psi_{k_i}^{-1}(f_{k_i})\le f\}$$

where the $\bigvee$ is taken over every finite index $K=\{k_1,\dots,k_n\}\subset\Gamma$. Then:

(1) The structure U is the coarsest L-fuzzy (resp. quasi-)uniformity on X for which each ψ_k is LF-uniformly continuous.

(2) A map $f: (Z, W)\to(X, U)$ is LF-uniformly continuous iff for each $k\in\Gamma$, $\psi_k\circ f: (Z, W)\to(X_k, V_k)$ is LF-uniformly continuous.

Proof (1) First, we will show that U is an L-fuzzy (resp. quasi-)uniformity on X.

(QU1) Suppose there exists $f, g\in\Omega_X$ such that

$$U(f\nabla g)\not\ge U(f)\wedge U(g).$$

Since L is a completely distributive lattice, by the definition of $U(f)$, there exists a finite index set $K=\{k_1,\dots,k_n\}\subset\Gamma$ such that

$$U(f\nabla g)\not\ge(\bigwedge_{i=1}^{n} V_{k_i}(f_{k_i}))\wedge U(g),\quad \nabla_{i=1}^{n}\psi_{k_i}(f_{k_i})\le f.$$

Also, by definition of $U(g)$, there exists a finite index set $L=\{p_1,\dots,p_m\}\subset\Gamma$ such that

$$U(f\nabla g)\not\ge(\bigwedge_{i=1}^{n} V_{k_i}(f_{k_i}))\wedge(\bigwedge_{j=1}^{m} V_{p_j}(g_{p_j})),$$
$$\nabla_{j=1}^{m}\psi_{p_j}(g_{p_j})\le g.$$

Since $\nabla_{i=1}^{n}\psi_{k_i}(f_{k_i}))\nabla(\nabla_{j=1}^{m}\psi_{p_j}^{-1}(g_{p_j}))\le f\nabla g$, by Lemma 2.3 (5), we have

$$U(f\nabla g)\ge(\bigwedge_{i=1}^{n} V_{k_i}(f_{k_i}))\wedge(\bigwedge_{j=1}^{m} V_{p_j}(g_{p_j})).$$

It is a contradiction. Hence $U(f\nabla g)\ge U(f)\wedge U(g)$ for all $f, g\in\Omega_X$.

(QU2) Suppose there exists $f\in\Omega_X$ such that

$$\bigvee\{U(f_1)\mid f_1\circ f_1\le f\}\not\ge U(f).$$

By the definition of $U(f)$, there exists a finite index $K=\{k_1,\dots,k_n\}\subset\Gamma$ such that

$$\bigvee\{U(f_1)\mid f_1\circ f_1\le f\}\not\ge\bigwedge_{i=1}^{n} V_{k_i}(f_{k_i}),$$
$$\nabla_{i=1}^{n}\psi_{k_i}(f_{k_i})\le f.$$

For each $k_i\in K$, since (X_{k_i}, V_{k_i}) is an L-fuzzy (resp. quasi-)uniform space, by (QU2),

$$\bigvee\{V_{k_i}(h)\mid h\circ h\le f_{k_i}\}\ge V_{k_i}(f_{k_i}).$$

Since L is a completely distributive lattice, for each $k_i\in K$, there exists $g_{k_i}\in\Omega_{X_{k_i}}$ with $g_{k_i}\circ g_{k_i}\le f_{k_i}$ such that

$$\bigvee\{U(f_1)\mid f_1\circ f_1\le f\}\not\ge\bigwedge_{i=1}^{n} V_{k_i}(g_{k_i}).$$

Put $g=\nabla_{i=1}^{n}\psi_{k_i}^{-1}(g_{k_i})$. For each $k_i\in K$, we have

$$g\circ g\le\psi_{k_i}^{-1}(g_{k_i})\circ\psi_{k_i}^{-1}(g_{k_i}).$$

Hence, by Lemma 2.1 (1) and (2),

$$\begin{aligned} g\circ g&\le\nabla_{i=1}^{n}(\psi_{k_i}^{-1}(g_{k_i})\circ\psi_{k_i}^{-1}(g_{k_i}))\\ &\le\nabla_{i=1}^{n}(\psi_{k_i}^{-1}(g_{k_i}\circ g_{k_i}))\ \text{(by Lemma 2.3(3))}\\ &\le\nabla_{i=1}^{n}\psi_{k_i}(f_{k_i})\le f.\end{aligned}$$

Then we have $g\circ g\le f$ and

$$U(g)\ge\bigwedge_{i=1}^{n} V_{k_i}(g_{k_i}).$$

It is a contradiction.

Hence $\bigvee\{U(f_1)\mid f_1\circ f_1\le f\}\ge U(f)$, for all $f\in\Omega_X$.

(QU3) It is trivial.

(QU4) Since each (X_k, V_k) is an L-fuzzy (resp. quasi-)uniform space, by (QU4), there exists $f_k \in \Omega_{X_k}$ such that $V_k(f_k)=1$. For all finite indices $K=\{k_1,\ldots,k_n\}\subset \Gamma$, put $f=\nabla_{i=1}^n \psi_{k_i}(f_{k_i})$. Then there exists $f\in\Omega_X$ such that $U(f)=1$.

(U) Let $\{(X_k, U_k) \mid k\in\Gamma\}$ be a family of L-fuzzy uniform spaces. Suppose that there exists $f\in\Omega_X$ such that

$$\bigvee\{U(f_1) \mid f_1\le f^{-1}\}\not\ge U(f).$$

By the definition of U, there exists a finite index $K=\{k_1,\ldots,k_n\}\subset\Gamma$ such that

$$\bigvee\{U(f_1) \mid f_1\le f^{-1}\}\not\ge \bigwedge_{i=1}^n V_{k_i}(f_{k_i})$$
$$\nabla_{i=1}^n \psi_{k_i}(f_{k_i})\le f.$$

For each $k_i\in K$, since (X_{k_i}, V_{k_i}) is an L-fuzzy uniform space, by (U),

$$\bigvee\{V_{k_i}(h) \mid h\le f_{k_i}^{-1}\}\ge V_{k_i}(f_{k_i}).$$

For each $k_i\in K$, there exists $g_{k_i}\in\Omega_{X_{k_i}}$ with $g_{k_i}\le f_{k_i}^{-1}$ such that

$$\bigvee\{U(f_1) \mid f_1\le f^{-1}\}\not\ge \bigwedge_{i=1}^n V_{k_i}(g_{k_i}).$$

On the other hand, we have

$$\begin{aligned}\nabla_{i=1}^n \psi_{k_i}^{-1}(g_{k_i}) &\le \nabla_{i=1}^n \psi_{k_i}^{-1}(f_{k_i}^{-1}) \text{ (by Lemma 2. 3(2))}\\ &= \nabla_{i=1}^n (\psi_{k_i}^{-1}(f_{k_i}))^{-1} \text{ (by Lemma 2.3(4))}\\ &= (\nabla_{i=1}^n \psi_{k_i}^{-1}(f_{k_i}))^{-1} \text{ (by Lemma 2.1 (6))}\\ &\le f^{-1}. \text{ (by Lemma 2.1 (4))}\end{aligned}$$

Put $g=\nabla_{i=1}^n \psi_{k_i}(g_{k_i})$. Then there exists $g\in\Omega_X$ such that

$$g\le f^{-1},\quad U(g)\ge \bigwedge_{i=1}^n V_{k_i}(g_{k_i}).$$

Thus

$$\bigvee\{U(f_1) \mid f_1\le f^{-1}\}\ge U(g)\ge \bigwedge_{i=1}^n V_{k_i}(g_{k_i}).$$

It is a contradiction. Hence $\bigvee\{U(f_1) \mid f_1\le f^{-1}\}\ge U(f)$, for all $f\in\Omega_X$.

Second, by the definition of U, for all $k\in\Gamma$, $f_k\in\Omega_{X_k}$, $U(\psi_k^{-1}(f_k))\ge V_k(f_k)$. Hence each $\psi_k: (X,U)\to(X_k, V_k)$ is LF-uniformly continuous.

Finally, if $\psi_k: (X,U^*)\to(X_k,V_k)$ is LF-uniformly continuous ,that is, $U^*(\psi_k^{-1}(f))\ge V_k(f)$ for all $k\in\Gamma$, then it is proved that $U^*\ge U$ from the following:

for all $f\in\Omega_X$

$$\begin{aligned}U(f) &= \bigvee\{\bigwedge_{i=1}^n V_{k_i}(f_{k_i}) \mid \nabla_{i=1}^n \psi_{k_i}^{-1}(f_{k_i})\le f\}\\ &\le \bigvee\{\bigwedge_{i=1}^n U^*(\psi_{k_i}^{-1}(f_{k_i})) \mid \nabla_{i=1}^n \psi_{k_i}^{-1}(f_{k_i})\le f\}\\ &\le \bigvee\{U^*(\nabla_{i=1}^n \psi_{k_i}^{-1}(f_{k_i})) \mid \nabla_{i=1}^n \psi_{k_i}^{-1}(f_{k_i})\le f\}\\ &\le U^*(f).\end{aligned}$$

(2) Necessity of the composition condition is clear since the composition of LF-uniformly continuous maps is LF-uniformly continuous.

Conversely, suppose that $\psi: (Z, W)\to(X, U)$ is not LF-uniformly continuous. There exists $f\in\Omega_X$ such that

$$W(\psi^{-1}(f))\not\ge U(f).$$

By the definition of U, there exists a finite index set $K=\{k_1,\ldots,k_n\}\subset\Gamma$ such that

$$W(\psi^{-1}(f))\not\ge \bigwedge_{i=1}^n V_{k_i}(f_{k_i}),\quad \nabla_{i=1}^n \psi_{k_i}^{-1}(f_{k_i})\le f.$$

On the other hand, for each $k_i\in K$, since $\psi_{k_i}\circ\psi$ is LF-uniformly continuous, we have

$V_{k_i}(f_{k_i})\le W(\psi_{k_i}\circ\psi)^{-1}(f_{k_i})) = W(\psi^{-1}(\psi_{k_i}^{-1}(f_{k_i})))$.

It follows that

$$\begin{aligned}\bigwedge_{i=1}^n V_{k_i}(f_{k_i}) &\le \bigwedge_{i=1}^n W(\psi^{-1}\circ\psi_{k_i}^{-1}(f_{k_i}))\\ &\le W(\nabla_{i=1}^n \psi^{-1}(\psi_{k_i}^{-1}(f_{k_i})))\\ &= W(\psi^{-1}(\nabla_{i=1}^n \psi_{k_i}^{-1}(f_{k_i})))\\ &\qquad \text{(by Lemma 2. 3(5))}\\ &\le W(\psi^{-1}(f)).\end{aligned}$$

It is a contradiction.

The category of LF-fuzzy uniform spaces and LF-uniformly continuous maps is denoted by **L-UNIF**.

Theorem 3.2 The forgetful functor W: **L-UNIF** →**Set** defined by $W(X,U)=X$ and $W(\psi)=\psi$ is topological.

Proof From Theorem 3.1, every W-structured source $(\psi_i: X\to W(X_i, U_i))_{i\in\Gamma}$ has an unique W-initial lift $(\psi_i: (X,U)\to(X_i,U_i))_{i\in\Gamma}$.

Using Theorems 3.1 and 3.2 and Definition 2.5, we obtain the following definition.

Definition 3.3 Let $\{(X_i, U_i)\}_{i\in\Gamma}$ be a family of L-fuzzy (resp. quasi-)uniform spaces, X a set and $\psi_i: X\to X_i$ a function, for each $i\in\Gamma$. The initial L-fuzzy (resp. quasi-) uniform structure U on X with respect to $(X, \psi_i, (X_i, U_i), \Gamma)$ is the coarsest L-fuzzy (resp. quasi-) uniform structure on X for which all $\psi_i, i\in\Gamma$, are LF-uniformly continuous maps.

Definition 3.4 Let (X, V) be an L-fuzzy (resp. quasi-)uniform space and A a subset of X. The pair (A, V_A) is said to be a subspace of (X, V) if it is endowed with the initial L-fuzzy (resp. quasi-)uniformity structure with respect to $(A, i, (X, V))$ where i is the inclusion map.

Definition 3.5 Let $X=\prod_{i\in\Gamma} X_i$ be the product of sets from the family $\{(X_i, U_i) \mid i\in\Gamma\}$ of L-fuzzy (resp. quasi-)uniform spaces. The initial L-fuzzy (resp. quasi-)uniformity structure $U=\otimes U_i$ on X with respect to the family $\{\pi_i: X\to(X_i, U_i) \mid i\in\Gamma\}$ of all projection maps is called the product L-fuzzy (resp. quasi-)uniformity structure of $\{U_i \mid i\in\Gamma\}$, and $(X, \otimes U_i)$ is called the product L-fuzzy (resp. quasi-)uniform space.

Corollary 3.6 Let $(X, U_i)_{i\in\Gamma}$ be a family of L-fuzzy (resp. quasi-)uniform spaces. We define, for $f\in\Omega_X$,

$$U(f)=\bigvee\{\bigwedge_{i=1}^{n} U_{k_i}(f_{k_i}) \mid \nabla_{i=1}^{n} f_{k_i}\le f\},$$

where the $\bigvee$ is taken over every finite index $K=\{k_1,\dots,k_n\}\subset\Gamma$. Then the structure U is the coarsest L-fuzzy (resp. quasi-)uniformity on X finer than U_i.

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