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# Convergence of Interval-valued Choquet integrals

**Dug Hun Hong* and Kyung Tae Kim****

***Department of Mathematics, Myongji University, Yongin, Kyonggi, Korea**
****Department of Electronics and Electrical Information Engineering, Kyungwon University, Sungnam Kyunggido, South Korea**

**Abstract**

Recently, many types of set-valued fuzzy integrals are studied by many authors. In this paper, we consider various types of convergence theorems of Choquet integrals of interval-valued function with respect to an autocontinuous fuzzy measure.

**Key Words** : Fuzzy measure, Choquet integral, autocontinuity

## 1. Introduction

It is well known that set-valued functions have been used repeatedly in Economics. Set-valued functions and their integration have been studied by many authors[2, 7-10, 17]. Recently, Jang et al. [10] considered a convergence theorem for inter-valued Choquet integrals under very restrictive condition. In this paper, we consider various types of convergence theorems of Choquet integrals of inter-valued function with respect to an autocontinuous fuzzy measure under very mild conditions. We also prove the result of Jang et al.[10] under simple condition.

## 2. Preliminaries

**Definition 2.1** [9, 13] (1) A fuzzy measure on a measurable space $(X, \Im)$ is an extended real-valued function $\mu: \Im \rightarrow [0, \infty]$ satisfying

(*i*) $\mu(\emptyset)=0$
(*ii*) $\mu(A) \leq \mu(B)$, whenever $A, B \in \Im$, $A \subset B$.

(2) A fuzzy measure $\mu$ is said to be autocontinuous from above [resp.,below] if $\mu(A \cup B_n) \rightarrow \mu(A)$

$$[resp., \mu(A \sim B_n) \rightarrow \mu(A)]$$

whenever $A \in \Im$, $\{B_n\} \subset \Im$ and $\mu(B_n) \rightarrow 0$.
(3) If $\mu$ is autocontinuous both from above and from below, it is said to be autocontinuous.
Recall that a function $f: X \rightarrow [0, \infty]$ is said to be measurable if $\{x \mid f(x) > \alpha\} \in \Im$ for all $\alpha \in (-\infty, \infty)$.
**Definition 2.2** [13] (1) A sequence $\{f_n\}$ of measurable functions is said to converge to $f$ in measure, in symbols $f_n \rightarrow_M f$ if for every $\varepsilon > 0$,

$$\lim_{n\rightarrow\infty} \mu(\{x \mid |(f_n(x) - f(x))| > \varepsilon\}) = 0.$$

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(2) A sequence $\{f_n\}$ of measurable functions is said to converge to $f$ in distribution, in symbols $f_n \rightarrow_D f$ if for every $\varepsilon > 0$,

$$\lim_{n\rightarrow\infty} \mu_{f_n}(r) = \mu_f(r) \ e.c.,$$

where $\mu_f(r) = \mu(\{x \mid f(x) > r\})$ and "e.c." stands for "except at most countably many values of $r$"

**Definition 2.3** [11-12] (1) The Choquet integral of a measurable function $f$ with respect to a fuzzy measure $\mu$ is defined by

$$(C)\int f \, d\mu = \int_0^\infty \mu_f(r) dr,$$

where the integral on the right-hand side is an ordinary one.
(2) A measurable function $f$ is called integrable if the Choquet integral of $f$ can be defined and its value is finite.
Throughout this paper, $R^+$ will denote the interval $[0, \infty)$, $I(R^+) = \{[a, b] | a, b \in R^+ \text{ and } a \leq b\}$. Then an element in $I(R^+)$ is called an interval number. On the interval number set, we define for each pair $[a, b], [c, d] \in I(R^+)$ and $k \in R^+$,

$[a, b] + [c, d] = [a+c, b+d]$,
$[a, b] \cdot [c, d] = [a \cdot c, b \cdot d]$,
$k[a, b] = [ka, kb]$
$[a, b] \leq [c, d]$ if and *only* if $a \leq c$ and $b \leq d$.

Then $(I(R^+), d_H)$ is a metric space, where $d_H$ is the Hausdorff metric defined by

$$d_H(A, B) = \max\{\sup_{x\in A} \inf_{y\in B} |x-y|, \ \sup_{y\in B} \inf_{x\in A} |x-y|\}$$

for all $A, B \in I(R^+)$. By the definition of the Hausdorff metric, we have immediately the following proposition.

**Proposition 2.4** For each pair
$[a, b], [c, d] \in I(R^+)$,
$d_H([a, b], [c, d]) = \max\{|a-c|, |b-d|\}$.

Let $C(R^+)$ be the class of closed subsets of $R^+$. Throughout this paper, we consider a closed set-valued function $F: X \to C(R^+)\backslash\{\emptyset\}$ and an interval number-valued function
$F: X \to I(R^+)\backslash\{\emptyset\}$. We denote that $d_H - \lim_{n\to\infty} A_n = A$ if and only if
$\lim_{n\to\infty} d_H(A_n, A) = 0$, where $A \in I(R^+)$ and $\{A_n\} \subset I(R^+)$.

**Definition 2.5** [1 ,6, 7] A closed set-valued function $F$ is said to be measurable if for each open set $O \subset R^+$,

$$F^{-1}(O) = \{x \in X \mid F(x) \cap O \neq \emptyset\} \in \mathfrak{I}$$

**Definition 2.6** [1] Let $F$ be a closed set-valued function. a measurable function $f: X \to R^+$ satisfying $f(x) \in F(x)$ for all $x \in X$ is call a measurable selection of $F$.
We say $f: X \to R^+$ is in $L_c^1(\mu)$ if and only if $f$ is measurable and $(C)\int f\, d\mu < \infty$. We note that " $x \in X\ \mu-a.e.$" stands for "$x \in X\ \mu-$almost everywhere". The property $p(x)$ holds for $x \in X\ \mu-a.e.$ means that there is a measurable set $A$ such that $\mu(A) = 0$ and the property $p(x)$ holds for all $x \in A^c$, where $A^c$ is the complement of $A$

**Definition 2.7** [6, 7] (1) Let $F$ be a closed set-valued function and $A \in \mathcal{A}$ . The Choquet integral of $F$ on $A$ is defined by

$$(C)\int_A F\, d\mu = \{(C)\int_A f\, d\mu \mid f \in S_c(F)\},$$

where $S_c(F)$ is the family of $\mu-a.e.$ . Choquet integrable selection of $F$, that is,

$$S_c(F) = \{f \in L_c^1(\mu) \mid f(x) \in F(x),\ x \in X \quad \mu-a.e.\}.$$

(2) A closed set-valued function $F$ is said to be Choquet integrable if $(C)\int F d\mu \neq \emptyset$.

(3) A closed set-valued function $F$ is said to be Choquet integrably bounded if there is a function $g \in L_c^1(\mu)$ such that

$$\|F(x)\| = \sup_{r \in F(x)} |r| \le g(x) \text{ for all } x \in X.$$

Instead of $(C)\int_X F\, d\mu$ , we will write $(C)\int F\, d\mu$.

Recall that a measurable closed set-valued function is said to e convex-valued if $F(x)$ is convex for all $x \in X$ and that a set $A$ is an interval number if and only if it is closed and convex.

**Theorem 2.8** [9] If $F$ is a measurable closed set-valued function and Choquet integrably bounded and if we define $f^*(x) = \sup\{r \mid r \in F(x)\}$ and $f_*(x) = \inf\{r \mid r \in F(x)\}$ for all $x \in X$ , then $f^*$ and $f_*$ are Choquet integrable selection of $F$.

## 3. Convergence of interval-valued Choquet integral

In this section, we consider Fatou's lemma, the Lebesgue convergence theorem, monotone convergence theorem and uniform integrability related convergence theorem. We begin with the concept of convergence of a sequence of elements in $I(R^+)$.
Let $\{A_n\} \subset I(R^+)$ be a sequence. We define
$\limsup A_n = \{x \mid x = \lim_{k\to\infty} x_{n_k},\ x_n \in A_n,\ n \ge 1\}$
and
$\liminf A_n = \{x \mid x = \lim x_n,\ x_n \in A_n,\ n \ge 1\}$

If $\limsup A_n = \liminf A_n = A$ , we say that $\{A_n\}$ is convergent to $A$, and it is simply written as $\lim A_n = A$ or $A_n \to A$. .

We note that if $\lim A_n = A$ and $A = [a, b]$ then $d_H - \lim_{n\to\infty} A_n = A$.

**Lemma 3.1** If $\{F_n\}$ is a sequence of interval-valued Choquet integrably bounded functions, then

(1) $\lim\ \sup(C)\int F_n\, d\mu$
$= [\liminf(C)\int f_{n*}\, d\mu,\ \limsup(C)\int f_n^*\, d\mu]$,

(2) $\lim\ \inf(C)\int F_n d\mu$
$= [\limsup(C)\int f_{n*}\, d\mu,\ \liminf(C)\int f_n^* d\mu]$,

**Proof.** Let $y \in \limsup(C)\int F_n\, d\mu$. Then we have that $y = \lim_{k\to\infty} y_{n_k}$ and
$y_n \in (C)\int F_n\, d\mu,\ n \ge 1$. So
$y_{n_k} = (C)\int f_{n_k}\, d\mu \to y$ as $k \to \infty$ where
$f_{n_k} \in S_c(F_{n_k})$. Since $f_{n*}(x) \le f_n(x) \le f_n^*(x)$ for all $x \in X$,

$$(C)\int f_{n*}\, d\mu \le (C)\int f_n\, d\mu \le (C)\int f_n^*\, d\mu,$$

and hence

$$y \in [\liminf(C)\int f_{n*}\, d\mu,\ \limsup(C)\int f_n^*\, d\mu].$$

Conversely, if $y < \liminf(C)\int f_{n^*}\, d\mu$, then for some small $\varepsilon > 0$, there exist $N > 0$ such that for any $n \ge N$, $y \le (C)\int f_{n^*}\, d\mu - \varepsilon$. Then clearly $y \notin \limsup(C)\int F_n d\mu$. Similarly, if $y > \limsup(C)\int f_n^*\, d\mu$ , we can easily prove that $y \notin \limsup(C)\int F_n\, d\mu$, which completes (1). The proof of (2) is similar.

**Theorem 3.2** (Fatou's lemma) Let $\mu$ be finite and autocontinuous. Let $\{F_n\}$ be a sequence of interval-valued measurable functions. If there exist Choquet integrable functions $G$ and $H$ such that $G \le F_n \le H$ for all $n$,

$$\limsup (C)\int F_n d\mu \subset (C)\int \limsup F_n d\mu.$$

**Proof.** It is easy to check that $(C)\int \limsup F_n d\mu = [(C)\int \liminf f_{n*}\, d\mu,\ (C)\int \limsup f_n^*\, d\mu]$. Hence by Lemma 3.1 and Fatou's lemma( Theorem 3.2 [6], Corollary 3.3 [6] ), noting that $\inf_{k\ge n} f_k \to_D \liminf f_n$ and $\sup_{k\ge n} f_k \to_D \limsup f_n$ by the assumption that $\mu$ be finite and autocontinuous., we get the result.

Noting that

$$(C)\int \lim\ \inf\ F_n\, d\mu = [(C)\int \limsup f_{n^*}\, d\mu,\ (C)\int \liminf f_n^*\, d\mu],$$

we similarly have the following result.

**Theorem 3.3** (Fatou's lemma) Let $\mu$ be finite and autocontinuous. If there exist Choquet integrable functions $G$ and $H$ such that $G \le F_n \le H$ for all $n$,

$$(C)\int \liminf\ F_n\, d\mu \subset \liminf (C)\int F_n\, d\mu.$$

We next consider Lebesgue Dominate Convergence Theorem.

**Theorem 3.4** If $\{F_n\}$ is a sequence of interval-valued measurable function such that $f_{n*} \to_D f_*$ and $f_n^* \to_D f^*$ and if $G$ and $H$ are Choquet integrably bounded such that $G \le F_n \le H$ for all $n$, then

$$\lim (C)\int F_n\, d\mu = (C)\int F\, d\mu.$$

**Proof.** By Theorem 3.2 [13], We have

$$\lim_{n\to\infty}(C)\int f_{n^*}\, d\mu = \int f_*\, d\mu \text{ and } \lim (C)\int f_n^* d\mu = \int f^* d\mu.$$

Hence by Lemma 3.1,

$$\limsup (C)\int F_n\, d\mu = \liminf (C)\int F_n\, d\mu = [(C)\int F_*\, d\mu,\ (C)\int F^*\, d\mu]$$

which means $\lim (C)\int F_n\, d\mu = (C)\int F\, d\mu.$

**Note.** If $\sup_{x\in X} d_H(F_n(x), F(x)) \to 0$, as $n \to \infty$ then we easily see that $f_{n^*} \to_D f_*$ and $f_n^* \to_D f^*$. Hence Theorem 3.5 of Jang et al. [10] is an easy consequence of Theorem 3.4.

**Definition 3.5** A sequence interval-valued $\{F_n\}$ is uniformly integrable if $\{\|F_n\|\}$ is uniformly integrable, i.e. ,

$$\lim_{\alpha\to\infty} \sup_n (C)\int \|F_n\| 1_{[\|F_n\|>\alpha]}\, d\mu = 0.$$

**Theorem 3.6** Let $\mu$ be finite and let be a sequence of interval-valued uniformly integrable measurable functions such that $f_{n^*} \to_D f_*$ and $f_n^* \to_D f^*$.

Them we have

$$\lim_{n\to\infty}(C)\int F_n\, d\mu = (C)\int F\, d\mu.$$

**Proof.** Since $\{F_n\}$ is uniformly integrable, both $\{f_{n^*}\}$ and $\{f_n^*\}$ are uniformly integrable. Then by Theorem 3.4 [6], we have

$$\lim_{n\to\infty}(C)\int f_{n^*}\, d\mu = (C)\int f_*\, d\mu$$

and

$$\lim_{n\to\infty}(C)\int f_n^* d\mu = (C)\int f^* d\mu.$$

Hence by Lemma 3.1, the result follows.

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**Dug Hun Hong** received the B.S., M.S. degrees in mathematics from Kyungpook National University , Taegu, Korea and Ph. D degree in mathematics from University of Minnesota, Twin City in 1981, 1983 and 1990, respectively. From 1991 to 2003, he worked with department of Statistics and School of Mechanical and Automotive Engineering, Catholic University of Daegu, Daegu, Korea, Since 2004, he has been a Professor in Department of Mathematics, Myongji University, Korea. His research interests include general fuzzy theory with application and probability theory.

Phone : 031-330-6156
Fax : 031-330-9528
e-mail : dhhong@mju.ac.kr

**Kyung Tae Kim** received the B.S. degree in Electrical Engineering from Kyungpook National University, Taegu, Korea and M.E and Ph. D degrees in Electrical Engineering from Yonsei University, Seoul, Korea in 1978, 1980 and 1987, respectively. Since 1987, he has been a Professor in Department of Electronics and Electrical Information Engineering, Kyungwon University, Korea. His research interests include mobile communications and optical communications.

e-mail : ktkim@kyungwon.ac.kr